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On a new class of impulsive fractional evolution equations

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Abstract

This paper is concerned with the existence of *PC*-mild solutions for Cauchy and nonlocal problems of impulsive fractional evolution equations for which the impulses are not instantaneous. By using the theory of operator semigroups, probability density functions, and some suitable fixed point theorems, we establish some existence results for these types of problems.

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Keywords: fractional evolution equations; impulses; *PC*-mild solutions; Cauchy problems; nonlocal problems

1 Introduction

Recently, much attention has been paid to the study of fractional differential equations due to the fact that they have been proved to be valuable tools in the mathematical modeling of many phenomena in physics, biology, mechanics, *etc.* (see [1–3]).

Impulsive differential equations of integer order have found extensive applications in realistic mathematical modeling of a wide variety of practical situations, such as biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, and frequency modulated systems. For the general theory and relevant developments of impulsive differential equations, please see [4–9] and the references therein. Usually the impulses of the evolution process described by impulsive differential equations are assumed to be abrupt and instantaneous. That is to say, the perturbations (impulses) start abruptly and the duration of them is negligible in comparison with the duration of the process.

However, in [10], the authors introduced a new class of abstract impulsive differential equations for which the impulses are not instantaneous. Specifically, they studied the existence of solutions for the following impulsive problem:

$$\begin{cases} u'(t) = Au(t) + f(t,u(t)), & t \in (s_i, t_{i+1}], i = 0,1,2,\dots,N, \\ u(t) = g_i(t,u(t)), & t \in (t_i, s_i], i = 1,2,\dots,N, \\ u(0) = x_0, \end{cases}$$

where $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, $x_0 \in X$, $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots <$

$t_N \le s_N < t_{N+1} = a$ are pre-fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$ for $i = 1, 2, \ldots, N$ and $f : [0, a] \times X \to X$ is a suitable function. The impulses start abruptly at the points t_i and their action continues on the interval $[t_i, s_i]$. As a motivation for the study of such systems, see [10], where an example of the hemodynamical equilibrium of a person was given.

Impulsive differential equations of fractional order have been studied by some authors, for example [11–17]. As for the study of impulsive fractional evolution equations, to the best of our knowledge, there are few papers [18–20] on this topic.

Motivated by [10], in this paper we consider a class of impulsive fractional evolution equations of the form

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, \\ x(t) = I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ x(0) = x_0, \end{cases} \tag{1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$ with the lower limit zero, $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $\{T(t)\}_{t\ge0}$ on a Banach space $(X, \|\cdot\|)$, $x_0 \in X$, $0 = t_0 = s_0 < t_1 \le s_1 < t_2 \le s_2 < \cdots < t_m \le s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$, $I_i : X \to X$ for $i = 1, 2, \ldots, m$ and $f : [0, T] \times X \to X$ is a nonlinear function.

The impulses in problem (1) start abruptly at the points t_i and their action continues on the interval $[t_i, s_i]$. To be precise, the function x takes an abrupt impulse at t_i and follows different rules in the two subintervals $(t_i, s_i]$ and $(s_i, t_{i+1}]$ of the interval $(t_i, t_{i+1}]$. At the point s_i, the function x is continuous. The term $I_i(x(t_i))$ means that the impulses are also related to the value of $x(t_i) = x(t_i^-)$.

From the results obtained in the papers [21–24], we know that the definition of mild solutions for fractional evolution equations is more involved than integer order evolution equations. Moreover, to construct solutions for impulsive fractional differential equations, we should properly handle the fractional derivative and impulsive conditions due to the memory property of fractional calculus (see [11–13]).

We remark that if $t_i = s_i$ and the second equation of (1) takes the form of $\Delta x(t_i) = I_i(x(t_i)) = x(t_i^+) - x(t_i^-)$ with $x(t_i^+) = \lim_{\epsilon\to0^+} x(t_i + \epsilon)$, $x(t_i^-) = \lim_{\epsilon\to0^-} x(t_i + \epsilon)$ representing the right and left limits of $x(t)$ at $t = t_i$, problem (1) reduces to the case considered in [20] (with the fixed impulses).

We also study the nonlocal Cauchy problems for impulsive fractional evolution equations

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, \\ x(t) = I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ x(0) = x_0 + b(x), \end{cases} \tag{2}$$

where A, f, I_i, g_i are the same as above, b is a given function; this constitutes a nonlocal Cauchy problem. It is well known that the nonlocal condition has a better effect on the solution and is more precise for physical measurements than the classical initial condition alone.

The rest of the paper is organized as follows. In Section 2 we present the notations, definitions and preliminary results needed in the following sections. In Section 3, a suitable concept of PC-mild solutions for our problems is introduced. Section 4 is concerned with the existence results of problems (1) and (2). An example is given in Section 5 to illustrate the results.

2 Preliminaries

Let us set $J = [0, T], J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, t_{m+1}]$ and introduce the space $PC(J, X) := \{u : J \to X | u \in C(J_k, X),\ k = 0, 1, 2, \ldots, m,$ and there exist $u(t_k^+)$ and $u(t_k^-)$, $k = 1, 2, \ldots, m$, with $u(t_k^-) = u(t_k)\}$. It is clear that $PC(J, X)$ is a Banach space with the norm $\|u\|_{PC} = \sup\{\|u(t)\| : t \in J\}$.

Lemma 2.1 (Theorem 2.1 in [8]) *Suppose $W \subseteq PC(J, X)$. If the following conditions are satisfied*:

(1) *W is a uniformly bounded subset of $PC(J, X)$;*
(2) *W is equicontinuous in (t_i, t_{i+1}), $i = 0, 1, 2, \ldots, m$, where $t_0 = 0$, $t_{m+1} = T$;*
(3) *$W(t) = \{u(t) : u \in W, t \in J \backslash \{t_1, t_2, \ldots, t_m\}\}$, $W(t_i^+) = \{u(t_i^+) : u \in W\}$ and $W(t_i^-) = \{u(t_i^-) : u \in W\}$, $i = 1, 2, \ldots, m$, are relatively compact subsets of X.*

Then W is a relatively compact subset of $PC(J, X)$.

