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Oscillation criteria for a class of higher odd order neutral difference equations with continuous variable

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Abstract
In this paper, we are mainly concerned with oscillatory behavior of solutions for a class of higher odd order nonlinear neutral difference equations with continuous variable. By converting the above difference equations to the corresponding differential equations and inequalities, the oscillatory criteria are obtained. In addition, examples are given to illustrate the obtained criteria, respectively.

Keywords: neutral difference equations; oscillation; continuous variable

1 Introduction

Difference equations have attracted a great deal of attention of researchers in mathematics, biology, physics, and economy. This is specially due to the applications in various problems of biology, physics, economy. Among the topics studied for oscillation of the solutions has been investigated intensively. Please see [1–18].

In this paper, we deal with the nonlinear neutral difference equation with continuous variable of the form

$$\Delta_\tau^m\big(x(t)-px(t-r)\big)+f\big(t,x\big(g(t)\big)\big)=0, \tag{1.1}$$

where $m\ge 3$, $p\ge 0$, τ and r are positive constants, $\Delta_\tau x(t)=x(t+\tau)-x(t)$, $0<g(t)<t$, $g\in C^1([t_0,\infty),R_+)$, $g'(t)>0$, and $f\in C([t_0,\infty)\times R,R)$. Throughout this paper we assume that

$$g(t+\tau)\ge g(t)+\tau \quad \text{for } t\ge t_0 \tag{1.2}$$

and

$$f(t,u)/u\ge q(t)>0 \quad \text{for } u\ne 0 \text{ and some } q\in C(R,R_+). \tag{1.3}$$

Let $t_0'=\min\{g(t_0),t_0-r\}$ and $I_0=[t_0',t_0]$. A function x is called the *solution* of (1.1) with $x(t)=\varphi(t)$ for $t\in I_0$ and $\varphi\in C(I_0,R)$ if it satisfies (1.1) for $t\ge t_0$.

A solution x is said to be *oscillatory* if it is neither eventually positive nor eventually negative; it is called *nonoscillatory* if it is not oscillatory.

The organization of this paper is as follows. We will give the main results in Section 2 and leave the proofs to Section 5. Three demonstrated examples will be presented in Section 3. In Section 4, some lemmas will be given to prove the main results.

2 Statement of the main results

For later convenience, let

$$\bar{q}(t) = \alpha \min_{t \le s \le t+m\tau} \{q(s)\} \Big(\min_{g(t) \le s \le g(t)+m\tau} \{ (g^{-1}(s))' \} \Big)^m, \tag{2.1}$$

where $0 < \alpha < 1$. Throughout this paper, the function $\bar{q}$ will play an important role in the oscillatory criteria for (1.1). Let

$$\beta_1 = \inf_{t \ge T} \left\{ \frac{(g^{-1}(t) - t)^{m-1} \bar{q}(g^{-1}(t))}{(m-1)!\tau^m} \right\} \tag{2.2}$$

and

$$\beta_2 = \inf_{t \ge T} \left\{ \frac{(g^{-1}(t) - t)^{m-1} \bar{q}(t)}{(m-1)!\tau^m} \right\}, \tag{2.3}$$

where $T \geq t_0$ is sufficiently large.

Theorem 2.1 *Assume that* (1.1) *with* $0 < p < 1$ *satisfies*

$$r\beta_1 \sum_{i=1}^{n} ip^i \ge 1 \tag{2.4}$$

and

$$0 \le \liminf_{t \to \infty} \int_{t-r}^{t} \big(g^{-1}(s) - s\big)^{m-1} \bar{q}\big(g^{-1}(s)\big)\, ds \le \frac{(m-1)!\tau^m (1-p) e^{-1}}{p - p^{n+1}} \tag{2.5}$$

for some integer $n \geq 1$. *Also assume that* $\bar{q}(t)$ *given by* (2.1) *is nonincreasing. Then, for every bounded solution* $x(t)$ *of* (1.1), *either* $x(t)$ *is oscillatory or* $\liminf_{t\to\infty}(|x(t)| - p|x(t-r)|) < 0$.

Corrollary 2.2 *The conclusion of Theorem* 2.1 *still holds if* (2.5) *is replaced by*

$$0 \le \liminf_{t \to \infty} \int_{t-r}^{t} \big(g^{-1}(s) - s\big)^{m-1} \bar{q}\big(g^{-1}(s)\big)\, ds \le \frac{(m-1)!\tau^m}{ep}. \tag{2.6}$$

Corrollary 2.3 *Assume that* (1.1) *with* $0 < p < 1$ *satisfies*

$$r\beta_2 \sum_{i=1}^{n} ip^i \ge 1 \tag{2.7}$$

and

$$0 \le \liminf_{t \to \infty} \int_{t-r}^{t} \big(g^{-1}(s) - s\big)^{m-1} \bar{q}(s)\, ds \le \frac{(m-1)!\tau^m (1-p) e^{-1}}{p - p^{n+1}} \tag{2.8}$$

for some integer $n \geq 1$. Also assume that $\bar{q}(t)$ given by (2.1) *is nondecreasing. Then the conclusion of Theorem* 2.1 *holds.*

Corollary 2.4 *The conclusion of Corollary* 2.3 *still holds if* (2.8) *is replaced by*

$$0 \leq \liminf_{t\to\infty} \int_{t-r}^{t} \left(g^{-1}(s) - s\right)^{m-1} \bar{q}(s)\,ds \leq \frac{(m-1)!\tau^m}{ep}. \tag{2.9}$$

Corollary 2.5 *Assume $0 < p < 1$ and $r = k\tau$. Under the assumptions of either Theorem* 2.1 *or Corollary* 2.2 *or Corollary* 2.3 *or Corollary* 2.4, *every bounded solution $x(t)$ of* (1.1) *is oscillatory.*

The following results are for the bounded solutions of (1.1) with $p > 1$.

Theorem 2.6 *Assume that $p > 1$, $r = k\tau$, $k \in N$, $r \geq t + m\tau - g(t)$, and*

$$r\beta_2 \sum_{i=1}^{n} \frac{(i-1)}{p^i} \geq 1 \tag{2.10}$$

for some integer $n \geq 2$. Also assume that $\bar{q}(t)$ given by (2.1) *is nondecreasing. Then every bounded solution $x(t)$ of* (1.1) *is oscillatory.*

Corollary 2.7 *Assume that $p > 1$, $r = k\tau$, $k \in N$, $r \geq t + m\tau - g(t)$, and*

$$r\beta_1 \sum_{i=1}^{n} \frac{(i-1)}{p^i} \geq 1 \tag{2.11}$$

for some integer $n \geq 2$. Also assume that $\bar{q}(t)$ given by (2.1) *is nonincreasing. Then every bounded solution $x(t)$ of* (1.1) *is oscillatory.*

3 Examples

Three examples will be given in this section to demonstrate the applications of the obtained results. From (2.2) and (2.3) it is clear that both β_1 and β_2 are nondecreasing functions of T. The following examples show that β_1 and β_2 may be independent of T or increasing functions of T.

