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Oscillation of forced second-order neutral delay differential equations

Ying Jiang[1], Youliang Fu[1], Haixia Wang[2] and Tongxing Li[3,4*]

*Correspondence: litongx2007@163.com
[3]LinDa Institute of Shandong Provincial Key Laboratory of Network Based Intelligent Computing, Linyi University, Linyi, Shandong 276005, P.R. China
[4]School of Informatics, Linyi University, Linyi, Shandong 276005, P.R. China
Full list of author information is available at the end of the article

Abstract
The objective of this paper is to study oscillation of a forced second-order neutral differential equation. By using the generalized Riccati substitution and integral technique, a new sufficient condition is obtained which insures that all solutions to the studied equation are oscillatory. An illustrative example is included.

MSC: 34C10; 34K11

Keywords: oscillation; second-order; forced term; neutral differential equation

1 Introduction

In this paper, we are concerned with the oscillation of a forced second-order nonlinear neutral differential equation

$$\bigl(r(t)\bigl[x(t)+P(t)x\bigl(\tau(t)\bigr)\bigr]'\bigr)'+\sum_{i=1}^{m}Q_i(t)f_i\bigl(x(t)\bigr)+\sum_{j=1}^{l}R_j(t)g_j\bigl(x\bigl(\tau(t)\bigr)\bigr)=F(t), \tag{1.1}$$

where $t \geq t_0 > 0$, $m \geq 1$, and $l \geq 1$ are integers. We suppose that the following assumptions are satisfied:

(A$_1$) $r \in \mathrm{C}^1([t_0,\infty),(0,\infty))$, $P, Q_i, R_j \in \mathrm{C}([t_0,\infty),[0,\infty))$, $f_i, g_j \in \mathrm{C}(\mathbb{R},\mathbb{R})$, $yf_i(y) > 0$, and $yg_j(y) > 0$ for $y \neq 0$, $i = 1,2,\ldots,m$, and $j = 1,2,\ldots,l$;

(A$_2$) $\tau \in \mathrm{C}([t_0,\infty),\mathbb{R})$, $\tau(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \infty$;

(A$_3$) there exist constants $\alpha_i > 0$ and $\beta_j > 0$ such that $f_i(y)/y \geq \alpha_i$ and $g_j(y)/y \geq \beta_j$ for $y \neq 0$, $i = 1,2,\ldots,m$, and $j = 1,2,\ldots,l$;

(A$_4$) for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that

$$F(t)\begin{cases}\leq 0, & t \in [s_1,t_1],\\ \geq 0, & t \in [s_2,t_2],\end{cases}$$

and

$$\sum_{j=1}^{l}\beta_j R_j(t) \geq \sum_{i=1}^{m}\alpha_i Q_i(t)P(t), \qquad t \in [s_1,t_1]\cup[s_2,t_2]. \tag{1.2}$$

Throughout the paper, we define

$$z(t) := x(t) + P(t)x\big(\tau(t)\big). \tag{1.3}$$

By a solution of (1.1) we mean a function $x \in C([T_x,\infty),\mathbb{R})$, $T_x \geq t_0$, which has the property $rz' \in C^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. We consider only those solutions x of (1.1) which satisfy condition $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on the interval $[T_x,\infty)$; otherwise, it is termed nonoscillatory.

As is well known, the study of qualitative theory of differential equations is of importance both in theory and applications. For instance, the problems of oscillatory behavior of neutral differential equations have a number of practical applications in the study of distributed networks containing lossless transmission lines which arise in high-speed computers where the lossless transmission lines are used to interconnect switching circuits. For some related contributions on oscillation of various classes of neutral differential equations, we refer the reader to [1–23] and the references cited therein.

In what follows, we provide some background details that motivated our study. El-Sayed [4] and Wong [19] investigated the second-order forced linear differential equation

$$\big(p(t)x'\big)' + q(t)x = f(t).$$

Zhang *et al.* [22] studied a second-order neutral differential equation

$$\big(r(t)\big[x(t) + p(t)x(t-\tau)\big]'\big)' + Q_1(t)f\big(x(t)\big) + Q_2(t)g\big(x(t-\tau)\big) = H(t), \tag{1.4}$$

where Q_1 and Q_2 are nonnegative functions. Equation (1.4) is a special case of (1.1). In the sequel, using a generalized Riccati substitution which differs from those exploited in [4, 19, 22], a new oscillation criterion for (1.1) is presented. Furthermore, an illustrative example is provided.