Let us recall the following well-known definitions.

Definition 2.1 ([3]) The Riemann-Liouville fractional integral of order q with the lower limit zero for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds, \quad q > 0,$$

provided the integral exists, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([3]) The Riemann-Liouville derivative of order q with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$${}^{\mathrm{L}}D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-q-1} f(s)\, ds, \quad n-1 < q < n, t > 0.$$

Definition 2.3 ([3]) The Caputo derivative of order q for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$${}^{\mathrm{c}}D^q f(t) = {}^{\mathrm{L}}D^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad n-1 < q < n, t > 0.$$

Remark 2.1

(a) If $f \in C^n[0, \infty)$, then, for $n-1 < q < n$,

$${}^{\mathrm{c}}D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)\, ds = I^{n-q} f^{(n)}(t), \quad t > 0.$$

(b) If f is an abstract function with values in X, then the integrals in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Let us recall the following definition of mild solutions for fractional evolution equations involving the Caputo fractional derivative.

Definition 2.4 ([22, 23]) A function $x \in C(J,X)$ is said to be a mild solution of the following problem:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = Ax(t) + y(t), & t \in (0,T], \\ x(0) = x_0, \end{cases}$$

if it satisfies the integral equation

$$x(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)y(s)\,ds.$$

Here

$$P_\alpha(t) = \int_0^\infty \xi_\alpha(\theta)T\big(t^\alpha\theta\big)\,d\theta, \qquad Q_\alpha(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T\big(t^\alpha\theta\big)\,d\theta, \tag{3}$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha\big(\theta^{-\frac{1}{\alpha}}\big) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty), \tag{4}$$

and ξ_α is a probability density function defined on $(0,\infty)$ [25], that is,

$$\xi_\alpha(\theta) \geq 0, \theta \in (0,\infty), \qquad \int_0^\infty \xi_\alpha(\theta)\,d\theta = 1.$$

It is not difficult to verify that

$$\int_0^\infty \theta\xi_\alpha(\theta)\,d\theta = \frac{1}{\Gamma(1+\alpha)}. \tag{5}$$

Remark 2.2 By applying the Laplace transform and probability density functions, Zhou and Jiao [22, 23] introduced the above definition of mild solutions for fractional evolution equations. For pioneering work on Caputo fractional evolution equations, we refer the readers to [26, 27].

We make the following assumption on A in the whole paper.

H(A): The operator A generators a strongly continuous semigroup $\{T(t): t \geq 0\}$ in X, and there is a constant $M_A \geq 1$ such that $\sup_{t\in[0,\infty)}\|T(t)\|_{L(X)} \leq M_A$. For any $t > 0$, $T(t)$ is compact.

Lemma 2.2 (see [22, 23]) *Let* H(A) *hold, then the operators P_α and Q_α have the following properties*:

(1) *For any fixed $t \geq 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are linear and bounded operators, and for any $x \in X$,*

$$\big\|P_\alpha(t)x\big\| \leq M_A\|x\|, \qquad \big\|Q_\alpha(t)x\big\| \leq \frac{\alpha M_A}{\Gamma(1+\alpha)}\|x\|;$$

(2) *$\{P_\alpha(t), t \geq 0\}$ and $\{Q_\alpha(t), t \geq 0\}$ are strongly continuous*;

(3) *for every $t > 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are compact operators.*

Finally we recall a fixed point theorem which will be needed in the sequel.

Theorem 2.1 (Krasnoselskii fixed point theorem) *Let M be a closed, convex, and non-empty subset of a Banach space X. Let A, B be the operators such that*: (a) $Ax + By \in M$ *for all* $x, y \in M$, (b) A *is compact and continuous*, (c) B *is a contraction. Then there exists a* $x \in M$ *such that* $x = Ax + Bx$.

3 The construction of mild solutions

Let $y \in PC(J, X)$. We first consider the following fractional impulsive problem:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = Ax(t) + y(t), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, \\ x(t) = I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ x(0) = x_0. \end{cases} \tag{6}$$

From the property of the Caputo derivative, a general solution of problem (6) can be written as

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(Ax(s) + y(s))\,ds, & t \in [0, t_1), \\ I_1(x(t_1)) + g_1(t, x(t)), & t \in (t_1, s_1], \\ d_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(Ax(s) + y(s))\,ds, & t \in (s_1, t_2), \\ \ldots, \\ I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ d_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(Ax(s) + y(s))\,ds, & t \in (s_i, t_{i+1}), \end{cases} \tag{7}$$

where d_i, $i = 1, 2, \ldots, m$, are elements of X. By (7) and the function x is continuous at the points s_i, we have, for $i = 0, 1, 2, \ldots, m$,

$$x(t) = d_i \chi_{[s_i, t_{i+1})}(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\bigl(Ax(s) + y(s)\bigr)\,ds, \quad t \in [s_i, t_{i+1}), \tag{8}$$

with $d_0 = x_0$ and $\chi_{[s_i, t_{i+1})}(t)$ is the characteristic function of $[s_i, t_{i+1})$, *i.e.*

$$\chi_{[s_i, t_{i+1})}(t) = \begin{cases} 1, & t \in [s_i, t_{i+1}), \\ 0, & \text{otherwise.} \end{cases}$$