Example 1 Consider the difference equation

$$\Delta_1^m \left(x(t) - \frac{1}{2}x(t-1) \right) + \left((m-1)! + \frac{1}{t} \right) x(t-1) = 0 \tag{3.1}$$

for $t > 0$, where m is an odd positive integer $m \geq 3$. Viewing (3.1) as (1.1), we have $\tau = 1$, $0 < p = 1/2 < 1$, $r = 1$, $q(t) = (m-1)! + 1/t$ and $g(t) = t - 1$. Then, according to (2.1),

$$\bar{q}(t) = \alpha \left((m-1)! + \frac{1}{t+m} \right).$$

So

$$\beta_1 = \inf_{t\geq T} \left\{ \frac{(t+1-t)^{m-1} \cdot \alpha((m-1)! + \frac{1}{t+m+1})}{(m-1)! \cdot 1^m} \right\} = \alpha$$

with $T \geq 3$. Since

$$\beta_1 \sum_{i=1}^{3} irp^i = \alpha \cdot \left(\frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8}\right) = \frac{11\alpha}{8} \geq 1$$

holds for $\alpha \in [8/11, 1)$ and

$$0 \leq \liminf_{t\to\infty} \int_{t-1}^{t} (s+1-s)^{m-1} \cdot \alpha\left((m-1)! + \frac{1}{s+m+1}\right) ds = \alpha \cdot (m-1)! \leq \frac{2 \cdot (m-1)!}{e}$$

holds for any $\alpha \in (0, 2/e]$, (2.4) and (2.6) are satisfied for $n = 3$ and $\alpha \in [8/11, 2/e]$. Since $r = 1 = \tau$, by Corollaries 2.2 and 2.5, every bounded solution $x(t)$ of (3.1) is oscillatory.

Example 2 Consider the difference equation

$$\Delta^m_{\frac{\pi}{m}}\big(x(t) - 2x(t-4\pi)\big) + 8x(t-\pi) + \frac{8\sigma}{1+t^2}x^3(t-\pi) = 0, \tag{3.2}$$

for $t > 0$, where m is an odd positive integer with $m \geq 3$ and σ is a positive real number. Regarding (3.2) as (1.1), we have $\tau = \pi/m$, $p = 2$, $r = 4\pi$, $g(t) = t - \pi$ and $q(t) = 8$. Then, for some $\alpha \in (0,1)$, $\bar{q} = 8\alpha$ by (2.1). Moreover, $r \geq t + m\tau - g(t)$ and $r = k\tau$ are satisfied. In addition,

$$\beta_2 = \inf_{t\geq T}\left\{\frac{8\alpha \cdot (t+\pi-t)^{m-1}}{(m-1)!(\frac{\pi}{m})^m}\right\} = \frac{8m^m\alpha}{\pi(m-1)!},$$

where $T \geq 12\pi$. So (2.10) is satisfied since

$$\beta_2 \sum_{i=1}^{3} \frac{4\pi(i-1)}{p^i} = \frac{8m^m\alpha}{\pi(m-1)!} \times 4\pi \times \left(\frac{1}{2^2} + \frac{2}{2^3}\right) = \alpha\frac{16m^m}{(m-1)!} \geq 1$$

holds for $\alpha \in [(m-1)!/(16m^m), 1)$. By Theorem 2.6, every bounded solution $x(t)$ of (3.2) is oscillatory.

Example 3 Consider the difference equation

$$\Delta^m_{\frac{\pi}{m}}\big(x(t) - 2x(t-2\pi)\big) + e^{-\frac{\sigma}{t}}x(t-\pi) = 0, \tag{3.3}$$

for $t > 0$, where m is an odd positive integer with $m \geq 3$ and σ is a positive constant. Regarding (3.3) as (1.1), we have $\tau = \pi/m$, $p = 2$, $r = 2\pi$, $g(t) = t - \pi$, and $q(t) = e^{-\frac{\sigma}{t}}$. Then, for some $\alpha \in (0,1)$, $\bar{q} = \alpha e^{-\frac{\sigma}{t}}$ by (2.1). Moreover, $r \geq t + m\tau - g(t)$ and $r = k\tau$ are satisfied. In addition,

$$\beta_2 = \inf_{t\geq T}\left\{\frac{\alpha e^{-\frac{\sigma}{t}} \cdot (t+\pi-t)^{m-1}}{(m-1)!(\frac{\pi}{m})^m}\right\} = \frac{m^m\alpha}{e^{\sigma/T}\pi(m-1)!} \to \frac{m^m\alpha}{\pi(m-1)!},$$

as $T \to \infty$. So (2.10) is satisfied when T is large enough since

$$\beta_2 \sum_{i=1}^{3} \frac{2\pi(i-1)}{p^i} \to \frac{\alpha m^m}{\pi(m-1)!} \times 2\pi \times \left(\frac{1}{2^2} + \frac{2}{2^3}\right) = \alpha\frac{m^m}{(m-1)!} > 1$$

as $T \to \infty$ for $\alpha \in ((m-1)!/(m^m), 1)$. By Theorem 2.6, every bounded solution $x(t)$ of (3.3) is oscillatory.

4 Related lemmas

To prove the main results, we need to prove the following lemmas first. The first lemma is about a function $x(t)$ satisfying the differential inequality

$$x'(t) + q(t)x\bigl(\tau(t)\bigr) \leq 0, \tag{4.1}$$

where $q, \tau \in C([t_0, \infty), R_+)$, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$. Let

$$\eta = \liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)\,ds.$$

Lemma 4.1 *Assume that τ is nondecreasing, $0 \leq \eta \leq e^{-1}$, and $x(t)$ is an eventually positive function satisfying* (4.1). *Set*

$$r = \liminf_{t\to\infty} \frac{x(t)}{x(\tau(t))}.$$

Then r satisfies

$$\frac{1 - \eta - \sqrt{1 - 2\eta - \eta^2}}{2} \leq r \leq 1.$$

The above lemma can be found in [6], p.18.

