Jang et al. *Advances in Difference Equations* (2015) 2015:162
DOI 10.1186/s13662-015-0495-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Some identities of higher order Barnes-type q-Bernoulli polynomials and higher order Barnes-type q-Euler polynomials

Lee-Chae Jang[1*], Sang-Ki Choi[2] and Hyuck In Kwon[3]

*Correspondence: lcjang@konkuk.ac.kr
[1]Graduate School of Education, Konkuk University, Seoul, 143-701, Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we consider higher order Barnes-type q-Bernoulli polynomials and numbers and investigate some identities of them. Furthermore, we discuss some identities of higher order Barnes-type q-Euler polynomials and numbers.

MSC: 11B68; 11S40

Keywords: p-adic invariant integral; Bernoulli polynomials; Euler polynomials; higher order Barnes-type q-Bernoulli polynomials and numbers

1 Introduction

Let p be a given odd prime number. Throughout this paper, we assume that $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the rings of p-adic integers, the fields of p-adic numbers and the completion of algebraic closure of $\mathbb{Q} - p$. The p-adic norm $|p|_p = \frac{1}{p}$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined as

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \quad \text{(see [1–12])}. \tag{1}$$

It is well known that an integral equation of the bosonic p-adic integral I_0 on $\mathbb{Z}_p$,

$$I_0(f_1) - I_0(f) = f'(0), \tag{2}$$

where $f_1(x) = f(x+1)$. Higher order Bernoulli polynomials are defined by Kim to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!} \quad \text{(see [5, 13–16])}. \tag{3}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ is called higher order Bernoulli numbers. Higher order Barnes-type Bernoulli polynomials are defined by Kim to be

$$\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t} - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x|a_1,\ldots,a_r) \frac{t^n}{n!} \quad \text{(see [11–15, 17–21])}. \tag{4}$$

When $x = 0$, $B_n^{(r)}(a_1, \ldots, a_r) = B_n^{(r)}(0|a_1, \ldots, a_r)$ is called higher order Barnes-type Bernoulli numbers.

In this paper we consider higher order Barnes-type q-Bernoulli polynomials and numbers and investigate some identities of them. We also discuss some identities of higher order Barnes-type q-Euler polynomials and numbers.

2 Higher order Barnes-type *q*-Bernoulli polynomials and numbers

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. By (2), if we take $f(x) = q^y e^{(x+y)t}$, then we get

$$\int_{\mathbb{Z}_p} q^y e^{(x+y)t}\, d\mu_0(y) = \frac{t + \log q}{qe^t - 1} e^{xt}, \tag{5}$$

where $f_1(x) = f(x+1)$. q-Bernoulli polynomials are defined by Kim to be

$$\frac{t + \log q}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} \quad \text{(see [13–15, 17, 19–21])}. \tag{6}$$

When $x = 0$, $B_{n,q} = B_{n,q}(0)$ is called q-Bernoulli numbers.

Higher order q-Bernoulli polynomials are defined as

$$\left(\frac{t + \log q}{qe^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $B_{n,q}^{(r)} = B_{n,q}^{(r)}(0)$ is called higher order q-Bernoulli numbers.

We define higher order Barnes-type q-Bernoulli polynomials as follows:

$$\frac{(t + \log q)^r}{(q^{a_1} e^{a_1 t} - 1) \cdots (q^{a_r} e^{a_r t} - 1)} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x|a_1, \ldots, a_r) \frac{t^n}{n!}. \tag{8}$$

When $x = 0$, $B_{n,q}(a_1, \ldots, a_r) = B_{n,q}(0|a_1, \ldots, a_r)$ is called higher order Barnes-type q-Bernoulli numbers. By (5), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{a_1 x_1 + \cdots + a_r t_r} e^{(a_1 x_1 + \cdots + a_r x_r + x)t}\, d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&\quad = \left(\prod_{i=1}^{r} a_i\right) \frac{(t + \log q)^r}{(q^{a_1} e^{a_1 t} - 1) \cdots (q^{a_r} e^{a_r t} - 1)} e^{xt}.
\end{aligned} \tag{9}$$

