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Existence and uniqueness results for q-fractional difference equations with p-Laplacian operators

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Abstract

In this paper, we consider the following two-point boundary value problem for q-fractional p-Laplace difference equations. New results on the existence and uniqueness of solutions for q-fractional boundary value problem are obtained. These results extend the corresponding ones of ordinary differential equations of integer order. Finally, an example is presented to illustrate the validity and practicability of our main results.

Keywords: fractional differential equations; existence; fixed point

1 Introduction

Fractional q-difference (q-fractional difference) equations are regarded as the fractional analog of q-difference equations. The topic of q-fractional equations has attracted the attention of many researchers. The details of some recent development of the subject can be found in [1–18], whereas the background material on q-fractional calculus can be found in [19, 20]. The study of boundary value problems of fractional q-difference equations is in its infancy.

In 2010, Ferreira [1] considered the existence of nontrivial solutions to the fractional q-difference equation

$$\begin{aligned} &D^{\alpha}_{q,0^+}x(t) = f\bigl(t,x(t)\bigr), \quad 0 \le t \le 1, \\ &x(0) = x(1) = 0, \end{aligned}$$

where $1 < \alpha \le 2$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function.

In 2011, El-Shahed and Al-Askar [2] studied the existence of a positive solution for the boundary value problem of the nonlinear factional q-difference equation

$$\begin{aligned} &{}^{C}D^{\alpha}_{q,0^+}x(t) + a(t)f\bigl(t,x(t)\bigr) = 0, \quad 0 \le t \le 1, 2 < \alpha \le 3, \\ &x(0) = D^2_q x(0) = 0, \qquad \gamma D_q x(1) + \beta D^2_q x(1) = 0. \end{aligned}$$

In 2012, Liang and Zhang [3] studied the existence and uniqueness of positive solutions for the three-point boundary problem of fractional q-differences

$${}^{C}D^{\alpha}_{q,0^+}x(t) + f\bigl(t,x(t)\bigr) = 0, \quad 0 \le t \le 1, 2 < \alpha \le 3,$$

$$x(0) = D_q x(0) = 0, \qquad D_q x(1) - \beta D_q x(\eta) = 0,$$

where $0 < \beta\eta^{\alpha-2} < 1$.

In 2013, Zhao *et al.* [4] studied the existence results for fractional q-difference equations with nonlocal q-integral boundary conditions,

$$\begin{aligned} &D_{q,0^+}^{\alpha} x(t) + f\bigl(t, x(t)\bigr) = 0, \quad 0 \le t \le 1, 2 < \alpha \le 3, \\ &x(0) = 0, \qquad x(1) = \mu x(\eta) = \mu \int_0^{\eta} \frac{(\eta \ominus_q qs)^{\beta-1}}{\Gamma_q(\beta)} u(s)\, ds. \end{aligned}$$

For some recent work on q-difference equations with p-Laplacian, we refer the reader to [15–18].

In [15], Aktuğlu and Özarslan dealt with the following Caputo q-fractional boundary value problem involving the p-Laplacian operator:

$$\begin{aligned} &D_q\bigl(\varphi_p\bigl({}^{\mathrm{C}}D_q^{\alpha} x(t)\bigr)\bigr) = f\bigl(t, x(t)\bigr), \quad 0 < t < 1, \\ &D_q^k x(0) = 0, \quad k = 2, 3, \ldots, n-1, \qquad x(0) = a_0 x(1), \qquad D_q x(0) = a_1 D_q x(1), \end{aligned}$$

where $a_0, a_1 \neq 0$, $\alpha > 1$, and $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. Under some conditions, the authors obtained the existence and uniqueness of the solution for the above boundary value problem by using the Banach contraction mapping principle.

In [16], Miao and Liang studied the following three-point boundary value problem with p-Laplacian operator:

$$\begin{aligned} &{}^{\mathrm{C}}D_q^{\beta}\bigl(\varphi_p\bigl({}^{\mathrm{C}}D_q^{\alpha} x(t)\bigr)\bigr) = f\bigl(t, x(t)\bigr), \quad 0 < t < 1, 2 < \alpha < 3, \\ &x(0) = D_q x(0) = 0, \qquad D_q x(1) = 0, \qquad D_q^{\beta} u(0) = 0, \end{aligned}$$

where $0 < \beta\gamma^{\alpha-2} < 1$. The authors proved the existence and uniqueness of a positive and nondecreasing solution for the boundary value problems by using a fixed point theorem in partially ordered sets.