Lemma 4.2 *Let $0 \leq p < 1$. Assume that $x(t)$ is a bounded and eventually positive* (*negative*) *solution of* (1.1) *with $z(t) = x(t) - px(t-r)$ and* $\liminf_{t\to\infty} z(t) \geq 0$ ($\limsup_{t\to\infty} z(t) \leq 0$). *Let*

$$y(t) = \int_{t}^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} z(\theta)\,d\theta.$$

Then $y(t) > 0$ (< 0), $(-1)^k y^{(k)}(t) > 0$ (< 0) for $1 \leq k \leq m$ eventually. Moreover,

$$\Delta_\tau^m y(t) + \bar{q}(t) \sum_{i=0}^{n} p^i y\bigl(g(t) - ir\bigr) < 0 \ (> 0) \tag{4.2}$$

holds for any fixed natural number n and for all large enough t.

Proof Suppose $x(t)$ is a bounded and eventually positive solution. Notice that $g(t) < t$ and $g'(t) > 0$ for all $t \geq t_0$. So there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t \geq t_1$. From (1.1) it follows that

$$\Delta_\tau^m z(t) + f\bigl(t, x\bigl(g(t)\bigr)\bigr) = 0.$$

By (1.3), we have $f(t, x(g(t))) \geq q(t)x(g(t)) > 0$ for $t \geq t_1$. Therefore,

$$y^{(m)}(t) + q(t)x\bigl(g(t)\bigr) \leq 0 \tag{4.3}$$

for $t \geq t_1$. According to $q(t)x(g(t)) > 0$, $y^{(m)}(t) < 0$ for all $t \geq t_1$. Thus, $y^{(m-1)}(t)$ is decreasing so either $y^{(m-1)}(t) > 0$ for all $t \geq t_1$ or $y^{(m-1)}(t) \leq y^{(m-1)}(t_2) < 0$ for some $t_2 > t_1$ and for all $t \geq t_2$. If the latter holds, then

$$y^{(m-k)}(t) \to -\infty, \quad k = 2, 3, \ldots, m,$$

as $t \to \infty$, a contradiction to the boundedness of x and z. Therefore we have $y^{(m-1)}(t) > 0$ for all $t \geq t_1$. Thus, $y^{(m-2)}(t)$ is increasing so either $y^{(m-2)}(t) < 0$ for all $t \geq t_1$ or $y^{(m-2)}(t) \geq y^{(m-2)}(t_3) > 0$ for some $t_3 \geq t_1$ and all $t \geq t_3$. If the latter holds, then

$$y^{(m-k)}(t) \to \infty, \quad k = 3, 4, \ldots, m,$$

as $t \to \infty$, a contradiction again to the boundedness of x and z. Hence, we must have $y^{(m-2)}(t) < 0$ for all $t \geq t_1$. Repeating the above process, we obtain $(-1)^k y^{(k)}(t) > 0$ for $1 \leq k \leq m$ and all $t \geq t_1$. Therefore, $y(t)$ is decreasing so either $y(t) > 0$ for all $t \geq t_1$ or there is a $t_4 \geq t_1$ such that $y(t) \leq y(t_4) < 0$ for $t \geq t_4$. Suppose the latter case holds. Then

$$\begin{aligned}
&\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta)\,d\theta \\
&\quad = y(t) + p \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta - r)\,d\theta \\
&\quad \leq y(t_4) + p \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta - r)\,d\theta \\
&\quad \cdots \\
&\quad \leq y(t_4) \sum_{i=0}^{s-1} p^i + p^s \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta - sr)\,d\theta \\
&\quad \leq \frac{y(t_4)(1-p^s)}{1-p} + p^s M \tau^m
\end{aligned}$$

for $t \geq t_4 + sr$, where $M = \sup_{t \geq t_0} x(t)$ and s is any positive integer. Let $s \to \infty$ so $t \to \infty$ as well, $p^s M \tau^m$ then is arbitrarily small due to $0 \leq p < 1$. Thus,

$$\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta)\,d\theta < 0,$$

which contradicts the assumption that $x(t)$ is eventually positive. Therefore, we must have $y(t) > 0$ for all $t \geq t_1$.

From (4.3) it follows that

$$\Delta_\tau^m z(t) + q(t)z\bigl(g(t)\bigr) + pq(t)x\bigl(g(t) - r\bigr) \leq 0.$$

According to the definition of $z(t)$, the above inequality becomes

$$\Delta_\tau^m z(t) + q(t)z\bigl(g(t)\bigr) + pq(t)z\bigl(g(t) - r\bigr) + p^2 q(t)x\bigl(g(t) - 2r\bigr) \leq 0.$$

Proceeding in the same way as the above, we have

$$\Delta_\tau^m z(t) + q(t)\sum_{i=0}^{n} p^i z\bigl(g(t) - ir\bigr) + p^{n+1}q(t)x\bigl(g(t) - (n+1)r\bigr) \le 0.$$

Since $q(t)p^{n+1}x(g(t) - (n+1)r) > 0$ when t is large enough, the above inequality implies that

$$\Delta_\tau^m z(t) + q(t)\sum_{i=0}^{n} p^i z\bigl(g(t) - ir\bigr) < 0.$$

In order to integrate the above inequality, we need to show that $z(t)$ is positive. If $p = 0$, then $z(t) = x(t) > 0$ holds eventually. Now suppose $0 < p < 1$. Since $y^{(m)}(t) = \Delta_\tau^m z(t) < 0$ for $t \ge t_1$,

$$\Delta_\tau^{m-1} z\bigl(t + (h+1)\tau\bigr) - \Delta_\tau^{m-1} z(t + h\tau) = \Delta_\tau^m z(t + h\tau) < 0$$

so $\Delta_\tau^{m-1} z(t + h\tau)$ is decreasing as h increases. By the boundedness of $x(t)$ we know that $\lim_{h\to\infty} \Delta_\tau^{m-1} z(t + h\tau)$ exists. If $\lim_{h\to\infty} \Delta_\tau^{m-1} z(t + h\tau) = S(t) \neq 0$, then

$$\Delta_\tau^{m-2} z\bigl(t + (h+1)\tau\bigr) \to -\infty \text{ or } \infty$$

as $h \to \infty$, a contradiction to the boundedness of $\Delta_\tau^{m-2} z(t)$. Thus, for each $t \ge t_1$, $\Delta_\tau^{m-1} z(t + h\tau)$ is decreasing and tends to 0 as $h \to \infty$. Repeating the same procedure, we see that $\Delta_\tau z(t + h\tau)$ is increasing as h increases and $\Delta_\tau z(t + h\tau) \to 0$ as $h \to \infty$; $z(t + h\tau)$ is decreasing as h increases so $\lim_{h\to\infty} z(t + h\tau)$ exists for each $t \ge t_1$. By assumption, $\liminf_{t\to\infty} z(t) \ge 0$. Then $z(t + h\tau)$ is decreasing and $\lim_{h\to\infty} z(t + h\tau) \ge 0$ so $z(t + h\tau) > 0$ for all $t \ge t_1$ and $h \ge 1$. Integrating $q(t)z(g(t) - ir)$, by the assumptions on g and q, we obtain