Now we follow the idea used in the papers [20, 22] and apply the Laplace transformation for (8) to obtain

$$u(\lambda) = \frac{e^{-\lambda s_i} - e^{-\lambda t_{i+1}}}{\lambda} d_i + \frac{1}{\lambda^{\alpha}} Au(\lambda) + + \frac{1}{\lambda^{\alpha}} \nu(\lambda),$$

where $u(\lambda) = \int_0^{\infty} e^{-\lambda s} x(s)\,ds$ and $\nu(\lambda) = \int_0^{\infty} e^{-\lambda s} y(s)\,ds$, $\lambda > 0$. Then

$$u(\lambda) = \lambda^{\alpha-1}\bigl(\lambda^{\alpha} I - A\bigr)^{-1} e^{-\lambda s_i} d_i - \lambda^{\alpha-1}\bigl(\lambda^{\alpha} I - A\bigr)^{-1} e^{-\lambda t_{i+1}} d_i + \bigl(\lambda^{\alpha} I - A\bigr)^{-1} \nu(\lambda),$$

where I is the identity operator defined on X. Note that the Laplace transform of $\varpi_{\alpha}(\theta)$ defined by (4) is given by

$$\int_0^{\infty} e^{-\lambda\theta} \varpi_{\alpha}(\theta)\,d\theta = e^{-\lambda^{\alpha}}.$$

Then by the same computations in [22, 23] and the properties of Laplace transform (translation theorem and linearity of the inverse Laplace transform), we obtain

$$x(t) = \chi_{[s_i,\infty)} P_\alpha(t-s_i)d_i - \chi_{[t_{i+1},\infty)} P_\alpha(t-t_{i+1})d_i + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)y(s)\,ds.$$

Here P_α and Q_α are given by (3). Hence we get

$$x(t) = P_\alpha(t-s_i)d_i + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)y(s)\,ds, \quad t \in [s_i, t_{i+1}).$$

Now it is time to determine the values of d_i, $i = 1, 2, \ldots, m$. Using the fact that x is continuous at the points s_i, we have

$$I_i\bigl(x(t_i)\bigr) + g_i\bigl(s_i, x(s_i)\bigr) = d_i + \int_0^{s_i} (s_i-s)^{\alpha-1} Q_\alpha(s_i-s)y(s)\,ds.$$

So we obtain

$$d_i = I_i\bigl(x(t_i)\bigr) + g_i\bigl(s_i, x(s_i)\bigr) - \int_0^{s_i} (s_i-s)^{\alpha-1} Q_\alpha(s_i-s)y(s)\,ds. \tag{9}$$

Therefore, a mild solution of problem (6) is given by

$$x(t) = \begin{cases} P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)y(s)\,ds, & t \in [0, t_1], \\ I_1(x(t_1)) + g_1(t, x(t)), & t \in (t_1, s_1], \\ P_\alpha(t-s_1)d_1 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)y(s)\,ds, & t \in (s_1, t_2], \\ \ldots, & \\ I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ P_\alpha(t-s_i)d_i + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)y(s)\,ds, & t \in (s_i, t_{i+1}], \end{cases}$$

where, for $i = 1, 2, \ldots, m$,

$$d_i = I_i\bigl(x(t_i)\bigr) + g_i\bigl(s_i, x(s_i)\bigr) - \int_0^{s_i} (s_i-s)^{\alpha-1} Q_\alpha(s_i-s)y(s)\,ds.$$

Next, by using the above results, we introduce the following definition of the mild solution for problem (1).

Definition 3.1 A function $x \in PC(J, X)$ is said to be a PC-mild solution of problem (1) if it satisfies the following relation:

$$x(t) = \begin{cases} P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)f(s, x(s))\,ds, & t \in [0, t_1], \\ I_1(x(t_1)) + g_1(t, x(t)), & t \in (t_1, s_1], \\ P_\alpha(t-s_1)d_1 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)f(s, x(s))\,ds, & t \in [s_1, t_2], \\ \ldots, & \\ I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ P_\alpha(t-s_i)d_i + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)f(s, x(s))\,ds, & t \in [s_i, t_{i+1}], \end{cases}$$

where, for $i = 1, 2, \ldots, m$,

$$d_i = I_i\big(x(t_i)\big) + g_i\big(s_i, x(s_i)\big) - \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) f\big(s, x(s)\big)\,ds. \tag{10}$$

Remark 3.1 For treating the mild solutions for abstract fractional differential equations, we can also refer to [21].

4 Existence results

This section deals with the existence results for problems (1) and (2).

From Definition 3.1, we define an operator $S : PC(J, X) \to PC(J, X)$ as

$$(Sx)(t) = \begin{cases} P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x(s))\,ds, & t \in [0, t_1], \\ I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i], \\ P_\alpha(t - s_i)d_i + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x(s))\,ds, & t \in [s_i, t_{i+1}] \end{cases}$$

with d_i, $i = 1, 2, \ldots, m$, defined by (10).

To prove our first existence result we introduce the following assumptions.

- $H(f)_1$: The function $f \in C(J \times X; X)$ and there exists $L_f \in L^{\frac{1}{\tau}}(J, \mathbb{R}^+)$ with $\tau \in (0, \alpha)$ such that $\|f(t, x) - f(t, y)\| \le L_f(t)\|x - y\|$ for all $x, y \in X$ and every $t \in J$.
- H(I): For $i = 1, 2, \ldots, m$, $I_i \in C(X, X)$ and there is a constant $L_I > 0$ such that $\|I_i(x) - I_i(y)\| \le L_I\|x - y\|$ for all $x, y \in X$.
- H(g): For $i = 1, 2, \ldots, m$, the functions $g_i \in C([t_i, s_i] \times X; X)$ and there exists $L_g \in C(J, \mathbb{R}^+)$ such that $\|g_i(t, x) - g_i(t, y)\| \le L_g(t)\|x - y\|$ for all $x, y \in X$ and $t \in [t_i, s_i]$.