In [17], Yang investigated the following fractional q-difference boundary value problem with p-Laplacian operator:

$$\begin{aligned} &{}^{\mathrm{C}}D_q^{\beta}\bigl(\varphi_p\bigl({}^{\mathrm{C}}D_q^{\alpha} x(t)\bigr)\bigr) = f\bigl(t, x(t)\bigr), \quad 0 < t < 1, 2 < \alpha < 3, \\ &x(0) = x(1) = 0, \qquad {}^{\mathrm{C}}D_q^{\alpha} x(1) = {}^{\mathrm{C}}D_q^{\alpha} x(0) = 0, \end{aligned}$$

where $1 < \alpha, \beta \le 2$. The existence results for the above boundary value problem were obtained by using the upper and lower solutions method associated with the Schauder fixed point theorem.

Very recently, in [18], Yuan and Yang considered a class of four-point boundary value problems of fractional q-difference equations with p-Laplacian operator

$$\begin{aligned} &D_q^{\beta}\bigl(\varphi_p\bigl(D_q^{\alpha} x(t)\bigr)\bigr) = f\bigl(t, x(t)\bigr), \quad 0 < t < 1, 2 < \alpha < 3, \\ &x(0) = 0, \qquad x(1) = ax\xi, \qquad D_q^{\alpha} x(0) = 0, \qquad D_q^{\alpha} x(1) = bD_q^{\alpha} x(\eta), \end{aligned}$$

where D_q^β, D_q^α are the fractional q-derivative of the Riemann-Liouville type with $1 < \alpha$, $\beta \le 2$. By applying the upper and lower solutions method associated with the Schauder fixed point theorem, the existence results of at least one positive solution for the above fractional q-difference boundary value problem with p-Laplacian operator are established.

However, the theory of boundary value problems for nonlinear q-difference equations is still in the initial stages and many aspects of this theory need to be explored. To the best of our knowledge, the theory of boundary value problems for nonlinear q-difference equations with p-Laplacian is yet to be developed.

Motivated by the previously mentioned works, we will consider the existence of solutions of q-fractional p-Laplacian BVP with two-point boundary conditions. The main difficulty is that, for $p \neq 2$, it is impossible for us to find a Green's function in the equivalent integral operator since the differential operator $D_{q,0^+}^\beta \varphi_p(D_{q,0^+}^\alpha)$ is nonlinear.

This paper is concerned with the BVP

$$\begin{cases} {}^{\mathrm{C}}D_{q,0^+}^\beta \varphi_p({}^{\mathrm{C}}D_{q,0^+}^\alpha x)(t) = f(t, x(t)), \quad t \in [0,1], \\ x(0) = \gamma x(1), \\ {}^{\mathrm{C}}D_q^\alpha x(0) = \eta \, {}^{\mathrm{C}}D_{q,0^+}^\alpha x(1), \end{cases} \tag{1}$$

where $\varphi_p(s) := |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_\nu$, $\frac{1}{p} + \frac{1}{\nu} = 1$, $0 < \alpha, \beta \le 1$, $1 < \alpha + \beta \le 2$, $0 < \gamma, \eta < 1$, by using some known fixed point theorems.

To make this paper self-contained, below we recall some well-known facts on q-calculus (see [19, 20] and references therein) and on fractional q-calculus.

In what follows, q is a real number satisfying $0 < q < 1$. We define the q-derivative of a real valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

Higher order q-derivatives are given by

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, t]$ is given by

$$I_q f(t) = \int_0^t f(s) \, d_q s = \sum_{n=0}^\infty t(1-q)q^n f\bigl(xq^n\bigr),$$

provided the series converges. If $0 < a < b$ and f is defined on the interval $[0, b]$, then

$$\int_a^b f(s) \, d_q s = \int_0^b f(s) \, d_q s - \int_0^a f(s) \, d_q s.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \qquad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

Observe that the operators I_q and D_q are inverses of each other, in the sense that

$$D_q I_q f(t) = f(t), \tag{2}$$

and if f is continuous at $t = 0$, then

$$I_q D_q f(t) = f(t) - f(0).$$

In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = D_q g(t)h(t) + g(qt)D_q h(t), \tag{3}$$

$$\int_0^x f(t)D_q g(t)\,d_q t = \big[f(t)g(t)\big]_0^x - \int_0^x D_q f(t)g(qt)\,d_q t. \tag{4}$$

In the limit $q \to 1$ the above results correspond to their counterparts in standard calculus.

A q-number denoted by $[a]_q$ is defined by

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-shifted factorial is defined as

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{j=0}^{n-1}\big(1-aq^j\big), \quad n \in \mathbb{N} \cup \{\infty\}.$$

The q-analog of the $(x-y)^n$ is

$$(x \ominus_q y)^0 := 1; \qquad (x \ominus_q y)^n := \prod_{j=0}^{n-1}\big(x - yq^j\big), \quad n \in \mathbb{N}, x, y \in \mathbb{R},$$

$$(x \ominus_q y)^\alpha := x^\alpha \prod_{j=0}^{\infty} \frac{x - yq^j}{x - yq^{\alpha+j}}, \quad \alpha \in \mathbb{R}.$$

The q-gamma function $\Gamma_q(x)$ is defined as

$$\Gamma_q(x) = \frac{(1 \ominus_q q)^{x-1}}{(1-q)^{x-1}}, \quad y \in \mathbb{R}\backslash\{0,-1,-2,\dots\},$$

and it satisfies $[x]_q\Gamma_q(x) = \Gamma_q(x+1)$.