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-1}}^{s_{m-1}+\tau} z\bigl(g(\theta) - ir\bigr)q(\theta)\, d\theta \\
&\quad \ge \min_{t\le s\le t+m\tau}\bigl\{q(s)\bigr\} \int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-1}}^{s_{m-1}+\tau} z\bigl(g(\theta) - ir\bigr)\, d\theta \\
&\quad \ge \min_{t\le s\le t+m\tau}\bigl\{q(s)\bigr\} \int_{g(t)}^{g(t+\tau)} \bigl(g^{-1}(s_1)\bigr)'\, ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} \bigl(g^{-1}(s_2)\bigr)'\, ds_2 \cdots \\
&\qquad \times \int_{s_{m-1}}^{g(g^{-1}(s_{m-1})+\tau)} z(\theta - ir)\bigl(g^{-1}(\theta)\bigr)'\, d\theta \\
&\quad \ge \min_{t\le s\le t+m\tau}\bigl\{q(s)\bigr\} \Bigl(\min_{g(t)\le s\le g(t)+m\tau} \bigl(g^{-1}(s)\bigr)'\Bigr)^m \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \\
&\qquad \times \int_{s_{m-1}}^{s_{m-1}+\tau} z(\theta - ir)\, d\theta \\
&\quad \ge \min_{t\le s\le t+m\tau}\bigl\{q(s)\bigr\} \Bigl(\min_{g(t)\le s\le g(t)+m\tau} \bigl(g^{-1}(s)\bigr)'\Bigr)^m y\bigl(g(t) - ir\bigr) \\
&\quad \ge \bar{q}(t)y\bigl(g(t) - ir\bigr).
\end{aligned}$$

Therefore,

$$\Delta_\tau^m y(t) + \bar{q}(t)\sum_{i=0}^{n} p^i y\big(g(t) - ir\big) < 0$$

holds for any fixed natural number n and for all large enough t. If $x(t)$ is a bounded and eventually negative solution, then the above proof with obvious changes shows the conclusion within brackets. □

Lemma 4.3 *Let* $0 \le p < 1$ *and* $r = k\tau$. *Assume that* $x(t)$ *is a bounded and eventually positive* (*negative*) *solution of* (1.1). *Let*

$$z(t) = x(t) - px(t - r),$$
$$y(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} z(\theta)\, d\theta.$$

Then the conclusion of Lemma 4.2 *holds.*

Proof The proof is the same as that of Lemma 4.2 until $\lim_{h\to\infty} z(t + h\tau)$ exists for each $t \ge t_1$. Suppose there is a $t' > t_1$ such that $\lim_{h\to\infty} z(t' + h\tau) = \delta < 0$. Then $z(t' + h\tau) \le \delta/2 < 0$ for $h \ge h_1 > 0$ so $z(t' + hr + h_1\tau) = z(t' + (kh + h_1)\tau) \le \delta/2$ for $h \ge 0$. Thus, for $h > 0$,

$$\begin{aligned} x\big(t' + hr + h_1\tau\big) &\le \delta/2 + px\big(t' + (h-1)r + h_1\tau\big) \\ &\le \delta\big(1 + p + \cdots + p^{h-1}\big)/2 + p^h x\big(t' + h_1\tau\big). \end{aligned}$$

This implies $x(t' + k(h + h_1)\tau) < 0$ for large h, a contradiction to the assumption that x is eventually positive. Therefore $\lim_{h\to\infty} z(t + h\tau) \ge 0$ for $t \ge t_1$. Since $z(t + h\tau)$ is decreasing as h increases, $z(t) > 0$ for all $t \ge t_1$. The rest of the proof of Lemma 4.2 is still valid here. □

Lemma 4.4 *Under the assumptions of Lemma* 4.2 *or Lemma* 4.3, *let*

$$v(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta)\, d\theta.$$

Then $v(t) > 0$ (< 0), $(-1)^k v^{(k)}(t) > 0$ (< 0) *for* $1 \le k \le m$ *eventually. Moreover,*

$$v^{(m)}(t) + \frac{1}{\tau^m}\bar{q}(t)\sum_{i=0}^{n} p^i v\big(g(t) - ir\big) < 0 \ (> 0) \tag{4.4}$$

holds for any fixed natural number n *and for all large enough* t.

Proof By the definition of $v(t)$, $v(t)$ has the same sign as $y(t)$ for all $t \ge t_1$. Furthermore, we have

$$v'(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y'(\theta)\, d\theta.$$

Then $v'(t)$ has the same sign as $y'(t)$. Similarly, $v^{(j)}(t)$ has the same sign as $y^{(j)}(t)$ for all $j = 1, 2, \ldots, m$. Notice also that $v^{(m)}(t) = \Delta_\tau^m y(t)$. If $y'(t) < 0$, then

$$\begin{aligned} v\bigl(g(t) - ir\bigr) &= \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta - ir)\, d\theta \\ &\le \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(t_{m-1} - ir)\, d\theta \\ &\le \tau \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-2}}^{t_{m-2}+\tau} y(t_{m-1} - ir)\, dt_{m-1} \\ &\cdots \\ &\le \tau^{m-1} \int_{g(t)}^{g(t)+\tau} y(t_1 - ir)\, dt_1 \\ &\le \tau^m y\bigl(g(t) - ir\bigr). \end{aligned}$$