2 Main results

Theorem 2.1 *Assume that conditions* (A_1)-(A_4) *are satisfied and let* $B_k = \{u \in C^1[s_k,t_k] : u(t) \not\equiv 0, u(s_k) = u(t_k) = 0\}$, $k = 1, 2$. *If there exist functions* $u \in B_k$, $\rho \in C^1([t_0,\infty),(0,\infty))$, *and* $\sigma \in C^1([t_0,\infty),\mathbb{R})$ *such that, for* $k = 1, 2$,

$$\begin{aligned} &J_k(u,\rho,\sigma) \\ &\quad= \int_{s_k}^{t_k} \left\{\rho\left[u^2\left(\sum_{i=1}^{m}\alpha_i Q_i + r\sigma^2 - (r\sigma)'\right) - r\left(u' + \frac{u\rho'}{2\rho} + u\sigma\right)^2\right]\right\}(t)\,\mathrm{d}t > 0, \end{aligned} \tag{2.1}$$

then every solution of (1.1) *is oscillatory.*

Proof Suppose that x is a nonoscillatory solution of (1.1) which is eventually positive. Then z defined by (1.3) is also eventually positive. Using (A_4), for any $T \geq t_0$, there exist $t_1 > s_1 \geq T$ such that $F(t) \leq 0$ for $t \in [s_1,t_1]$. From (A_3), (1.1), (1.2), and (1.3), we have

$$\begin{aligned}\bigl(rz'\bigr)'(t) &= F(t) - \sum_{i=1}^{m} Q_i(t)f_i\bigl(x(t)\bigr) - \sum_{j=1}^{l} R_j(t)g_j\bigl(x\bigl(\tau(t)\bigr)\bigr)\\ &\le -\sum_{i=1}^{m}\alpha_i Q_i(t)x(t) - \sum_{j=1}^{l}\beta_j R_j(t)x\bigl(\tau(t)\bigr)\\ &\le -\left[\sum_{i=1}^{m}\alpha_i Q_i(t)x(t) + \sum_{i=1}^{m}\alpha_i Q_i(t)P(t)x\bigl(\tau(t)\bigr)\right]\\ &= -\sum_{i=1}^{m}\alpha_i Q_i(t)z(t). \end{aligned} \tag{2.2}$$

For $t \ge T$, we define a generalized Riccati substitution by

$$V(t) := -\rho(t)\left[\frac{r(t)z'(t)}{z(t)} + r(t)\sigma(t)\right]. \tag{2.3}$$

Then we have

$$\begin{aligned} V' &= -\rho'\left(\frac{rz'}{z} + r\sigma\right) - \rho\left(\frac{rz'}{z} + r\sigma\right)'\\ &= \frac{\rho'}{\rho}V - \rho\left(\frac{rz'}{z}\right)' - \rho(r\sigma)'\\ &= \frac{\rho'}{\rho}V - \rho\frac{(rz')'}{z} + \rho\frac{r(z')^2}{z^2} - \rho(r\sigma)'. \end{aligned} \tag{2.4}$$

By virtue of (2.3), we obtain

$$\left(\frac{z'}{z}\right)^2 = \left(\frac{V}{-\rho r} - \sigma\right)^2 = \left(\frac{V}{\rho r}\right)^2 + \sigma^2 + 2\frac{V\sigma}{\rho r}. \tag{2.5}$$

For $t \in [s_1, t_1]$, substituting (2.2) and (2.5) into (2.4), we conclude that

$$\begin{aligned} V' &= \frac{\rho'}{\rho}V - \rho\frac{(rz')'}{z} + \rho r\left(\frac{V^2}{\rho^2 r^2} + \sigma^2 + 2\frac{V\sigma}{\rho r}\right) - \rho(r\sigma)'\\ &= -\rho\frac{(rz')'}{z} + \rho\bigl[r\sigma^2 - (r\sigma)'\bigr] + \left(\frac{\rho'}{\rho} + 2\sigma\right)V + \frac{V^2}{\rho r}\\ &\ge \rho\left[\sum_{i=1}^{m}\alpha_i Q_i + r\sigma^2 - (r\sigma)'\right] + \left(\frac{\rho'}{\rho} + 2\sigma\right)V + \frac{V^2}{\rho r}. \end{aligned} \tag{2.6}$$

Let $u \in B_1$ be given as in the hypothesis. Multiplying (2.6) by u^2 and integrating the resulting inequality from s_1 to t_1, we have

$$\begin{aligned} \int_{s_1}^{t_1} u^2 V'\,\mathrm{d}t \ge{}& \int_{s_1}^{t_1} u^2\rho\left[\sum_{i=1}^{m}\alpha_i Q_i + r\sigma^2 - (r\sigma)'\right]\mathrm{d}t + \int_{s_1}^{t_1}\left(\frac{\rho'}{\rho} + 2\sigma\right)Vu^2\,\mathrm{d}t\\ &+ \int_{s_1}^{t_1}\frac{V^2}{\rho r}u^2\,\mathrm{d}t. \end{aligned} \tag{2.7}$$