Definition 1 Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\alpha \geq 0$ is $I_q^0 f(t) = f(t)$ and

$$I_{q,0^+}^\alpha f(t) := \int_0^t \frac{(t \ominus_q qs)^{\alpha-1}}{\Gamma_q(\alpha)} f(s)\,d_q s, \quad \alpha > 0, t \in [0,1].$$

The q-fractional integration possesses the semigroup property:

$$I_{q,0^+}^\beta I_{q,0^+}^\alpha f(t) = I_{q,0^+}^{\beta+\alpha} f(t), \quad \alpha, \beta \in \mathbb{R}^+.$$

Further,

$$I_{q,0^+}^\beta x^\sigma = \frac{\Gamma_q(\sigma+1)}{\Gamma_q(\beta+\sigma+1)} x^{\beta+\sigma}.$$

Definition 2 The Caputo fractional q-derivative of order $\beta > 0$ is defined by

$$ {}^{\mathrm{C}}D^{\beta}_{q,0^+}f(t) = I^{\lceil\beta\rceil-\beta}_{q,0^+}D^{\lceil\beta\rceil}_{q,0^+}f(t), $$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Next we recall some properties involving Riemann-Liouville q-fractional integral and Caputo fractional q-derivative ([20], Theorem 5.2):

$$ I^{\beta}_{q,0^+}{}^{\mathrm{C}}D^{\beta}_{q,0^+}f(t) = f(t) - \sum_{k=0}^{\lceil\beta\rceil-1}\frac{t^k}{\Gamma_q(k+1)}\bigl(D^k_q f\bigr)\bigl(0^+\bigr), \quad t \in (0,a], \beta > 0; \tag{5} $$

$$ {}^{\mathrm{C}}D^{\beta}_{q,0^+}I^{\beta}_{q,0^+}f(t) = f(t), \quad t \in (0,a], \beta > 0. \tag{6} $$

The following properties of the p-Laplacian operator will be used in the rest of the paper.

(L1) If $1 < p < 2$, $uv > 0$; $\|u\|, \|v\| \geq r > 0$, then

$$ \bigl|\varphi_p(u) - \varphi_p(v)\bigr| \leq (p-1)r^{p-2}|u-v|. $$

(L2) If $p > 2$, $|u|, |v| \leq R$, then

$$ \bigl|\varphi_p(u) - \varphi_p(v)\bigr| \leq (p-1)R^{p-2}|u-v|. $$

Next we present the fixed point theorems that will be used in the proofs of our main results.

Theorem 3 (Banach fixed point theorem) *Let (X,d) be a complete metric space, and let $\Psi : X \to X$ be a contraction mapping. Then Ψ admits a unique fixed point X.*

Theorem 4 *Let E be a Banach space, C a closed, convex subset of E and $U \subset C$ an open subset with $0 \in U$. Let $F : \overline{U} \to C$ be a continuous function such that $F(\overline{U})$ is contained in a compact set. Then either*

1. *F has a fixed point in $\overline{U}$, or*
2. *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

2 Main results

As mentioned before, we will discuss the existence (and uniqueness) of solutions for the nonlinear q-fractional p-Laplacian BVP with two-point boundary conditions. In what follows we assume that $1 < p < 2$, $\nu > 2$.

Lemma 5 *Given $f \in C[0,1]$, the unique solution of*

$$ \begin{cases} {}^{\mathrm{C}}D^{\beta}_{q,0^+}\varphi_p({}^{\mathrm{C}}D^{\alpha}_{q,0^+}x)(t) = f(t), & t \in [0,1], \\ x(0) = \gamma x(1), \\ {}^{\mathrm{C}}D^{\alpha}_{q,0^+}x(0) = \eta\, {}^{\mathrm{C}}D^{\alpha}_{q,0^+}x(1), \end{cases} \tag{7} $$

is

$$\begin{aligned} x(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t \ominus_q qs)^{\alpha-1} \\ &\quad \times \varphi_\nu \left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s \ominus_q q\tau)^{\beta-1} f(\tau)\,d\tau + \frac{\varphi_p(\eta)}{1-\varphi_p(\eta)} I_{q,0^+}^{\beta} f(1) \right) d_q s \\ &\quad + \frac{\gamma}{1-\gamma} \int_0^1 (1 \ominus_q qs)^{\alpha-1} \\ &\quad \times \varphi_\nu \left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s \ominus_q q\tau)^{\beta-1} f(\tau)\,d\tau + \frac{\varphi_p(\eta)}{1-\varphi_p(\eta)} I_{q,0^+}^{\beta} f(1) \right) d_q s. \end{aligned}$$