Hence, from (4.2) it follows that

$$v^{(m)}(t) + \frac{1}{\tau^m} \bar{q}(t) \sum_{i=0}^{n} p^i v\bigl(g(t) - ir\bigr) < 0$$

holds for any fixed natural number n and for all large enough t. If $y'(t) > 0$, then $v(g(t) - ir) \ge \tau^m y(g(t) - ir)$ so

$$v^{(m)}(t) + \frac{1}{\tau^m} \bar{q}(t) \sum_{i=0}^{n} p^i v\bigl(g(t) - ir\bigr) > 0.$$

□

Lemma 4.5 *Under the assumptions of Lemma* 4.4, *for each* $t \ge t_1$ *there is a* $\theta \in (g(t), t)$ *such that*

$$\bigl|v'\bigl(g(t)\bigr)\bigr| > \frac{(t - g(t))^{m-1}}{(m-1)!} \bigl|v^{(m)}(\theta)\bigr|. \tag{4.5}$$

Proof Under the assumptions of Lemma 4.4, we know that $(-1)^j v^{(j)}(t)$ for $j = 1, 2, \ldots, m$ have the same sign. According to Taylor's formula, we have

$$\begin{aligned} v'\bigl(g(t)\bigr) &= v'(t) + v''(t)\bigl(g(t) - t\bigr) + \frac{1}{2} v^{(3)}(t)\bigl(g(t) - t\bigr)^2 + \cdots \\ &\quad + \frac{1}{(m-1)!} v^{(m)}(\theta)\bigl(g(t) - t\bigr)^{m-1} \end{aligned}$$

for some $\theta \in (g(t), t)$ and (4.5) follows immediately. □

The next lemmas are for the bounded solutions of (1.1) with $p > 1$.

Lemma 4.6 *Let* $p > 1$ *and* $r = k\tau$, $k \in N$. *Assume that* $x(t)$ *is a bounded and eventually positive* (*negative*) *solution of* (1.1). *Let*

$$\begin{aligned} z(t) &= x(t) - px(t - r), \\ y(t) &= \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} z(\theta)\, d\theta. \end{aligned}$$

Then $y(t) < 0$ (> 0), $(-1)^k y^{(k)}(t) > 0$ (< 0) for $1 \leq k \leq m$ eventually. Moreover,

$$\Delta_\tau^m y(t) - \bar{q}(t) \sum_{i=1}^{n} \frac{1}{p^i} y\bigl(g(t) + ir\bigr) < 0 \ (> 0) \tag{4.6}$$

holds for any fixed integer $n \geq 1$ and for all large enough t.

Proof Suppose $x(t)$ is a bounded and eventually positive solution. Since $g(t) < t$ and $g'(t) > 0$, from the assumptions, there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t \geq t_1$. Notice also that

$$\Delta_\tau^m z(t) + f\bigl(t, x\bigl(g(t)\bigr)\bigr) = 0.$$

According to (1.3), we have $f(t, x(g(t))) \geq q(t)x(g(t)) > 0$ for $t \geq t_1$. Therefore

$$\Delta_\tau^m z(t) + q(t)x\bigl(g(t)\bigr) \leq 0 \tag{4.7}$$

for $t \geq t_1$. By the definition of $y(t)$, $y^{(m)}(t) = \Delta_\tau^m z(t)$. Thus, from (4.7) it follows that

$$y^{(m)}(t) + q(t)x\bigl(g(t)\bigr) \leq 0 \tag{4.8}$$

for $t \geq t_1$. Due to $q(t)x(g(t)) > 0$, $y^{(m)}(t) < 0$ for all $t \geq t_1$. From the proof of Lemma 4.2 we know that $(-1)^k y^{(k)}(t) > 0$ holds for $1 \leq k \leq m$ and all $t \geq t_1$. Thus, $y(t)$ is decreasing. We now prove that $y(t) < 0$ for all $t \geq t_1$. Since $y^{(m)}(t) = \Delta_\tau^m z(t)$ for all $t \geq t_1$, from the proof of Lemma 4.2 we know that $z(t + h\tau)$ is decreasing for each fixed $t \geq t_1$ as h increases. Next we show that $z(t) < 0$, so that $y(t) < 0$ for some $t_2 \geq t_1$ and all $t \geq t_2$. Suppose there is a $t' > t_1$ such that $z(t' + h\tau) > 0$ for all $h \geq 1$. Under $r = k\tau$, we then have $z(t' + hr) > 0$ for all $h \geq 1$ so $x(t' + hr) > p^h x(t')$ for all $h \geq 1$. So $x(t' + hr) \to \infty$ as $h \to \infty$, a contradiction to the boundedness of x. Therefore, for each $t \in [t_1, t_1 + \tau]$, $z(t + h\tau)$ is decreasing as h increases and there is an integer $H(t) > 0$ such that $z(t + h\tau) < z(t + H(t)\tau) < 0$ for all $h > H(t)$. Since $z(t)$ is continuous for each $t' \in [t_1, t_1 + \tau]$, there is an open interval $I(t')$ such that $z(t + h\tau) < z(t + H(t')\tau) < 0$ hold for all $t \in I(t')$ and $h > H(t')$. Since $[t_1, t_1 + \tau]$ is compact and $\{I(t') : t' \in [t_1, t_1 + \tau]\}$ is an open cover of $[t_1, t_1 + \tau]$, there is a finite subset of $\{I(t') : t' \in [t_1, t_1 + \tau]\}$ covering $[t_1, t_1 + \tau]$. Therefore, there is a $K > 0$ such that

$$z(t + h\tau) \leq z(t + K\tau) < 0$$

for all $t \in [t_1, t_1 + \tau]$ and all $h \geq K$. Hence, there is a $t_3 > t_1$ such that $z(t) < 0$, so that $y(t) < 0$ for all $t \geq t_3$.