Integrating (2.7) by parts and using the fact that $u(s_1) = u(t_1) = 0$, we deduce that

$$-\int_{s_1}^{t_1} 2uu'V\,\mathrm{d}t \geq \int_{s_1}^{t_1} u^2\rho\left[\sum_{i=1}^{m}\alpha_i Q_i + r\sigma^2 - (r\sigma)'\right]\mathrm{d}t + \int_{s_1}^{t_1}\left(\frac{\rho'}{\rho} + 2\sigma\right)Vu^2\,\mathrm{d}t + \int_{s_1}^{t_1}\frac{V^2}{\rho r}u^2\,\mathrm{d}t.$$

That is,

$$\int_{s_1}^{t_1}\left[\frac{u^2V^2}{\rho r} + 2uV\left(u' + u\left(\frac{\rho'}{2\rho} + \sigma\right)\right)\right]\mathrm{d}t + \int_{s_1}^{t_1} u^2\rho\left[\sum_{i=1}^{m}\alpha_i Q_i + r\sigma^2 - (r\sigma)'\right]\mathrm{d}t \leq 0.$$

Hence,

$$\int_{s_1}^{t_1}\left[\frac{uV}{\sqrt{\rho r}} + \sqrt{\rho r}\left(u' + \frac{u\rho'}{2\rho} + u\sigma\right)\right]^2\mathrm{d}t + \int_{s_1}^{t_1}\left[u^2\rho\left(\sum_{i=1}^{m}\alpha_i Q_i + r\sigma^2 - (r\sigma)'\right) - \rho r\left(u' + \frac{u\rho'}{2\rho} + u\sigma\right)^2\right]\mathrm{d}t \leq 0,$$

which is equivalent to

$$\int_{s_1}^{t_1}\left[\frac{uV}{\sqrt{\rho r}} + \sqrt{\rho r}\left(u' + \frac{u\rho'}{2\rho} + u\sigma\right)\right]^2\mathrm{d}t + J_1(u,\rho,\sigma) \leq 0, \tag{2.8}$$

where $J_1(u,\rho,\sigma)$ is as in (2.1). Since $J_1(u,\rho,\sigma) > 0$, inequality (2.8) yields

$$\int_{s_1}^{t_1}\left[\frac{uV}{\sqrt{\rho r}} + \sqrt{\rho r}\left(u' + \frac{u\rho'}{2\rho} + u\sigma\right)\right]^2\mathrm{d}t \leq -J_1(u,\rho,\sigma) < 0,$$

which is a contradiction. This contradiction proves that x is not eventually positive.

When x is eventually negative, we use $u \in B_2$ and $F(t) \geq 0$ on $[s_2, t_2]$ to arrive at a similar contradiction. The proof is complete. □

Example 2.1 For $t \geq 1$, consider the forced second-order neutral delay differential equation

$$\left(x(t) + \frac{1}{2}x\left(\frac{t}{2}\right)\right)'' + 8x(t) + 4t^2x\left(\frac{t}{2}\right) = \sin t. \tag{2.9}$$

Let $r(t) = 1$, $P(t) = 1/2$, $\tau(t) = t/2$, $m = l = 1$, $Q_1(t) = 8$, $R_1(t) = 4t^2$, $f_1(y) = g_1(y) = y$, $\alpha_1 = \beta_1 = 1$, $u = \sin t$, $\rho(t) = 1$, and $\sigma(t) = 0$. Set $s_1 = (2n+1)\pi$, $t_1 = (2n+2)\pi$, $s_2 = (2n+3)\pi$, and $t_2 = (2n+4)\pi$. Then

$$J_1(u,\rho,\sigma) = \int_{s_1}^{t_1}\left\{\rho\left[u^2\left(\sum_{i=1}^{m}\alpha_i Q_i + r\sigma^2 - (r\sigma)'\right) - r\left(u' + \frac{u\rho'}{2\rho} + u\sigma\right)^2\right]\right\}(t)\,\mathrm{d}t = \int_{(2n+1)\pi}^{(2n+2)\pi}\left(8\sin^2 t - \cos^2 t\right)\mathrm{d}t = \frac{7}{2}\pi.$$

Similarly, $J_2(u,\rho,\sigma) = 7\pi/2$. Hence, by Theorem 2.1, every solution of (2.9) is oscillatory.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

Author details
[1]Qingdao Technological University, Feixian, Shandong 273400, P.R. China. [2]College of Economics, Ocean University of China, Qingdao, Shandong 266100, P.R. China. [3]LinDa Institute of Shandong Provincial Key Laboratory of Network Based Intelligent Computing, Linyi University, Linyi, Shandong 276005, P.R. China. [4]School of Informatics, Linyi University, Linyi, Shandong 276005, P.R. China.

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