Theorem 4.1 *Assume* $H(f)_1$, H(I), *and* H(g) *are satisfied and*

$$M_A\big(L_I + \|L_g\|_{C(J)}\big) + (1 + M_A)\frac{\alpha M_A}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} T^{\alpha-\tau}\|L_f\|_{L^{\frac{1}{\tau}}(J)} < 1. \tag{11}$$

Then there exists a unique PC-mild solution of problem (1).

Proof From the assumptions it is easy to show that the operator S is well defined on $PC(J, X)$.

Let $x, y \in PC(J, X)$. For $t \in [0, t_1]$, from Lemma 2.2, we have

$$\begin{aligned} \big\|(Sx)(t) - (Sy)(t)\big\| &\le \int_0^t (t-s)^{\alpha-1}\big\|Q_\alpha(t-s)\big(f\big(s, x(s)\big) - f\big(s, y(s)\big)\big)\big\|\,ds \\ &\le \frac{\alpha M_A}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} t_1^{\alpha-\tau}\|L_f\|_{L^{\frac{1}{\tau}}([0,t_1])}\|x - y\|_{PC}. \end{aligned}$$

Similarly, we have, for $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$,

$$\begin{aligned} \big\|(Sx)(t) - (Sy)(t)\big\| &\le \big\|I_i\big(x(t_i)\big) - I_i\big(y(t_i)\big)\big\| \\ &\quad + \big\|g_i\big(t, x(t)\big) - g_i\big(t, y(t)\big)\big\| \\ &\le \big(L_I + \|L_g\|_{C(J)}\big)\|x - y\|_{PC}, \end{aligned}$$

and, for $t \in [s_i, t_{i+1}]$, $i = 1, 2, \dots, m$,

$$\begin{aligned}
&\big\|(Sx)(t) - (Sy)(t)\big\| \\
&\quad\leq \bigg\| P_\alpha(t - s_i)\bigg[I_i\big(x(t_i)\big) - I_i\big(y(t_i)\big) + g_i\big(s_i, x(s_i)\big) - g_i\big(s_i, y(s_i)\big) \\
&\qquad - \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s)\big(f\big(s, x(s)\big) - f\big(s, y(s)\big)\big)\, ds\bigg]\bigg\| \\
&\qquad + \int_0^t (t - s)^{\alpha-1} \big\| Q_\alpha(t - s)\big(f\big(s, x(s)\big) - f\big(s, y(s)\big)\big)\big\|\, ds \\
&\quad\leq M_A\left(L_I + \|L_g\|_{C(J)} + \frac{\alpha M_A}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} s_i^{\alpha-\tau} \|L_f\|_{L^{\frac{1}{\tau}}([0,s_i])}\right)\|x - y\|_{PC} \\
&\qquad + \frac{\alpha M_A}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} t_{i+1}^{\alpha-\tau} \|L_f\|_{L^{\frac{1}{\tau}}([0,t_{i+1}])} \|x - y\|_{PC}.
\end{aligned}$$

From the above we can deduce that (since $M_A \geq 1$)

$$\begin{aligned}
&\big\|(Sx)(t) - (Sy)(t)\big\|_{PC} \\
&\quad\leq \bigg[M_A\big(L_I + \|L_g\|_{C(J)}\big) + (1 + M_A) \\
&\qquad \times \frac{\alpha M_A}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} T^{\alpha-\tau} \|L_f\|_{L^{\frac{1}{\tau}}(J)}\bigg] \|x - y\|_{PC}.
\end{aligned}$$

Then it follows from condition (11) that S is a contraction on the space $PC(J, X)$. Hence by the Banach contraction mapping principle, S has a unique fixed point $x \in PC(J, X)$ which is just the unique PC-mild solution of problem (1). The proof is now complete. □

The next result is based on the Krasnoselskii fixed point theorem.

H(f)$_2$: For any $x \in X$, the map $t \to f(t, x)$ is strongly measurable on J. For a.e. $t \in J$, the map $x \to f(t, x)$ is continuous. There exist $m_f \in L^{\frac{1}{\tau}}(J, \mathbb{R}^+)$ with $\tau \in (0, \alpha)$ and $\varphi_f \in C([0, \infty), \mathbb{R}^+)$ nondecreasing such that $\|f(t, x)\| \leq m_f(t)\varphi_f(\|x\|)$ for all $x \in X$ and $t \in J$.

H(Ig)$_2$: There exist $m_g \in C(J, \mathbb{R}^+)$ and $\varphi_I, \varphi_g \in C([0, \infty), \mathbb{R}^+)$ nondecreasing such that, for all $x \in X$, $i = 1, 2, \dots, m$,

$$\big\|I_i(x)\big\| \leq \varphi_I\big(\|x\|\big), \qquad \big\|g_i(t, x)\big\| \leq m_g(t)\varphi_g\big(\|x\|\big), \quad t \in (t_i, s_i].$$

Theorem 4.2 *Let* H(f)$_2$, H(I), H(g), *and* H(Ig)$_2$ *hold. Assume that*

$$M_A\big(L_I + \|L_g\|_{C(J)}\big) < 1 \tag{12}$$

and there exists a constant $r > 0$ *such that*

$$\begin{aligned}
&M_A\big[\varphi_I(r) + \|m_g\|_{C(J)}\varphi_g(r) + \|x_0\|\big] \\
&\quad + (1 + M_A)\frac{\alpha M_A \varphi_f(r)}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} T^{\alpha-\tau} \|m_f\|_{L^{\frac{1}{\tau}}(J)} \leq r.
\end{aligned}\tag{13}$$

Then there exists a PC-mild solution of problem (1).