From (4.7), we have

$$\Delta_\tau^m z(t) - \frac{q(t)}{p} z\bigl(g(t) + r\bigr) + \frac{q(t)}{p} x\bigl(g(t) + r\bigr) \leq 0.$$

According to the definition of $z(t)$, it follows from the above inequality that

$$\Delta_\tau^m z(t) - \frac{q(t)}{p} z\bigl(g(t) + r\bigr) + \frac{q(t)}{p}\left(-\frac{1}{p} z\bigl(g(t) + 2r\bigr) + \frac{1}{p} x\bigl(g(t) + 2r\bigr)\right) \leq 0.$$

Repeating the above procedure, we obtain

$$\Delta_\tau^m z(t) - q(t)\sum_{i=1}^{n}\frac{1}{p^i}z\big(g(t)+ir\big) + q(t)\frac{1}{p^n}x\big(g(t)+nr\big) \le 0.$$

Since $q(t)x(g(t)+nr) > 0$ for sufficiently large t, we have

$$\Delta_\tau^m z(t) - q(t)\sum_{i=1}^{n}\frac{1}{p^i}z\big(g(t)+ir\big) < 0.$$

Integrating $q(t)z(g(t)+ir)$, by the assumptions on p and g, we obtain

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-1}}^{s_{m-1}+\tau} z\big(g(\theta)+ir\big)q(\theta)\,d\theta \\
&\quad\le \min_{t\le l\le t+m\tau}\big\{q(l)\big\}\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-1}}^{s_{m-1}+\tau} z\big(g(\theta)+ir\big)\,d\theta \\
&\quad\le \min_{t\le l\le t+m\tau}\big\{q(l)\big\}\int_{g(t)}^{g(t+\tau)} \big(g^{-1}(s_1)\big)'\,ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} \big(g^{-1}(s_2)\big)'\,ds_2 \cdots \\
&\qquad\times \int_{s_{m-1}}^{g(g^{-1}(s_{m-1})+\tau)} z(\theta+ir)\big(g^{-1}(\theta)\big)'\,d\theta \\
&\quad\le \min_{t\le l\le t+m\tau}\big\{q(l)\big\}\Big(\min_{g(t)\le s\le g(t)+m\tau}\big(g^{-1}(s)\big)'\Big)^m \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \\
&\qquad\times \int_{s_{m-1}}^{s_{m-1}+\tau} z(\theta+ir)\,d\theta \\
&\quad\le \min_{t\le l\le t+m\tau}\big\{q(l)\big\}\Big(\min_{g(t)\le s\le g(t)+m\tau}\big(g^{-1}(s)\big)'\Big)^m y\big(g(t)+ir\big) \\
&\quad\le \bar{q}(t)y\big(g(t)+ir\big).
\end{aligned}$$

Therefore,

$$\Delta_\tau^m y(t) - \bar{q}(t)\sum_{i=1}^{n}\frac{1}{p^i}y\big(g(t)+ir\big) < 0$$

holds for any fixed integer $n \ge 1$ and for all large enough t. If $x(t)$ is a bounded and eventually negative solution, then the conclusion within brackets follows from the above proof with minor modification. □

Lemma 4.7 *Under the assumptions of Lemma* 4.6, *let*

$$v(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta)\,d\theta.$$

Then $v(t) < 0$ (> 0), $(-1)^k v^{(k)}(t) > 0$ (< 0) *for* $1 \le k \le m$ *eventually. Moreover,*

$$v^{(m)}(t) - \frac{1}{\tau^m}\bar{q}(t)\sum_{i=1}^{n}\frac{1}{p^i}v\big(g(t)-m\tau+ir\big) < 0\ (>0) \tag{4.9}$$

holds for any fixed integer $n \ge 1$ *and for all large enough* t.

Proof By the definition of $v(t)$, $v(t)$ has the same sign as $y(t)$. Further, we have

$$v'(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y'(\theta)\, d\theta.$$

Then $v'(t)$ has the same sign as $y'(t)$. Similarly, $(-1)^k v^{(k)}(t)$ for $1 \le k \le m$ and $(-1)^j y^{(j)}(t)$ for $1 \le j \le m$ all have the same sign. Note also that $v^{(m)}(t) = \Delta_\tau^m y(t)$. If $y'(t) < 0$, then

$$\begin{aligned}
v\bigl(g(t) + ir\bigr) &= \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta + ir)\, d\theta \\
&\ge \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(t_{m-1} + \tau + ir)\, d\theta \\
&\ge \tau \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-2}}^{t_{m-2}+\tau} y(t_{m-1} + \tau + ir)\, dt_{m-1} \\
&\cdots \\
&\ge \tau^{m-1} \int_{g(t)}^{g(t)+\tau} y\bigl(t_1 + (m-1)\tau + ir\bigr)\, dt_1 \\
&\ge \tau^m y\bigl(g(t) + m\tau + ir\bigr).
\end{aligned}$$

Hence, from (4.6) we have

$$v^{(m)}(t) - \frac{1}{\tau^m}\bar{q}(t) \sum_{i=1}^{n} \frac{1}{p^i} v\bigl(g(t) - m\tau + ir\bigr) < 0$$

for any fixed integer $n \ge 1$ and for all large enough t. If $y'(t) > 0$, then $0 < v(g(t) + ir) \le \tau^m y(g(t) + m\tau + ir)$ so

$$v^{(m)}(t) - \frac{1}{\tau^m}\bar{q}(t) \sum_{i=1}^{n} \frac{1}{p^i} v\bigl(g(t) - m\tau + ir\bigr) > 0.$$

□

Lemma 4.8 *Assume that $x(t)$ is an eventually positive* (*negative*) *and bounded solution of* (1.1). *Let $z(t)$ and $v(t)$ be defined as in Lemma* 4.6 *and Lemma* 4.7. *Then, under the assumptions of Lemma* 4.6, *for any given $t \ge t_1$, there is a $\theta \in (g(t), t)$ such that*

$$\bigl|v'\bigl(g(t)\bigr)\bigr| > \frac{(t - g(t))^{m-1}}{(m-1)!} \bigl|v^{(m)}(\theta)\bigr|. \tag{4.10}$$

Proof The proof of Lemma 4.5 is still valid for Lemma 4.8. □

5 Proofs of the main results

Here, the proofs of the main results will be presented.

Proof of Theorem 2.1 Suppose the conclusion is not true. Let $x(t)$ be an eventually positive and bounded solution of (1.1) with $\liminf_{t\to\infty}(x(t) - px(t-r)) \ge 0$. Let $y(t)$ be defined as in Lemma 4.2 and $v(t)$ be defined as in Lemma 4.4. By Lemma 4.4, we know that $v(t) > 0$,