By (9) and (8), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_{n,q}(x|a_1, \ldots, a_r) \frac{t^n}{n!} \\
&\quad = \frac{(t + \log q)^r}{(q^{a_1} e^{a_1 t} - 1) \cdots (q^{a_r} e^{a_r t} - 1)} e^{xt} \\
&\quad = \left(\prod_{i=1}^{r} a_i\right)^{-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{a_1 x_1 + \cdots + a_r x_r} e^{(a_1 x_1 + \cdots + a_r x_r + x)t}\, d\mu_0(x_1) \cdots d\mu_0(x_r)
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left(\left(\prod_{i=1}^{r} a_i\right)^{-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1x_1+\cdots+a_rx_r+x} \times (a_1x_1+\cdots+a_rx_r+x)^n\,d\mu_0(x_1)\cdots d\mu_0(x_r)\right)\frac{t^n}{n!}. \tag{10}$$

From (10), we obtain the following theorem.

Theorem 2.1 *Let $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$\begin{aligned} &B_{n,q}(x|a_1,\ldots,a_r) \\ &\quad= \left(\prod_{i=1}^{r} a_i\right)^{-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1x_1+\cdots+a_rx_r+x} \\ &\qquad\times (a_1x_1+\cdots+a_rx_r+x)^n\,d\mu_0(x_1)\cdots d\mu_0(x_r). \end{aligned} \tag{11}$$

From (1), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{dp^N-1} f(x) \\ &= \frac{1}{d}\lim_{N\to\infty}\frac{1}{p^N}\sum_{a=0}^{d-1}\sum_{x=0}^{p^N-1} f(a+dx) \\ &= \frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p} f(a+dx)\,d\mu_0(x). \end{aligned} \tag{12}$$

By (12), we have

$$\begin{aligned} &\left(\prod_{i=1}^{r} a_i\right)^{-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1x_1+\cdots+a_rx_r} e^{(a_1dx_1+\cdots+a_rdx_r+x)t}\,d\mu_0(x_1)\cdots d\mu_0(x_r) \\ &\quad= \frac{1}{d^r}\sum_{l_1,\ldots,l_r=0}^{d-1} q^{a_1l_1\cdots+a_rl_r}\left(\prod_{i=1}^{r} a_i\right)^{-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1\,dx_1+\cdots+a_r\,dx_r} \\ &\qquad\times e^{(a_1l_1\cdots+a_rl_r+x+a_1\,dx_1+\cdots+a_r\,dx_r)t}\,d\mu_0(x_1)\cdots d\mu_0(x_r) \\ &\quad= \frac{1}{d^r}\sum_{l_1,\ldots,l_r=0}^{d-1} q^{a_1l_1\cdots+a_rl_r}\left(\prod_{i=1}^{r} a_i\right)^{-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1\,dx_1+\cdots+a_r\,dx_r} \\ &\qquad\times e^{(\frac{a_1l_1+\cdots+a_rl_r+x}{d}+a_1x_1+\cdots+a_rx_r)\,dt}\,d\mu_0(x_1)\cdots d\mu_0(x_r) \\ &\quad= \sum_{l_1,\ldots,l_r=0}^{d-1} q^{a_1l_1\cdots+a_rl_r}\sum_{n=0}^{\infty}\frac{d^n}{d^r}\left(\prod_{i=1}^{r} a_i\right)^{-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1x_1+\cdots+a_rx_r} \\ &\qquad\times\left(\frac{a_1l_1+\cdots+a_rl_r+x}{d}+a_1x_1+\cdots+a_rx_r\right)^n d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!} \\ &\quad= \sum_{n=0}^{\infty} d^{n-r}\sum_{l_1,\ldots,l_r=0}^{d-1} q^{a_1x_1+\cdots+a_rx_r} B_{n,q}\left(\frac{a_1l_1+\cdots+a_rl_r+x}{d}\middle|a_1,\ldots,a_r\right)\frac{t^n}{n!}. \end{aligned} \tag{13}$$

By (8), (9), (11) and (13), we obtain the following theorem.