Proof We define two operators $S_1, S_2 : PC(J,X) \to PC(J,X)$ as

$$(S_1x)(t) = \begin{cases} P_\alpha(t)x_0, & t \in [0,t_1], \\ I_i(x(t_i)) + g_i(t,x(t)), & t \in (t_i,s_i], \\ P_\alpha(t-s_i)(I_i(x(t_i)) + g_i(s_i,x(s_i))), & t \in [s_i,t_{i+1}], \end{cases}$$

$$(S_2x)(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s,x(s))\,ds, & t \in [0,t_1], \\ 0, & t \in (t_i,s_i], \\ \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s,x(s))\,ds & \\ \quad - P_\alpha(t-s_i)\int_0^{s_i}(s_i-s)^{\alpha-1} Q_\alpha(s_i-s) f(s,x(s))\,ds, & t \in [s_i,t_{i+1}] \end{cases}$$

for $i = 1,2,\ldots,m$. Since $P_\alpha(0) = I$, it is easy to verify that for any $x \in PC(J,X)$, $S_1x, S_2x \in PC(J,X)$, hence they are well defined. We have $Sx = S_1x + S_2x$.

Let $r > 0$ satisfy condition (13). We set

$$M = \{u \in PC(J,X) : \|u\|_{PC} \le r\}.$$

Then M is a closed, convex, and nonempty subset of the Banach space $PC(J,X)$.

Next we will show that the operators S_1, S_2 satisfy the requirements of Theorem 2.1, *i.e.* S_1 is a contraction, S_2 is compact and continuous and $S_1x + S_2y \in M$ for all $x, y \in M$.

Step 1: $S_1x + S_2y \in M$ *for all* $x, y \in M$. For any $x, y \in M$. We have, for $t \in [0,t_1]$,

$$\begin{aligned} \|(S_1x)(t) + (S_2y)(t)\| &\le M_A\|x_0\| + \frac{\alpha M_A \varphi_f(r)}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1} m_f(s)\,ds \\ &\le M_A\|x_0\| + \frac{\alpha M_A \varphi_f(r)}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} t_1^{\alpha-\tau}\|m_f\|_{L^{\frac{1}{\tau}}(J)} \end{aligned}$$

for $t \in (t_i,s_i]$, $i = 1,2,\ldots,m$,

$$\begin{aligned} \|(S_1x)(t) + (S_2y)(t)\| &\le \|I_i(x(t_i))\| + \|g_i(t,x(t))\| \\ &\le \varphi_I(r) + \|m_g\|_{C(J)}\varphi_g(r), \end{aligned}$$

and for $t \in [s_i,t_{i+1}]$, $i = 1,2,\ldots,m$,

$$\begin{aligned} \|(S_1x)(t) + (S_2y)(t)\| &\le \biggl\| P_\alpha(t-s_i)\biggl[I_i(x(t_i)) + g_i(s_i,x(s_i)) \\ &\qquad - \int_0^{s_i}(s_i-s)^{\alpha-1} Q_\alpha(s_i-s) f(s,y(s))\,ds\biggr]\biggr\| \\ &\qquad + \int_0^t (t-s)^{\alpha-1}\|Q_\alpha(t-s) f(s,y(s))\|\,ds \\ &\le M_A[\varphi_I(r) + \|m_g\|_{C(J)}\varphi_g(r)] \\ &\qquad + M_A\frac{\alpha M_A \varphi_f(r)}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} s_i^{\alpha-\tau}\|m_f\|_{L^{\frac{1}{\tau}}(J)} \\ &\qquad + \frac{\alpha M_A \varphi_f(r)}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} t_{i+1}^{\alpha-\tau}\|m_f\|_{L^{\frac{1}{\tau}}(J)}. \end{aligned}$$

Then we get

$$\|S_1x + S_2y\|_{PC} \le M_A\big[\varphi_I(r) + \|m_g\|_{C(J)}\varphi_g(r) + \|x_0\|\big] \\ + (1 + M_A)\frac{\alpha M_A\varphi_f(r)}{\Gamma(\alpha + 1)}\left(\frac{1 - \tau}{\alpha - \tau}\right)^{1-\tau} T^{\alpha-\tau}\|m_f\|_{L^{\frac{1}{\tau}}(J)}.$$

It follows from (13) that $S_1x + S_2y \in M$ for all $x, y \in M$.

Step 2: S_1 *is a contraction*. Let $x, y \in PC(J, X)$. From H(I), H(g), and Lemma 2.2, we have, for $t \in [0, t_1]$,

$$(S_1x)(t) - (S_1y)(t) = 0$$

for $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$,

$$\big\|(S_1x)(t) - (S_1y)(t)\big\| \le \big(L_I + \|L_g\|_{C(J)}\big)\|x - y\|_{PC},$$

and for $t \in [s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$,

$$\big\|(S_1x)(t) - (S_1y)(t)\big\| \le M_A\big(L_I + \|L_g\|_{C(J)}\big)\|x - y\|_{PC}.$$

Therefore we deduce that

$$\|S_1x - S_1y\|_{PC} \le M_A\big(L_I + \|L_g\|_{C(J)}\big)\|x - y\|_{PC}.$$

In view of (12), the operator S_1 is a contraction on $PC(J, X)$.

Step 3: S_2 *is compact and continuous*. Firstly, we will prove that S_2 is continuous. Let $x_n \to x$ in $PC(J, X)$ as $n \to \infty$. We can assume without any loss of generality that $\|x_n\|_{PC} \le R$ for some $R > 0$ and $n \ge 1$. By H(f)$_2$, we have

$$f\big(t, x_n(t)\big) \to f\big(t, x(t)\big) \quad \text{a.e. } t \in J, \tag{14}$$

$$\big\|f\big(t, x_n(t)\big)\big\| \le m_f(t)\varphi_f(R) \quad \text{for } t \in J, n \ge 1. \tag{15}$$