Proof Assume that $x(t)$ satisfies (7). Then from (5) we have

$$\varphi_p\bigl({}^{\mathrm{C}}D_{q,0^+}^{\alpha} x(t)\bigr) = I_{q,0^+}^{\beta} f(t) + c_0, \quad c_0 \in \mathbb{R}.$$

From the boundary condition ${}^{\mathrm{C}}D_{q,0^+}^{\alpha} x(0) = \eta\, {}^{\mathrm{C}}D_{q,0^+}^{\alpha} x(1)$, one can see that

$$\begin{aligned} &\varphi_p\bigl({}^{\mathrm{C}}D_{q,0^+}^{\alpha} x(0)\bigr) = c_0, \qquad \varphi_p(\eta) = |\eta|^{p-2}\eta, \\ &\varphi_p\bigl({}^{\mathrm{C}}D_{q,0^+}^{\alpha} x(1)\bigr) = I_{q,0^+}^{\beta} f(1) + c_0, \\ &\varphi_p\bigl(\eta\, {}^{\mathrm{C}}D_{q,0^+}^{\alpha} x(1)\bigr) = \varphi_p(\eta) I_{q,0^+}^{\beta} f(1) + c_0 \varphi_p(\eta), \\ &\varphi_p(\eta) I_{q,0^+}^{\beta} f(1) + c_0 \varphi_p(\eta) = c_0, \\ &c_0 = \frac{\varphi_p(\eta)}{1-\varphi_p(\eta)} I_{q,0^+}^{\beta} f(1). \end{aligned}$$

Thus

$$x(t) = I_{q,0^+}^{\alpha} \varphi_\nu \bigl(I_{q,0^+}^{\beta} f(\cdot) + c_0 \bigr)(t) + c_1,$$

which together with the boundary value condition $x(0) = \gamma x(1)$ yields

$$\begin{aligned} &c_1 = \gamma \bigl(I_{q,0^+}^{\alpha} \varphi_\nu \bigl(I_{q,0^+}^{\beta} f(1) + c_0 \bigr) + c_1 \bigr), \\ &c_1 = \frac{\gamma I_{q,0^+}^{\alpha} \varphi_\nu (I_{q,0^+}^{\beta} f + c_0)(1)}{1-\gamma}. \end{aligned}$$

It follows that

$$\begin{aligned} x(t) &= I_{q,0^+}^{\alpha} \varphi_\nu \left(I_{q,0^+}^{\beta} f + \frac{\varphi_p(\eta)}{1-\varphi_p(\eta)} I_{q,0^+}^{\beta} f(1) \right)(t) \\ &\quad + \frac{\gamma}{1-\gamma} I_{q,0^+}^{\alpha} \varphi_\nu \left(I_{q,0^+}^{\beta} f + + \frac{\varphi_p(\eta)}{1-\varphi_p(\eta)} I_{q,0^+}^{\beta} f(1) \right)(1). \end{aligned}$$

The converse is clear. The proof is completed. □

We denote by $C[0,1]$ the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by

$$\|x\| := \max\bigl\{ \bigl|x(t)\bigr| : 0 \le t \le 1 \bigr\}.$$

We use Lemma 5 to define an operator $\Psi : C[0,1] \to C[0,1]$ by

$$\begin{aligned}(\Psi x)(t) = {}& \frac{1}{\Gamma_q(\alpha)} \int_0^t (t \ominus_q qs)^{\alpha-1} \\ & \times \varphi_\nu\left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s \ominus_q q\tau)^{\beta-1} f(\tau)\,d\tau + \frac{\varphi_p(\eta)}{1-\varphi_p(\eta)} I_{q,0^+}^{\beta} f(1)\right) d_q s \\ & + \frac{\gamma}{1-\gamma} \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 \ominus_q qs)^{\alpha-1} \\ & \times \varphi_\nu\left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s \ominus_q q\tau)^{\beta-1} f(\tau)\,d\tau + \frac{\varphi_p(\eta)}{1-\varphi_p(\eta)} I_{q,0^+}^{\beta} f(1)\right) d_q s. \end{aligned} \tag{8}$$

Observe that the problem (1) has a (unique) solution if and only if the operator equation $\Psi x = y$ has a (unique) fixed point. In the sequel, we need the following operators:

$$\begin{aligned}(\Psi_0 x)(s) = {}& \varphi_\nu\left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s \ominus_q q\tau)^{\beta-1} f\big(\tau, x(\tau)\big)\,d\tau \right. \\ & \left. + \frac{\varphi_p(\eta)}{1-\varphi_p(\eta)} \frac{1}{\Gamma_q(\beta)} \int_0^1 (1 \ominus_q q\tau)^{\beta-1} f\big(\tau, x(\tau)\big)\,d\tau\right), \\ (\Psi_1 h)(t) = {}& \frac{1}{\Gamma_q(\alpha)} \int_0^t (t \ominus_q qs)^{\alpha-1} h(s)\,d_q s \\ & + \frac{\gamma}{1-\gamma} \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 \ominus_q qs)^{\alpha-1} h(s)\,d_q s.\end{aligned}$$