Theorem 2.2 *Let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$\begin{aligned} &B_{n,q}(x|a_1,\ldots,a_r) \\ &\quad = d^{n-r} \sum_{l_1,\ldots,l_r=0}^{d-1} q^{a_1x_1+\cdots+a_rx_r} B_{n,q}\left(\frac{l_1x_1+\cdots+l_rx_r+x}{d}\middle|a_1,\ldots,a_r\right). \end{aligned} \tag{14}$$

It is well known that an integral equation of the bosonic p-adic integral I_0 on $\mathbb{Z}_p$ satisfies the following integral equation:

$$I_0(f_n) - I_0(f) = \sum_{i=1}^{n-1} f'(i). \tag{15}$$

If we take $f(x_i) = q^{a_ix_i}e^{a_ix_it}$ for $i = 1,\ldots,r$, then we have

$$\int_{\mathbb{Z}_p} q^{a_ix_i}e^{a_ix_it}\,d\mu_0(x_i) = \frac{a_i(t+\log q)}{q^{a_in}e^{a_int}-1}\sum_{l=0}^{n-1} q^{a_il}e^{a_ilt}. \tag{16}$$

By (16), we get

$$\begin{aligned} &\left(\prod_{i=1}^{r} a_i\right)^{-1} \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1x_1+\cdots+a_rx_r} e^{(a_1x_1+\cdots+a_rx_r)t}\,d\mu_0(x_1)\cdots d\mu_0(x_r) \\ &\quad = \frac{(t+\log q)^r}{(q^{a_1n}e^{a_1nt}-1)\cdots(q^{a_rn}e^{a_rnt}-1)} \sum_{l_1,\ldots,l_r=0}^{n-1} q^{a_1l_1+\cdots+a_rl_r} e^{(a_1l_1+\cdots+a_rl_r)t} \\ &\quad = \left(\sum_{k=0}^{\infty} B_{k,q}(na_1,\ldots,na_r)\frac{t^k}{k!}\right) \sum_{l_1,\ldots,l_r=0}^{n-1} q^{a_1l_1+\cdots+a_rl_r} \sum_{j=0}^{\infty}(a_1l_1+\cdots+a_rl_r)^j\frac{t^j}{j!} \\ &\quad = \sum_{l_1,\ldots,l_r=0}^{n-1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} q^{a_1l_1+\cdots+a_rl_r}(a_1l_1+\cdots+a_rl_r)^j B_{k,q}(na_1,\ldots,na_r)\frac{t^{k+j}}{k!j!} \\ &\quad = \sum_{m=0}^{\infty}\sum_{l_1,\ldots,l_r=0}^{n-1}\sum_{j=0}^{m}\binom{m}{j} q^{a_1l_1+\cdots+a_rl_r}(a_1l_1+\cdots+a_rl_r)^j \\ &\qquad \times B_{m-j,q}(na_1,\ldots,na_r)\frac{t^m}{m!}. \end{aligned} \tag{17}$$

Thus, by (11) and (17), we obtain the following theorem.

Theorem 2.3 *Let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$\begin{aligned} &B_{n,q}(a_1,\ldots,a_r) \\ &\quad = \sum_{l_1,\ldots,l_r=0}^{n-1}\sum_{j=0}^{m}\binom{m}{j} q^{a_1l_1+\cdots+a_rl_r}(a_1l_1+\cdots+a_rl_r)^j B_{m-j,q}(na_1,\ldots,na_r). \end{aligned} \tag{18}$$