Since, for $t \in [0, t_1]$,

$$\big\|(S_2x_n)(t) - (S_2x)(t)\big\| \le \frac{\alpha M_A}{\Gamma(\alpha + 1)}\int_0^t (t - s)^{\alpha-1}\big\|f\big(s, x_n(s)\big) - f\big(s, x(s)\big)\big\|\,ds$$

for $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$,

$$\big\|(S_2x_n)(t) - (S_2x)(t)\big\| = 0,$$

and, for $t \in [s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$,

$$\big\|(S_2x_n)(t) - (S_2x)(t)\big\| \le \frac{\alpha M_A}{\Gamma(\alpha + 1)}\int_0^t (t - s)^{\alpha-1}\big\|f\big(s, x_n(s)\big) - f\big(s, x(s)\big)\big\|\,ds \\ + M_A\frac{\alpha M_A}{\Gamma(\alpha + 1)}\int_0^{s_i} (s_i - s)^{\alpha-1}\big\|f\big(s, x_n(s)\big) - f\big(s, x(s)\big)\big\|\,ds.$$

Then from (14), (15), and by means of the Lebesgue dominated convergence theorem, we obtain

$$\|S_2x_n - S_2x\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

This means that S_2 is continuous.

Next we shall show that S_2 maps bounded set into relatively compact set in $PC(J,X)$. Let B be any bounded subset of $PC(J,X)$ such that for $x \in B$, $\|x\|_{PC} \leq R$ for some $R > 0$, it suffices to show that the set of functions $S_2(B) = \{S_2x : x \in B\}$ satisfies the conditions of Lemma 2.1.

For the same reasons as in Step 1, the set $S_2(B)$ is uniformly bounded.

For any $x \in B$, if $0 \leq t' < t'' \leq t_1$, we have

$$\begin{aligned}
\|(S_2x)(t'') - (S_2x)(t')\| &= \bigg\| \int_0^{t''} (t''-s)^{\alpha-1} Q_\alpha(t''-s) f(s,x(s))\,ds \\
&\quad - \int_0^{t'} (t'-s)^{\alpha-1} Q_\alpha(t'-s) f(s,x(s))\,ds \bigg\| \\
&\leq I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \bigg\| \int_{t'}^{t''} (t''-s)^{\alpha-1} Q_\alpha(t''-s) f(s,x(s))\,ds \bigg\|, \\
I_2 &= \bigg\| \int_0^{t'} (t'-s)^{\alpha-1} \big(Q_\alpha(t''-s) - Q_\alpha(t'-s)\big) f(s,x(s))\,ds \bigg\|, \\
I_3 &= \bigg\| \int_0^{t'} \big((t''-s)^{\alpha-1} - (t'-s)^{\alpha-1}\big) Q_\alpha(t''-s) f(s,x(s))\,ds \bigg\|.
\end{aligned}$$

Repeating the discussion in [23] (see p.1072 of it), we find that I_1, I_2, I_3 tend to zero as $t'' \to t'$ independently of $x \in B$. If $t_i < t' < t'' \leq t_{i+1}$, $i = 1, 2, \ldots, m$, we have the following.

Case 1: $t_i < t' < t'' \leq s_i$,

$$\|(S_2x)(t'') - (S_2x)(t')\| = 0.$$

Case 2: $s_i \leq t' < t'' \leq t_{i+1}$,

$$\|(S_2x)(t'') - (S_2x)(t')\| \leq I_1 + I_2 + I_3 + \big\|\big(P_\alpha(t'-s_i) - P_\alpha(t''-s_i)\big)\Pi\big\|, \tag{16}$$

where $\Pi = \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) f(s,x(s))\,ds$. Since H(A) and the proof of Lemma 3.4 in [23] imply that the continuity of $P_\alpha(t)$ and $Q_\alpha(t)$ ($t > 0$) in t is in the uniform operator topology, we deduce that the right-hand side of (16) tends to zero independently of $x \in B$, as $t'' \to t'$.

Case 3: $t_i < t' < s_i < t'' \leq t_{i+1}$,

$$\begin{aligned}
&\|(S_2x)(t'') - (S_2x)(t')\| \\
&\quad = \bigg\| \int_0^{t''} (t''-s)^{\alpha-1} Q_\alpha(t''-s) f(s,x(s))\,ds \\
&\qquad - P_\alpha(t''-s_i) \int_0^{s_i} (s_i-s)^{\alpha-1} Q_\alpha(s_i-s) f(s,x(s))\,ds \bigg\| \to 0
\end{aligned}$$

independently of $x \in B$, as $t'' \to t'$ (we have $t'' \to s_i$). Hence $S_2(B)$ is equicontinuous in (t_i, t_{i+1}), $i = 0, 1, 2, \dots, m$.

Finally, let $S_2(B)(t)$ denote the set $\{(S_2x)(t) : x \in B\}$, $t \in J$, we shall prove that $S_2(B)(t)$ is relatively compact in X. Clearly, $S_2(B)(0) = \{0\}$ is compact.

Case 1′: $0 < t \le t_1$. For each $h \in (0, t)$ and $\delta > 0$, we define a set

$$S_2^{h,\delta}(B)(t) = \bigl\{(\mathcal{M}_{h,\delta}x)(t) : x \in B\bigr\}$$

with

$$\begin{aligned}(\mathcal{M}_{h,\delta}x)(t) &= \alpha \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\bigl((t-s)^\alpha \theta\bigr) f\bigl(s, x(s)\bigr)\, d\theta\, ds \\ &= \alpha \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) \bigl[T\bigl(h^\alpha \delta\bigr) T\bigl((t-s)^\alpha \theta - h^\alpha \delta\bigr)\bigr] f\bigl(s, x(s)\bigr)\, d\theta\, ds \\ &= \alpha T\bigl(h^\alpha \delta\bigr) \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\bigl((t-s)^\alpha \theta - h^\alpha \delta\bigr) f\bigl(s, x(s)\bigr)\, d\theta\, ds.\end{aligned}$$