$(-1)^k v^{(k)}(t) > 0$ for $1 \le k \le m$ and (4.4), *i.e.*,

$$v^{(m)}(t) + \frac{1}{\tau^m}\bar{q}(t)\sum_{i=0}^{n} p^i v\bigl(g(t) - ir\bigr) < 0$$

holds for any fixed natural number n and for all large enough t. By Lemma 4.5, we know that

$$v'\bigl(g(t)\bigr)\frac{(m-1)!}{(t-g(t))^{m-1}} < v^{(m)}(\theta)$$

for some $\theta \in (g(t), t)$. Since $\bar{q}(\theta) \ge \bar{q}(t)$ by assumption and

$$v\bigl(g(\theta) - ir\bigr) \ge v\bigl(g(t) - ir\bigr),$$

from (4.4) with t replaced by θ, it follows that

$$v'\bigl(g(t)\bigr)\frac{(m-1)!}{(t-g(t))^{m-1}} + \frac{1}{\tau^m}\bar{q}(t)\sum_{i=0}^{n} p^i v\bigl(g(t) - ir\bigr) < 0$$

i.e.,

$$v'\bigl(g(t)\bigr) + \frac{(t-g(t))^{m-1}}{(m-1)!\tau^m}\bar{q}(t)\sum_{i=0}^{n} p^i v\bigl(g(t) - ir\bigr) < 0. \tag{5.1}$$

With the replacement of t by $g^{-1}(t)$, (5.1) yields

$$v'(t) + \frac{(g^{-1}(t)-t)^{m-1}}{(m-1)!\tau^m}\bar{q}\bigl(g^{-1}(t)\bigr)\sum_{i=0}^{n} p^i v(t - ir) < 0. \tag{5.2}$$

Assume that $(-1)^k v^{(k)}(t) > 0$ and (5.2) hold for $0 \le k \le m$ and $t \ge t_1 \ge t_0$. Without loss of generality, we may assume $T \ge t_1 + nr$. Let

$$w(t) = \frac{-v'(t)}{v(t)}.$$

Note that $w(t) > 0$ and $v(t) = v(T)\exp\int_T^t -w(\theta)\,d\theta$ for all $t \ge T \ge t_1 + nr$. From (5.2) it follows that

$$w(t) > \frac{(g^{-1}(t)-t)^{m-1}\bar{q}(g^{-1}(t))}{(m-1)!\tau^m}\sum_{i=0}^{n} p^i \exp\int_{t-ir}^{t} w(s)\,ds, \tag{5.3}$$

i.e.,

$$w(t) > \frac{Q_1(t)}{(m-1)!\tau^m}\sum_{i=0}^{n} p^i \exp\int_{t-ir}^{t} w(s)\,ds \tag{5.4}$$

for all $t \ge Tx$, where $Q_1(t) = (g^{-1}(t) - t)^{m-1}\bar{q}x(g^{-1}(t)) > 0$.

Let $w_0(t) \equiv 0$ for $t \geq Tx - nr$ and let

$$w_{k+1}(t) = \frac{Q_1(t)}{(m-1)!\tau^m} \sum_{i=0}^{n} p^i \exp \int_{t-ir}^{t} w_k(s)\,ds$$

for each $k \in N$ and $t \geq Tx + nkr$. Let

$$\alpha_{1k} = \inf_{t \geq T+(k-1)nr} \{w_k(t)\}, \quad k \in \bar{N}.$$

Then

$$\alpha_{1k+1} \geq \inf_{t \geq T} \left\{ \frac{Q_1(t)}{(m-1)!\tau^m} \sum_{i=0}^{n} p^i e^{ir\alpha_{1k}} \right\} = \beta_1 \sum_{i=0}^{n} p^i e^{ir\alpha_{1k}}.$$

Now, (5.4), (2.4), and the definition of $\{\alpha_{1k}\}$ imply that $\{\alpha_{1k}\}$ is an increasing sequence. Suppose

$$\lim_{k \to \infty} \alpha_{1k} = \rho_1 < \infty.$$

So $\rho_1 \geq \beta_1 \sum_{i=0}^{n} p^i e^{ir\rho_1}$. Let

$$F_1(x) = \beta_1 \sum_{i=0}^{n} p^i e^{irx} - x.$$

Then $F_1'(x) = \beta_1 \sum_{i=1}^{n} irp^i e^{irx} - 1$ and $F_1''(x) > 0$, so $F_1'(x)$ is increasing. Since $F_1'(0) = \beta_1 \sum_{i=0}^{n} irp^i - 1 \geq 0$ by (2.4), then $F_1'(x) > 0$ for $x > 0$. Hence $F_1(x)$ is increasing. Thus, from $F_1(0) = \beta_1 \sum_{i=0}^{n} p^i > 0$ we have $F_1(x) > 0$ for all $x \geq 0$. This shows that no positive number ρ_1 satisfies $\rho_1 \geq \beta_1 \sum_{i=0}^{n} p^i e^{ir\rho_1}$. Therefore, we must have $\alpha_{1k} \to \infty$ as $k \to \infty$. Note that $w(t) \geq w_{k+1}(t) \geq \alpha_{1k+1}$ for $t \geq T_3 + nkr$. Thus $w(t) \to \infty$ as $t \to \infty$. Notice also that

$$w(t) \geq w_{k+1}(t) \geq \alpha_{1k+1} \quad \text{for } t \geq T + nkr.$$

Thus $w(t) \to \infty$ as $t \to \infty$, which implies

$$\frac{v(t)}{v(t+r)} = \exp \int_{t}^{t+r} w(s)\,ds \to \infty \quad \text{as } t \to \infty. \tag{5.5}$$

On the other hand, since $v'(t) < 0$ and $v(t) > 0$, (5.2) yields (by dropping the $i = 0$ term)

$$\begin{aligned} v'(t) &< -\frac{(g^{-1}(t)-t)^{m-1}}{(m-1)!\tau^m} \bar{q}\big(g^{-1}(t)\big) \sum_{i=1}^{n} p^i v(t-ir) \\ &< -\frac{(g^{-1}(t)-t)^{m-1}}{(m-1)!\tau^m} \cdot \frac{p-p^{n+1}}{1-p} \cdot \bar{q}\big(g^{-1}(t)\big) v(t-r). \end{aligned} \tag{5.6}$$

By (2.5) and Lemma 4.1,

$$\liminf_{t \to \infty} \frac{v(t)}{v(t-r)} \in (0,1].$$

Thus $v(t+r)/v(t)$ has a positive lower bound so $v(t)/v(t+r)$ has a positive upper bound. This contradicts (5.5). Assume that $x(t)$ is an eventually negative and bounded solution of (1.1) with $\limsup_{t\to\infty}(x(t)-px(t-r))\le 0$. Then the above proof with a minor modification also leads to a contradiction. Therefore, the conclusion of the theorem holds. □

Proof of Corollary 2.2 The proof is the same as that of Theorem 2.1 except (5.6). The conclusion still holds if (5.6) is replaced by

$$v'(t) < -\frac{(g^{-1}(t)-t)^{m-1}}{\tau^m(m-1)!}\bar{q}\bigl(g^{-1}(t)\bigr)pv(t-r). \qquad \square$$

Proof of Corollary 2.3 The proof of Theorem 2.1 is still valid after the replacement of $\bar{q}(\theta)\ge\bar{q}(t)$ by $\bar{q}(\theta)\ge\bar{q}(g(t))$. □

The proof of Corollary 2.4 is similar to that of Corollary 2.2.