By (16), we get

$$\begin{aligned}
&\left(\prod_{i=1}^{r} a_i\right)^{-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{a_1x_1+\cdots+a_rx_r} e^{(a_1x_1+\cdots+a_rx_r)t}\, d\mu_0(x_1)\cdots d\mu_0(x_r) \\
&= \frac{(t+\log q)^r}{(q^{a_1n}e^{a_1nt}-1)\cdots(q^{a_rn}e^{a_rnt}-1)} \sum_{l_1,\ldots,l_r=0}^{n-1} q^{a_1l_1+\cdots+a_rl_r} e^{(a_1l_1+\cdots+a_rl_r)t} \\
&= \sum_{l_1,\ldots,l_r=0}^{n-1} \frac{(t+\log q)^r}{(q^{a_1n}e^{a_1nt}-1)\cdots(q^{a_rn}e^{a_rnt}-1)} q^{a_1l_1+\cdots+a_rl_r} e^{(a_1l_1+\cdots+a_rl_r)t} \\
&= \sum_{l_1,\ldots,l_r=0}^{n-1} q^{a_1l_1+\cdots+a_rl_r} \frac{(t+\log q)^r}{(q^{a_1n}e^{a_1nt}-1)\cdots(q^{a_rn}e^{a_rnt}-1)} e^{\frac{a_1l_1+\cdots+a_rl_r}{n}nt} \\
&= \sum_{l_1,\ldots,l_r=0}^{n-1} q^{a_1l_1+\cdots+a_rl_r} \sum_{m=0}^{\infty} B_{m,q^n}\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big| a_1,\ldots,a_r\right)\frac{n^m t^m}{m!} \\
&= \sum_{m=0}^{\infty} n^m \sum_{l_1,\ldots,l_r=0}^{n-1} q^{a_1l_1+\cdots+a_rl_r} B_{m,q^n}\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big| a_1,\ldots,a_r\right)\frac{t^m}{m!}.
\end{aligned} \tag{19}$$

Thus, by (11) and (19), we obtain the following theorem.

Theorem 2.4 *Let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$B_{m,q}(a_1,\ldots,a_r) = n^m \sum_{l_1,\ldots,l_r=0}^{n-1} q^{a_1l_1+\cdots+a_rl_r} B_{m,q^n}\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big| a_1,\ldots,a_r\right). \tag{20}$$

3 Higher order Barnes-type q-Euler polynomials

Higher Euler polynomials are defined as

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad \text{(see [17–19, 22–24])}. \tag{21}$$

When $x = 0$, $E_n = E_n(0)$ is called higher Euler numbers. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \quad \text{(see [4])}. \tag{22}$$

It is well known that an integral equation of the fermionic p-adic integral on $\mathbb{Z}_p$ is

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{23}$$

where $f_1(x) = f(x+1)$.

Let $a_1,\ldots,a_r \in \mathbb{C}_p \setminus \{0\}$. Higher order Barnes-type Euler polynomials are defined as

$$\frac{2^r}{(e^{a_1t}+1)\cdots(e^{a_rt}+1)} e^{xt} = \sum_{n=0}^{\infty} E_n(x|a_1,\ldots,a_r)\frac{t^n}{n!} \quad \text{(see [18, 19, 23])}. \tag{24}$$

When $x = 0$, $E_n(a_1, \ldots, a_r) = E_n(0|a_1, \ldots, a_r)$ is called higher order Barnes-type Euler numbers. We define higher order Barnes-type q-Euler polynomials as follows:

$$\frac{2^r}{(q^{a_1}e^{a_1 t}+1)\cdots(q^{a_r}e^{a_r t}+1)}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x|a_1, \ldots, a_r)\frac{t^n}{n!}. \tag{25}$$

When $x = 0$, $E_{n,q}(a_1, \ldots, a_r) = E_{n,q}(0|a_1, \ldots, a_r)$ is called higher order Barnes-type q-Euler numbers.

By (23), if we take $f(x_i) = q^{a_i x_i}e^{a_i x_i t}$ for $i = 1, \ldots, r$, then we have

$$\int_{\mathbb{Z}_p} q^{a_i x_i}e^{a_i x_i t}\, d\mu_{-1}(x_i) = \frac{2}{q^{a_i x_i}e^{a_i x_i t}+1}. \tag{26}$$

By (26), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^{a_1 x_1+\cdots+a_r t_r}e^{(a_1 x_1+\cdots+a_r x_r+x)t}\, d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &\quad = \frac{2^r}{(q^{a_1}e^{a_1 t}+1)\cdots(q^{a_r}e^{a_r t}+1)}e^{xt}.\end{aligned} \tag{27}$$