It is obvious that

$$\begin{aligned}&\Psi_0 : C[0,1] \to C[0,1], \qquad \Psi_1 : C[0,1] \to C[0,1], \\ &(\Psi x)(t) = (\Psi_1 \circ \Psi_0) x(t).\end{aligned}$$

Set

$$B_r := \big\{x \in C[0,1] : \|x\| < r\big\}.$$

To state and prove the existence and uniqueness theorem we impose the following assumptions.

(A1) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\begin{aligned}&\big|f(t,x)\big| \le a(t), \quad a(t) \in C\big([0,1], \mathbb{R}^+\big), \\ &\big|f(t,x) - f(t,y)\big| \le L|x-y|, \quad t \in [0,1], x, y \in \mathbb{R}.\end{aligned}$$

(A2) The following inequality holds:

$$\Lambda := \frac{\nu-1}{1-\gamma} \frac{1}{\Gamma_q^{\nu-1}(\beta+1)\Gamma_q(\alpha+1)} \frac{1}{(1-\eta^{p-1})^{\nu-1}} \|a\|^{\nu-2} < 1.$$

Lemma 6 *Assume that the assumption* (A1) *holds.*

(i) *If* $\nu > 2$, *then the operator* Ψ_0 *satisfies the following conditions*:

$$\left|(\Psi_0 x)(t) - (\Psi_0 y)(t)\right| \le (\nu - 1)R^{\nu-2} \frac{1}{\Gamma_q(\beta + 1)} \frac{1}{1 - \eta^{p-1}} \|x - y\|,$$

$$\left|(\Psi_0 x)(t)\right| \le R^{\nu-1},$$

where

$$R := \frac{1}{\Gamma_q(\beta + 1)} \frac{1}{1 - \eta^{p-1}} \|a\|.$$

(ii) *For any* $x \in C[0,1]$ *the function* $(\Psi_0 x)(t)$ *is uniformly continuous on* $[0,1]$.

Proof (i) Set

$$M := \sup_{0\le s\le 1} \left|f(s,0)\right|, \qquad u(s) := I^{\beta}_{q,0^+} f\bigl(s,x(s)\bigr) + \frac{\eta^{p-1}}{1 - \eta^{p-1}} I^{\beta}_{q,0^+} f\bigl(1,x(1)\bigr),$$

$$v(s) := I^{\beta}_{q,0^+} f\bigl(s,y(s)\bigr) + \frac{\eta^{p-1}}{1 - \eta^{p-1}} I^{\beta}_{q,0^+} f\bigl(1,y(1)\bigr).$$

It is clear that

$$\begin{aligned}
\left|u(s)\right| &\le \left|I^{\beta}_{q,0^+} f\bigl(s,x(s)\bigr)\right| + \frac{\eta^{p-1}}{1 - \eta^{p-1}} \left|I^{\beta}_{q,0^+} f\bigl(1,x(1)\bigr)\right| \\
&\le \frac{1}{\Gamma_q(\beta)} \left| \int_0^s (s \ominus_q q\tau)^{\beta-1} f\bigl(\tau, x(\tau)\bigr) \right| d_q\tau \\
&\quad + \frac{1}{\Gamma_q(\beta)} \frac{\eta^{p-1}}{1 - \eta^{p-1}} \left| \int_0^1 (1 \ominus_q q\tau)^{\beta-1} f\bigl(\tau, x(\tau)\bigr) \right| d_q\tau \\
&\le \frac{1}{\Gamma_q(\beta)} \|a\| \int_0^s (s \ominus_q q\tau)^{\beta-1} \, d_q\tau \\
&\quad + \frac{1}{\Gamma_q(\beta)} \frac{\eta^{p-1}}{1 - \eta^{p-1}} \|a\| \int_0^1 (1 \ominus_q q\tau)^{\beta-1} \, d_q\tau \\
&= \frac{1}{\Gamma_q(\beta + 1)} \frac{1}{1 - \eta^{p-1}} \|a\| := R.
\end{aligned}$$