(Observe that $\theta \ge \delta$ and $t - h \ge s$, hence $(t-s)^\alpha \theta - h^\alpha \delta \ge 0$.) Since the operator $T(h^\alpha \delta)$ $(h^\alpha \delta > 0)$ is compact, the set $S_2^{h,\delta}(B)(t)$ is relatively compact in X. Moreover, for every $x \in B$, we have

$$\begin{aligned}&\bigl\|(S_2x)(t) - (\mathcal{M}_{h,\delta}x)(t)\bigr\| \\ &\quad = \alpha \biggl\| \int_0^t \int_0^\delta \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\bigl((t-s)^\alpha \theta\bigr) f\bigl(s, x(s)\bigr)\, d\theta\, ds \\ &\qquad + \int_0^t \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\bigl((t-s)^\alpha \theta\bigr) f\bigl(s, x(s)\bigr)\, d\theta\, ds \\ &\qquad - \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\bigl((t-s)^\alpha \theta\bigr) f\bigl(s, x(s)\bigr)\, d\theta\, ds \biggr\| \\ &\quad \le \alpha \biggl\| \int_0^t \int_0^\delta \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\bigl((t-s)^\alpha \theta\bigr) f\bigl(s, x(s)\bigr)\, d\theta\, ds \biggr\| \quad \text{(denoted by } G_1\text{)} \\ &\qquad + \alpha \biggl\| \int_{t-h}^t \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) T\bigl((t-s)^\alpha \theta\bigr) f\bigl(s, x(s)\bigr)\, d\theta\, ds \biggr\| \quad \text{(denoted by } G_2\text{)}.\end{aligned}$$

By the Hölder inequality and $\mathrm{H(f)}_2$, we get

$$\begin{aligned}G_1 &\le \alpha M_A \int_0^t (t-s)^{\alpha-1} \bigl\|f\bigl(s, x(s)\bigr)\bigr\|\, ds \int_0^\delta \theta \xi_\alpha(\theta)\, d\theta \\ &\le \alpha M_A \varphi_f(R) \biggl(\frac{1-\tau}{\alpha-\tau}\biggr)^{1-\tau} t_1^{\alpha-\tau} \|m_f\|_{L^{\frac{1}{\tau}}(J)} \int_0^\delta \theta \xi_\alpha(\theta)\, d\theta\end{aligned}$$

and

$$\begin{aligned}G_2 &\le \alpha M_A \int_{t-h}^t (t-s)^{\alpha-1} \bigl\|f\bigl(s, x(s)\bigr)\bigr\|\, ds \int_\delta^\infty \theta \xi_\alpha(\theta)\, d\theta \\ &\le \alpha M_A \varphi_f(R) \biggl(\frac{1-\tau}{\alpha-\tau}\biggr)^{1-\tau} h^{\alpha-\tau} \|m_f\|_{L^{\frac{1}{\tau}}(J)} \int_0^\infty \theta \xi_\alpha(\theta)\, d\theta.\end{aligned}$$

Therefore from the property of the probability density function ξ_α and (5), we obtain

$$\left\|(S_2x)(t) - (\mathcal{M}_{h,\delta}x)(t)\right\| \to 0 \quad \text{as } h \to 0, \delta \to 0.$$

This means that there are relatively compact sets arbitrarily close to the set $S_2(B)(t)$. Hence the set $S_2(B)(t)$ is also relatively compact in X.

Case 2′: $t_i < t \leq s_i$, $i = 1, 2, \ldots, m$. In such a case,

$$S_2(B)(t) = \{0\} \text{ is compact.}$$

Case 3′: $s_i < t \leq t_{i+1}$, $i = 1, 2, \ldots, m$,

$$\begin{aligned} S_2(B)(t) = \biggl\{ &\int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f\bigl(s, x(s)\bigr)\, ds \\ &- P_\alpha(t-s_i) \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) f\bigl(s, x(s)\bigr)\, ds : x \in B \biggr\}. \end{aligned}$$

By the same argument as in Case 1′ and $P_\alpha(t - s_i)$ is a compact operator (see Lemma 2.2), we know $S_2(B)(t)$ is relatively compact.

Therefore it follows from Lemma 2.1 that S_2 is compact and continuous.

As a consequence of Steps 1-3, we know that $S_1 + S_2$ satisfies all conditions of Krasnoselskii fixed point theorem (Theorem 2.1). Hence the operator S has a fixed point in $PC(J, X)$ which is a PC-mild solution of problem (1). The proof is complete. □

Finally in this section, we extend the results obtained above to nonlocal problems for impulsive fractional evolution equations. Specifically, we show study the existence and uniqueness of the mild solutions for problem (2). Here we only state the existence results for problem (2) without proofs since these are similar to the ones obtained for problem (1) above.

Definition 4.1 A function $x \in PC(J, X)$ is said to be a PC-mild solution of problem (2) if it satisfies the following relation:

$$x(t) = \begin{cases} P_\alpha(t)(x_0 + b(x)) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x(s))\, ds, & t \in [0, t_1], \\ I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ P_\alpha(t-s_i) d_i + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x(s))\, ds, & t \in [s_i, t_{i+1}], \end{cases}$$

with d_i, $i = 1, 2, \ldots, m$, defined by (10).

H(b): $b : PC(J, X) \to X$ and there exist a constant $L_b > 0$ and $\varphi_b \in C([0, \infty), \mathbb{R}^+)$ nondecreasing such that, for $x, y \in PC(J, X)$,

$$\left\|b(x) - b(y)\right\| \leq L_b \|x - y\|_{PC}, \qquad \left\|b(x)\right\| \leq \varphi_b\bigl(\|x\|_{PC}\bigr).$$

Theorem 4.3 *Assume* H(f)$_1$, H(I), H(g), *and* H(b) *are satisfied and*

$$M_A\bigl(L_I + \|L_g\|_{C(J)} + L_b\bigr) + (1 + M_A) \frac{\alpha M_A}{\Gamma(\alpha+1)} \left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} T^{\alpha-\tau} \|L_f\|_{L^{\frac{1}{\tau}}(J)} < 1.$$

Then there exists a unique PC-mild solution of problem (2).