By (24) and (27), we get

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,q}(x|a_1, \ldots, a_r)\frac{t^n}{n!}\\ &\quad = \frac{2^r}{(q^{a_1}e^{a_1 t}+1)\cdots(q^{a_r}e^{a_r t}+1)}e^{xt}\\ &\quad = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1 x_1+\cdots+a_r x_r}e^{(a_1 x_1+\cdots+a_r x_r+x)t}\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\quad = \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1 x_1+\cdots+a_r x_r}(a_1 x_1+\cdots+a_r x_r+x)^n\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^n}{n!}.\end{aligned} \tag{28}$$

From (28), we obtain the following theorem.

Theorem 3.1 *Let $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$\begin{aligned}&E_{n,q}(x|a_1, \ldots, a_r)\\ &\quad = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{a_1 x_1+\cdots+a_r x_r}(a_1 x_1+\cdots+a_r x_r+x)^n\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).\end{aligned} \tag{29}$$

From (22), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) &= \lim_{N\to\infty}\sum_{x=0}^{dp^N-1} f(x)(-1)^x\\ &= \frac{1}{d}\lim_{N\to\infty}\sum_{a=0}^{d-1}\sum_{x=0}^{p^N-1}(-1)^{a+x}f(a+dx)\\ &= \frac{1}{d}\sum_{a=0}^{d-1}(-1)^a\int_{\mathbb{Z}_p} f(a+dx)\, d\mu_{-1}(x).\end{aligned} \tag{30}$$

By (30), if we take $f(x_i) = q^{a_i x_i} e^{a_i x_i t}$ for $i = 1, \ldots, r$, then we have

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{a_i x_i} e^{a_i x_i t}\, d\mu_{-1}(x) &= \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} q^{a_i(a+dx-i)} e^{a_i(a+dx_i)t}\, d\mu_{-1}(x_i) \\ &= \sum_{a=0}^{d-1} (-1)^a q^{a_i a} e^{a_i a t} \int_{\mathbb{Z}_p} q^{a_i dx_i} e^{a_i dx_i t}\, d\mu_{-1}(x_i). \end{aligned} \tag{31}$$

By (31), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{a_1x_1+\cdots+a_rx_r} e^{(a_1x_1+\cdots+a_rx_r+x)t}\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} q^{a_1l_1+\cdots+a_rl_r} \\ &\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{a_1\,dx_1+\cdots+a_r\,dx_r} e^{(\frac{a_1l_1+\cdots+a_1l_r+x}{d}+a_1x_1+\cdots+a_rx_r)\,dt}\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \sum_{n=0}^{\infty} d^n \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} q^{a_1l_1+\cdots+a_rl_r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{a_1\,dx_1+\cdots+a_r\,dx_r} \\ &\quad \times \left(\frac{a_1l_1+\cdots+a_1l_r+x}{d} + a_1x_1+\cdots+a_rx_r\right)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} d^n \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} q^{a_1l_1+\cdots+a_rl_r} \\ &\quad \times E_{n,q^d}\left(\frac{a_1l_1+\cdots+a_rl_r+x}{d}\middle|a_1,\ldots,a_r\right)\frac{t^n}{n!}. \end{aligned} \tag{32}$$

By (27) and (32), we obtain the following theorem.

Theorem 3.2 *Let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$\begin{aligned}&E_{n,q}(x|a_1,\ldots,a_r) \\ &= d^n \sum_{l_1,\ldots,l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} q^{a_1l_1+\cdots+a_rl_r} E_{n,q^d}\left(\frac{a_1l_1+\cdots+a_rl_r+x}{d}\middle|a_1,\ldots,a_r\right).\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]Graduate School of Education, Konkuk University, Seoul, 143-701, Korea. [2]Department of Mathematics Education, Konkuk University, Seoul, 143-701, Korea. [3]Department of Mathematics, Kwangwoon University, Seoul, Korea.