It follows that

$$\left|(\Psi_0 x)(s)\right| = \left|\varphi_\nu\bigl(u(s)\bigr)\right| \le R^{\nu-1}$$

and

$$\begin{aligned}
&\left|(\Psi_0 x)(t) - (\Psi_0 y)(t)\right| \\
&\quad = \left|\varphi_\nu\bigl(u(s)\bigr) - \varphi_p\bigl(v(s)\bigr)\right| \\
&\quad \le (\nu - 1)R^{\nu-2} \left|u(s) - v(s)\right| \\
&\quad \le (\nu - 1)R^{\nu-2} \frac{1}{\Gamma_q(\beta)} \left| \int_0^s (s \ominus_q q\tau)^{\beta-1} f\bigl(\tau, x(\tau)\bigr) \, d\tau - \int_0^s (s \ominus_q q\tau)^{\beta-1} f\bigl(\tau, y(\tau)\bigr) \, d\tau \right|
\end{aligned}$$

$$
\begin{aligned}
&\quad + (\nu-1)R^{\nu-2}\frac{1}{\Gamma_q(\beta)}\frac{\eta^{p-1}}{1-\eta^{p-1}} \\
&\quad \times \left|\int_0^1 (1\ominus_q q\tau)^{\beta-1} f\big(\tau,x(\tau)\big)\,d\tau - \int_0^1 (1\ominus_q q\tau)^{\beta-1} f\big(\tau,y(\tau)\big)\,d\tau\right| \\
&\le (\nu-1)R^{\nu-2}\frac{1}{\Gamma_q(\beta+1)}\frac{1}{1-\eta^{p-1}}\|x-y\|.
\end{aligned}
$$

(ii) We have

$$
\begin{aligned}
&\big|(\Psi_0 x)(t_1) - (\Psi_0 x)(t_2)\big| \\
&\quad = \big|\varphi_\nu\big(u(t_1)\big) - \varphi_p\big(u(t_2)\big)\big| \\
&\quad \le (\nu-1)R^{\nu-2}\big|u(t_1)-u(t_2)\big| \\
&\quad \le (\nu-1)R^{\nu-2}\frac{1}{\Gamma_q(\beta)} \\
&\qquad \times \left|\int_0^{t_1} (t_1\ominus_q q\tau)^{\beta-1} f\big(\tau,x(\tau)\big)\,d\tau - \int_0^{t_2} (t_2\ominus_q q\tau)^{\beta-1} f\big(\tau,x(\tau)\big)\,d\tau\right| \\
&\quad = (\nu-1)R^{\nu-2}\frac{1}{\Gamma_q(\beta)} \\
&\qquad \times \left|t_1^\beta \int_0^1 (1\ominus_q qs)^{\beta-1} f\big(t_1 s,x(t_1 s)\big)\,ds - t_2^\beta \int_0^1 (1\ominus_q qs)^{\beta-1} f\big(t_2 s,x(t_2 s)\big)\,ds\right| \\
&\quad \le (\nu-1)R^{\nu-2}\frac{1}{\Gamma_q(\beta)}\left|\big(t_1^\beta - t_2^\beta\big)\int_0^1 (1\ominus_q qs)^{\beta-1} f\big(t_1 s,x(t_1 s)\big)\,ds\right| \\
&\qquad + (\nu-1)R^{\nu-2}\frac{1}{\Gamma_q(\beta)}\left|t_2^\beta \int_0^1 (1\ominus_q qs)^{\beta-1}\big(f\big(t_1 s,x(t_1 s)\big) - f\big(t_2 s,x(t_2 s)\big)\big)\,ds\right|.
\end{aligned}
$$

It follows that $(\Psi_0 x)(t)$ is uniformly continuous on $[0,1]$. □

The first theorem that we state follows from the Banach fixed point theorem.

Theorem 7 *Under the assumptions* (A1) *and* (A2) *the boundary value problem* (1) *has a unique solution in* $C[0,1]$.

Proof The idea of the proof is to show that Ψ as defined in (8) admits a unique fixed point in $C[0,1]$. For $x,y\in C[0,1]$ and for each $t\in[0,1]$, from the definition of Ψ and the assumptions (A1) and (A2), we obtain

$$
\begin{aligned}
&\big|(\Psi x)(t) - (\Psi y)(t)\big| \\
&\quad = \big|(\Psi_1\circ\Psi_0)x(t) - (\Psi_1\circ\Psi_0)y(t)\big| \\
&\quad \le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t\ominus_q qs)^{\alpha-1}\big|(\Psi_0 x)(s) - (\Psi_0 y)(s)\big|\,d_q s \\
&\qquad + \frac{\gamma}{1-\gamma}\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1\ominus_q qs)^{\alpha-1}\big|(\Psi_0 x)(s) - (\Psi_0 y)(s)\big|\,d_q s \\
&\quad \le \frac{1}{1-\gamma}\frac{1}{\Gamma_q(\alpha+1)}\|\Psi_0 x - \Psi_0 y\|.
\end{aligned}
$$

Consequently, by Lemma 6 we have

$$\left|(\Psi x)(t)-(\Psi y)(t)\right| \le \frac{\nu-1}{1-\gamma}\frac{1}{\Gamma_q^{\nu-1}(\beta+1)\Gamma_q(\alpha+1)}\frac{1}{(1-\eta^{p-1})^{\nu-1}}\|a\|^{\nu-2}\|x-y\|.$$

Taking into account that, by our assumption (A2), $\Lambda<1$, we conclude that the operator Ψ is a contraction. Therefore, by the Banach contraction principle, the problem (1) has a unique solution. This completes the proof of Theorem 7. □

For our next result, we use the Leray-Schauder alternative to ensure the existence of a solution for (1).