Theorem 4.4 *Let* H(f)$_2$, H(I), H(g), H(Ig)$_2$, *and* H(b) *hold. Assume that*

$$M_A\left(L_b + L_I + \|L_g\|_{C(J)}\right) < 1 \tag{17}$$

and there exists a constant $r > 0$ *such that*

$$\begin{aligned} &M_A\left[\varphi_I(r) + \|m_g\|_{C(J)}\varphi_g(r) + \varphi_b(r) + \|x_0\|\right] \\ &\quad + (1 + M_A)\frac{\alpha M_A \varphi_f(r)}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} T^{\alpha-\tau}\|m_f\|_{L^{\frac{1}{\tau}}(J)} \leq r. \end{aligned} \tag{18}$$

Then there exists a PC-mild solution of problem (2).

5 Examples

A simple example is given in this section to illustrate the results.

Let $X = L^2([0,\pi])$. Define an operator $A : D(A) \subseteq X \to X$ by $Ax = x''$ with $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ in X. Moreover, $T(t)$ is compact for $t > 0$ and $\|T(t)\|_{L(X)} \leq e^{-t} \leq 1 = M_A$, $t \geq 0$.

Consider the following impulsive problem:

$$\begin{cases} {}^{c}D_t^{\frac{3}{4}} u(t,y) = \frac{\partial^2}{\partial y^2} u(t,y) + f(t, u(t,y)), & t \in [0,\frac{1}{2}] \cup (\frac{2}{3}, 1], y \in [0,\pi], \\ u(t,y) = I(u(t_{\frac{1}{2}}, y)) + g(t, u(t,y)), & t \in (\frac{1}{2}, \frac{2}{3}], y \in [0,\pi], \\ u(t,y) = u_0(y), & y \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \in [0,1]. \end{cases} \tag{19}$$

Here ${}^{c}D_t^{\frac{1}{2}}$ means that the Caputo fractional derivative is taken for the time variable t with the lower limit zero.

Assumption 1 Let

$$\begin{aligned} &f\left(t, u(t,y)\right) = \frac{\cos t}{(t+6)^2}\left(u(t,y) + \arctan u(t,y)\right), \\ &I\left(u(t,y)\right) = \frac{|u(t,y)|}{4 + |u(t,y)|}, \qquad g\left(t, u(t,y)\right) = \frac{1}{3}\sin u(t,y) + e^t. \end{aligned}$$

Define $x(t)(y) = u(t,y)$, $(t,y) \in [0,1] \times [0,\pi]$. Then f, I, and g can be rewritten as

$$\begin{aligned} &f\left(t, x(t)\right) = \frac{\cos t}{(t+6)^2}\left(x(t) + \arctan x(t)\right), \\ &I\left(x(t)\right) = \frac{|x(t)|}{4 + |x(t)|}, \qquad g\left(t, x(t)\right) = \frac{1}{3}\sin x(t) + e^t. \end{aligned}$$

We can verify that H(f)$_1$, H(I), and H(g) hold by putting $L_f(t) = \frac{2\cos t}{(t+6)^2}$, $L_I = \frac{1}{4}$, and $L_g(t) \equiv \frac{1}{3}$. Moreover, since $\alpha = \frac{3}{4}$, let $\tau = \frac{1}{2}$, we have

$$\begin{aligned} &M_A\left(L_I + \|L_g\|_{C(J)}\right) + (1 + M_A)\frac{\alpha M_A}{\Gamma(\alpha+1)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} T^{\alpha-\tau}\|L_f\|_{L^{\frac{1}{\tau}}(J)} \\ &\quad \leq \frac{1}{4} + \frac{1}{3} + 2 \times 0.8160 \times 1.4142 \times \frac{1}{18} = 0.7116 < 1. \end{aligned}$$

Therefore by Theorem 4.1, we deduce that problem (19) has a unique PC-mild solution on $[0,1]$.

Assumption 2 Let

$$f\big(t,u(t,y)\big) = e^{-|u(t,y)|} + 6t^2 + \sin t + \frac{u(t,y)}{1+u^2(t,y)},$$
$$I\big(u(t,y)\big) = \frac{|u(t,y)|}{4+|u(t,y)|} + 5t, \qquad g\big(t,u(t,y)\big) = \frac{1}{3}\arctan u(t,y) + 3.$$

Similarly, the functions f, I, and g can be rewritten as

$$f\big(t,x(t)\big) = e^{-|x(t)|} + 6t^2 + \sin t + \frac{x(t)}{1+x^2(t)},$$
$$I\big(x(t)\big) = \frac{|x(t)|}{4+|x(t)|} + 5t, \qquad g\big(t,x(t)\big) = \frac{1}{3}\arctan x(t) + 3.$$

Put $m_f(t) \equiv 1$, $\varphi_f(\|x\|) \equiv 9\sqrt{\pi}$, $\varphi_I(\|x\|) \equiv \frac{1}{4}\sqrt{\pi}$, $m_g(t) \equiv 1$, $\varphi_g(\|x\|) \equiv (\frac{\pi}{6}+3)\sqrt{\pi}$, $L_I = \frac{1}{4}$, and $L_g(t) \equiv \frac{1}{3}$, then it is easy to show that $\mathrm{H(f)}_2$, H(I), H(g), and $\mathrm{H(Ig)}_2$ hold. We have

$$M_A\big(L_I + \|L_g\|_{C(J)}\big) = \frac{7}{12} < 1.$$

In such a case, obviously, we can choose a constant $r > 0$ such that condition (13) holds. Therefore it follows from Theorem 4.2 that problem (19) has a PC-mild solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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