Received: 11 March 2015 Accepted: 6 May 2015 Published online: 29 May 2015

References
1. Kim, T: Barnes' type multiple degenerate Bernoulli and Euler polynomials. Appl. Math. Comput. **258**, 556-564 (2015)
2. Kim, T, Kim, DS, Bayad, A, Rim, S-H: Identities on the Bernoulli and the Euler numbers and polynomials. Ars Comb. **107**, 455-463 (2012)

3. Kim, T, Dolgy, DV, Kim, DS, Rim, S-H: A note on the identities of special polynomials. Ars Comb. **113A**, 97-106 (2014)
4. Kim, T: On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$. J. Math. Anal. Appl. **331**(2), 779-792 (2007)
5. Kim, DS, Kim, T: A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials. Russ. J. Math. Phys. **22**(1), 26-33 (2015)
6. Lim, D, Do, Y: Some identities of Barnes-type special polynomials. Adv. Differ. Equ. **2015**, 42 (2015)
7. Luo, Q-M, Qi, F: Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials. Adv. Stud. Contemp. Math. **7**(1), 11-18 (2003)
8. Ozden, H: p-Adic distribution of the unification of the Bernoulli, Euler and Genocchi polynomials. Appl. Math. Comput. **218**(3), 970-973 (2011)
9. Park, J-W, Rim, S-H: On the modified q-Bernoulli polynomials with weight. Proc. Jangjeon Math. Soc. **17**(2), 231-236 (2014)
10. Ryoo, CS, Kwon, HI, Yoon, J, Jang, YS: Fourier series of the periodic Bernoulli and Euler functions. Abstr. Appl. Anal. **2014**, Article ID 856491 (2014)
11. Ding, D, Yang, J: Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials. Proc. Jangjeon Math. Soc. **17**(1), 115-123 (2014)
12. Jang, LC: A family of Barnes-type multiple twisted q-Euler numbers and polynomials related to fermionic p-adic invariant integrals on $\mathbb{Z}_p$. J. Comput. Anal. Appl. **13**(2), 376-387 (2011)
13. Andrews, GE, Askey, R, Roy, R: Special Functions. Encyclopedia of Mathematics and Its Applications, vol. 71. Cambridge University Press, Cambridge (1999). ISBN:0-521-62321-9; 0-521-78988-5
14. Bayad, A, Kim, T: Results on values of Barnes polynomials. Rocky Mt. J. Math. **43**(6), 1857-1869 (2013)
15. Bayad, A, Kim, T, Kim, W, Lee, SH: Arithmetic properties of q-Barnes polynomials. J. Comput. Anal. Appl. **15**(1), 111-117 (2013)
16. Kang, D, Jeong, J-J, Lee, BJ, Rim, S-H, Choi, SH: Some identities of higher order Genocchi polynomials arising from higher order Genocchi basis. J. Comput. Anal. Appl. **17**(1), 141-146 (2014)
17. Chen, C-P, Srivastava, HM: Some inequalities and monotonicity properties associated with the gamma and psi functions and the Barnes G-function. Integral Transforms Spec. Funct. **22**(1), 1-15 (2011)
18. Kim, T: p-Adic q-integrals associated with the Changhee-Barnes' q-Bernoulli polynomials. Integral Transforms Spec. Funct. **15**(5), 415-420 (2004)
19. Kim, T: Barnes-type multiple q-zeta functions and q-Euler polynomials. J. Phys. A **43**(25), 255201 (2010)
20. Kim, T: On the multiple q-Genocchi and Euler numbers. Russ. J. Math. Phys. **15**(4), 482-486 (2008)
21. Kim, T, Rim, S-H: On Changhee-Barnes' q-Euler numbers and polynomials. Adv. Stud. Contemp. Math. **9**(2), 81-86 (2004)
22. Can, M, Cenkci, M, Kurt, V, Simsek, Y: Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions. Adv. Stud. Contemp. Math. **18**(2), 135-160 (2009)
23. Jang, L, Kim, T, Kim, Y-H, Hwang, K-W: Note on the q-extension of Barnes' type multiple Euler polynomials. J. Inequal. Appl. **2009**, Article ID 13532 (2009)
24. Kim, T: On Euler-Barnes multiple zeta functions. Russ. J. Math. Phys. **10**(3), 261-267 (2003)