(A3) $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and there exist a function $l\in C([0,1],\mathbb{R}^+)$ and nondecreasing functions $\psi:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\left|f(s,x)\right| \le l(s)\psi\left(|x|\right), \quad (s,x)\in[0,1]\times\mathbb{R}.$$

(A4) There exists a constant $\omega>0$ such that

$$\omega > \frac{1}{1-\gamma}\frac{1}{\Gamma_q^{\nu-1}(\beta+1)\Gamma_q(\alpha+1)}\frac{1}{(1-\eta^{p-1})^{\nu-1}}\|l\|^{\nu-1}\psi^{\nu-1}(\omega).$$

Theorem 8 *Under conditions* (A3) *and* (A4), *the boundary value problem* (1) *has at least one solution in* $C[0,1]$.

Proof Consider the operator $\Psi: C[0,1]\to C[0,1]$ defined by (8). It is easy to show that Ψ is continuous. We complete the proof in the following steps.

Step 1: Ψ maps bounded sets into bounded sets in $C[0,1]$.

Indeed, for $x\in\overline{B}_r$ from (A3) and Lemma 6 ($a(s)=l(s)\psi(r)$) we have

$$\begin{aligned}
\left|(\Psi x)(t)\right| &= \left|(\Psi_1\circ\Psi_0)x(t)\right| \\
&\le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t\ominus_q qs)^{\alpha-1}\left|\Psi_0 x(s)\right|\,d_qs \\
&\quad + \frac{\gamma}{1-\gamma}\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1\ominus_q qs)^{\alpha-1}\left|\Psi_0 x(s)\right|\,d_qs \\
&\le \frac{1}{1-\gamma}\frac{1}{\Gamma_q(\alpha+1)}\|\Psi_0 x\| \\
&\le \frac{1}{1-\gamma}\frac{1}{\Gamma_q(\alpha+1)}R^{\nu-1} \\
&\le \frac{1}{1-\gamma}\frac{1}{\Gamma_q^{\nu-1}(\beta+1)\Gamma_q(\alpha+1)}\frac{1}{(1-\eta^{p-1})^{\nu-1}}\|l\|^{\nu-1}\psi^{\nu-1}(r)
\end{aligned}$$

and the result follows.

Step 2: Ψ maps bounded sets into equicontinuous sets of $C[0,1]$.

Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in\overline{B}_r$. Then we can write

$$\begin{aligned}
&\left|(\Psi x)(t_2)-(\Psi x)(t_1)\right| \\
&\quad = \left|\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1\ominus_q qs)^{\alpha-1}(\Psi_0 x)(s)\,d_qs - \frac{1}{\Gamma_q(\alpha)}\int_0^{t_2}(t_2\ominus_q qs)^{\alpha-1}(\Psi_0 x)(s)\,d_qs\right|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma_q(\alpha)}\left| t_1^\alpha \int_0^1 (1\ominus_q q\tau)^{\alpha-1}(\Psi_0 x)(t_1\tau)\,d_q\tau - t_2^\alpha \int_0^1 (1\ominus_q q\tau)^{\alpha-1}(\Psi_0 x)(t_2\tau)\,d_q\tau\right| \\
&\le \frac{1}{\Gamma_q(\alpha)}\left|\left(t_1^\alpha - t_2^\alpha\right)\int_0^1 (1\ominus_q q\tau)^{\alpha-1}(\Psi_0 x)(t_1\tau)\,d_q\tau\right| \\
&\quad + \frac{1}{\Gamma_q(\alpha)} t_2^\alpha \int_0^1 (1\ominus_q q\tau)^{\alpha-1}\left|(\Psi_0 x)(t_1\tau) - (\Psi_0 x)(t_2\tau)\right|\,d_q\tau := J.
\end{aligned}$$

By Lemma 6 the function $(\Psi_0 x)(t)$ is uniformly continuous on $[0,1]$ and uniformly bounded on $\overline{B}_r$, it follows that $\lim_{t_1\to t_2} J = 0$. Thus $\Psi(\overline{B}_r)$ is equicontinuous. It follows from the Arzelá-Ascoli theorem that the operator $\Psi : C[0,1] \to C[0,1]$ is compact.

Step 3: Ψ has a fixed point in $\overline{B}_\omega$.