Proof of Corollary 2.5 The proof of Theorem 2.1 is still valid after the replacement of Lemma 4.2 by Lemma 4.3. □

Proof of Corollary 2.6 Suppose the conclusion is not true. Without loss of generality, assume that (1.1) has an eventually positive and bounded solution $x(t)$. Let $y(t)$ be defined as in Lemma 4.6 and $v(t)$ be defined as in Lemma 4.7. By Lemma 4.7, we know that $v(t)<0$, $(-1)^k v^{(k)}(t)>0$ for $1\le k\le m$, and (4.9), *i.e.*,

$$v^{(m)}(t)-\frac{1}{\tau^m}\bar{q}(t)\sum_{i=1}^{n}\frac{1}{p^i}v\bigl(g(t)-m\tau+ir\bigr)<0$$

holds for any fixed integer $n\ge 1$ and for all large enough t. By Lemma 4.8, we know that

$$v'\bigl(g(t)\bigr)\frac{(m-1)!}{(t-g(t))^{m-1}}<v^{(m)}(\theta)$$

for some $\theta\in(g(t),t)$. Since $\bar{q}(\theta)\ge\bar{q}(g(t))$ and $v(g(\theta)-ir)\le v(g(g(t))-ir)$, with the replacement of t by θ, (4.9) yields

$$v'\bigl(g(t)\bigr)\frac{(m-1)!}{(t-g(t))^{m-1}}-\frac{1}{\tau^m}\bar{q}\bigl(g(t)\bigr)\sum_{i=1}^{n}\frac{1}{p^i}v\bigl(g\bigl(g(t)\bigr)-m\tau+ir\bigr)\le 0$$

i.e.,

$$v'\bigl(g(t)\bigr)-\frac{(t-g(t))^{m-1}}{(m-1)!\tau^m}\bar{q}\bigl(g(t)\bigr)\sum_{i=1}^{n}\frac{1}{p^i}v\bigl(g\bigl(g(t)\bigr)-m\tau+ir\bigr)\le 0. \tag{5.7}$$

With the replacement of t by $g^{-1}(t)$, (5.7) becomes

$$v'(t)-\frac{(g^{-1}(t)-t)^{m-1}}{(m-1)!\tau^m}\bar{q}(t)\sum_{i=1}^{n}\frac{1}{p^i}v\bigl(g(t)-m\tau+ir\bigr)\le 0. \tag{5.8}$$

Assume that $v(t)$, $(-1)^k v^{(k)}(t) > 0$ $(1 \leq k \leq m)$ and (5.8) hold for $t \geq t_1 \geq t_0$ and, without loss of generality, that $T \geq t_1 + nr$. Let

$$w(t) = \frac{v'(t)}{v(t)}.$$

Note that $w(t) > 0$ and $v(t) = v(t')\exp\int_{t'}^{t} w(\theta)\,d\theta$ for all $t, t' \geq T$. From (5.8), we have

$$w(t) \geq \frac{(g^{-1}(t)-t)^{m-1}\bar{q}(t)}{(m-1)!\tau^m}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-m\tau+ir} w(s)\,ds, \tag{5.9}$$

i.e.,

$$w(t) \geq \frac{Q_{m2}(t)}{(m-1)!\tau^m}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-m\tau+ir} w(s)\,ds \tag{5.10}$$

for all $t \geq T$, where $Q_2(t) = (g^{-1}(t)-t)^{m-1}\bar{q}(t) > 0$.

Let $w_0(t) \equiv 0$ for $t \geq T$. For each $k \in \bar{N}$ and $t \geq T$, let

$$w_{k+1}(t) = \frac{Q_2(t)}{(m-1)!\tau^m}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-m\tau+ir} w_k(s)\,ds$$

and

$$\alpha_{2k} = \inf_{t\geq T}\bigl\{w_k(t)\bigr\}, \quad k \in \bar{N}.$$

So $w(t) \geq w_{k+1}(t) \geq w_k(t) \geq \alpha_{2k}$ for all $k \in N$ and $t \geq T$. By assumption, we therefore have

$$\begin{aligned}\alpha_{2k+1} &\geq \inf_{t\geq T}\left\{Q_2(t)\cdot\frac{1}{(m-1)!\tau^m}\sum_{i=1}^{n}\frac{1}{p^i}e^{[g(t)-t-m\tau+ir]\alpha_{2k}}\right\}\\ &\geq \beta_2\sum_{i=1}^{n}\frac{1}{p^i}e^{(i-1)r\alpha_{2k}}.\end{aligned}$$

Note that $\{\alpha_{2k}\}$ is a bounded nondecreasing sequence and suppose that

$$\lim_{k\to\infty}\alpha_{2k} = \rho_2 < \infty. \tag{5.11}$$

So $\rho_2 \geq \beta_2\sum_{i=1}^{n}(e^{(i-1)r\rho_2}/p^i)$. Let

$$F_2(x) = \beta_2\sum_{i=1}^{n}\frac{e^{(i-1)rx}}{p^i} - x.$$

Then $F_2'(x) = \beta_2\sum_{i=2}^{n}((i-1)re^{(i-1)rx}/p^i) - 1$ and $F_2''(x) > 0$, so $F_2'(x)$ is increasing. Since $F_2'(0) = \beta_2\sum_{i=1}^{n}((i-1)r/p^i) - 1 \geq 0$ by (2.10), $F_2'(x) > 0$ for $x > 0$. Hence $F_2(x)$ is increasing. Thus, from $F_2(0) = \beta_2\sum_{i=1}^{n}1/p^i > 0$ we have $F_2(x) > 0$ for all $x \geq 0$. This shows that no positive number ρ_2 satisfies $\rho_2 \geq \beta_2\sum_{i=1}^{n}(e^{(i-1)r\rho_2}/p^i)$. This contradiction shows that the conclusion holds. □

Proof of Corollary 2.7 The proof of Theorem 2.6 is still valid after the replacement of $\bar{q}(\theta) \geq \bar{q}(g(t))$ by $\bar{q}(\theta) \geq \bar{q}(t)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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