Let x be a solution and $x = \lambda\Psi x$, $0 < \lambda < 1$. Using the arguments of the proof of boundedness of Ψ, for $0 \le t \le 1$ we can write

$$\begin{aligned}
\left|x(t)\right| &= \left|\lambda(\Psi x)(t)\right| \\
&\le \frac{1}{1-\gamma}\frac{1}{\Gamma_q^{\nu-1}(\beta+1)\Gamma_q(\alpha+1)}\frac{1}{(1-\eta^{p-1})^{\nu-1}}\|l\|^{\nu-1}\psi^{\nu-1}\left(\|x\|\right).
\end{aligned}$$

Consequently, in view of (A4), there exists $\omega > 0$ such that $\|x\| \neq \omega$. We observe that the operator $\Psi : \overline{B}_\omega \to C[0,1]$ is continuous and completely continuous. From the choice of B_ω, there is no $x \in \partial B_\omega$ such that $x = \lambda\Psi x$ for some $\lambda \in (0,1)$. Consequently, we can apply a nonlinear Leray-Schauder type alternative, to conclude that Ψ has a fixed point $x \in \overline{B}_\omega$ which is a solution of the problem (1). This completes the proof of Theorem 8. □

3 Examples

Example 1 Consider a two-point boundary value problem of nonlinear fractional q-difference equations given by

$$\begin{aligned}
&{}^{\mathrm{C}}D_{q,0^+}^{1/2}\varphi_{3/2}\left({}^{\mathrm{C}}D_{q,0^+}^{3/4}x(t)\right) = \tan^{-1}x(t) + \sin t, \\
&x(0) = \gamma x(1), \\
&{}^{\mathrm{C}}D_{q,0^+}^{3/4}x(0) = \eta\, {}^{\mathrm{C}}D_{q,0^+}^{3/4}x(1).
\end{aligned} \tag{9}$$

Corresponding to (1), we get $\beta = 1/2$, $p = 3/2$, $\nu = 3$, $\alpha = 3/4$, and $f(t,x) = \tan^{-1}x + \sin t$, $a(t) = 1 + \pi/2$. It is obvious that

$$\left|f(t,x)\right| \le 1 + \frac{\pi}{2}, \qquad \left|f(t,x) - f(t,y)\right| \le |x-y|.$$

We choose γ, η such that

$$\begin{aligned}
\Lambda &= \frac{\nu-1}{1-\gamma}\frac{1}{\Gamma_q^{\nu-1}(\beta+1)\Gamma_q(\alpha+1)}\left(1 + \left|\frac{\varphi_p(\eta)}{1-\varphi_p(\eta)}\right|\right)^{\nu-1}\|a\|^{\nu-2} \\
&= \frac{2}{1-\gamma}\frac{1}{\Gamma_q^2(3\pi/2)\Gamma_q(7/4)}\left(\frac{1}{1-\eta^{1/2}}\right)^2\left(1+\frac{\pi}{2}\right) < 1.
\end{aligned}$$

The above facts imply that the BVP (9) satisfies all assumptions of Theorem 7 and has a unique solution.

Example 2 Consider (9) with a different right-hand side $f(t,x)$:

$$
\begin{aligned}
&{}^{\mathrm{C}}D_{q,0^+}^{1/2}\varphi_{3/2}\bigl({}^{\mathrm{C}}D_{q,0^+}^{3/4}x(t)\bigr) = \frac{\cos(t^4+1)}{\sqrt{4+t}}\left(\bigl|x(t)\bigr| + \frac{|x(t)|^3}{1+|x(t)|^3} + \frac{1}{2}\right),\\
&x(0) = \gamma x(1), \\
&{}^{\mathrm{C}}D_{q,0^+}^{3/4}x(0) = \eta\, {}^{\mathrm{C}}D_{q,0^+}^{3/4}x(1).
\end{aligned} \tag{10}
$$

Choosing

$$
f(t,x) = \frac{\cos(t^4+1)}{\sqrt{4+t}}\left(|x| + \frac{|x|^3}{1+|x|^3} + \frac{1}{2}\right)
$$

one can see that

$$
\bigl|f(t,x)\bigr| \le l(t)\psi\bigl(|x|\bigr), \quad (t,x) \in [0,1]\times\mathbb{R},
$$

with

$$
l(t) = \frac{\cos(t^4+1)}{\sqrt{4+t}}, \qquad \psi\bigl(|x|\bigr) = |x| + \frac{3}{2}.
$$

We may choose γ, η such that

$$
0 < \rho := \frac{1}{1-\gamma}\frac{1}{\Gamma_q^{\nu-1}(\beta+1)\Gamma_q(\alpha+1)}\frac{1}{(1-\eta^{p-1})^{\nu-1}}\|l\|^{\nu-1} < \frac{1}{6}.
$$

Using this one can see that there is $\omega > 0$ such that

$$
\omega > \rho\left(\omega + \frac{3}{2}\right)^2.
$$

Thus all the conditions of Theorem 8 are satisfied. Hence there exists a solution of the problem (10).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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