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Some algebraic identities on quadra Fibona-Pell integer sequence

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Abstract

In this work, we define a quadra Fibona-Pell integer sequence $W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4}$ for $n \geq 4$ with initial values $W_0 = W_1 = 0$, $W_2 = 1$, $W_3 = 3$, and we derive some algebraic identities on it including its relationship with Fibonacci and Pell numbers.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Binet's formula; binary linear recurrences

1 Preliminaries

Let p and q be non-zero integers such that $D = p^2 - 4q \neq 0$ (to exclude a degenerate case). We set the sequences U_n and V_n to be

$$\begin{aligned} U_n &= U_n(p,q) = pU_{n-1} - qU_{n-2}, \\ V_n &= V_n(p,q) = pV_{n-1} - qV_{n-2} \end{aligned} \tag{1}$$

for $n \geq 2$ with initial values $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, and $V_1 = p$. The sequences U_n and V_n are called the (first and second) Lucas sequences with parameters p and q. V_n is also called the companion Lucas sequence with parameters p and q.

The characteristic equation of U_n and V_n is $x^2 - px + q = 0$ and hence the roots of it are $x_1 = \frac{p+\sqrt{D}}{2}$ and $x_2 = \frac{p-\sqrt{D}}{2}$. So their Binet formulas are

$$U_n = \frac{x_1^n - x_2^n}{x_1 - x_2} \quad \text{and} \quad V_n = x_1^n + x_2^n$$

for $n \geq 0$. For the companion matrix $M = \left[\begin{smallmatrix} p & -q \\ 1 & 0 \end{smallmatrix}\right]$, one has

$$\begin{bmatrix} U_n \\ U_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} p \\ 2 \end{bmatrix}$$

for $n \geq 1$. The generating functions of U_n and V_n are

$$U(x) = \frac{x}{1 - px + qx^2} \quad \text{and} \quad V(x) = \frac{2 - px}{1 - px + qx^2}. \tag{2}$$

Fibonacci, Lucas, Pell, and Pell-Lucas numbers can be derived from (1). Indeed for $p = 1$ and $q = -1$, the numbers $U_n = U_n(1,-1)$ are called the Fibonacci numbers (A000045 in

OEIS), while the numbers $V_n = V_n(1,-1)$ are called the Lucas numbers (A000032 in OEIS). Similarly, for $p = 2$ and $q = -1$, the numbers $U_n = U_n(2,-1)$ are called the Pell numbers (A000129 in OEIS), while the numbers $V_n = V_n(2,-1)$ are called the Pell-Lucas (A002203 in OEIS) (companion Pell) numbers (for further details see [1–6]).

2 Quadra Fibona-Pell sequence

In [7], the author considered the quadra Pell numbers $D(n)$, which are the numbers of the form $D(n) = D(n-2) + 2D(n-3) + D(n-4)$ for $n \geq 4$ with initial values $D(0) = D(1) = D(2) = 1$, $D(3) = 2$, and the author derived some algebraic relations on it.

In [8], the authors considered the integer sequence (with four parameters) $T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4}$ with initial values $T_0 = 0$, $T_1 = 0$, $T_2 = -3$, $T_3 = 12$, and they derived some algebraic relations on it.

In the present paper, we want to define a similar sequence related to Fibonacci and Pell numbers and derive some algebraic relations on it. For this reason, we set the integer sequence W_n to be

$$W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4} \tag{3}$$

for $n \geq 4$ with initial values $W_0 = W_1 = 0$, $W_2 = 1$, $W_3 = 3$ and call it a *quadra Fibona-Pell sequence*. Here one may wonder why we choose this equation and call it a quadra Fibona-Pell sequence. Let us explain: We will see below that the roots of the characteristic equation of W_n are the roots of the characteristic equations of both Fibonacci and Pell sequences. Indeed, the characteristic equation of (3) is $x^4 - 3x^3 + 3x + 1 = 0$ and hence the roots of it are

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}, \qquad \gamma = 1+\sqrt{2} \quad \text{and} \quad \delta = 1-\sqrt{2}. \tag{4}$$

(Here α, β are the roots of the characteristic equation of Fibonacci numbers and γ, δ are the roots of the characteristic equation of Pell numbers.) Then we can give the following results for W_n.

Theorem 1 *The generating function for W_n is*

$$W(x) = \frac{x^2}{x^4 + 3x^3 - 3x + 1}.$$

Proof The generating function $W(x)$ is a function whose formal power series expansion at $x = 0$ has the form

$$W(x) = \sum_{n=0}^{\infty} W_n x^n = W_0 + W_1 x + W_2 x^2 + \cdots + W_n x^n + \cdots.$$

Since the characteristic equation of (3) is $x^4 - 3x^3 + 3x + 1 = 0$, we get

$$\begin{aligned}\left(1 - 3x + 3x^3 + x^4\right)W(x) &= \left(1 - 3x + 3x^3 + x^4\right)\left(W_0 + W_1 x + \cdots + W_n x^n + \cdots\right)\\ &= W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1)x^2\end{aligned}$$

$$+ (W_3 - 3W_2 + 3W_0)x^3 + \cdots$$
$$+ (W_n - 3W_{n-1} + 3W_{n-3} + W_{n-4})x^n + \cdots.$$

Notice that $W_0 = W_1 = 0$, $W_2 = 1$, $W_3 = 3$, and $W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4}$. So $(1 - 3x + 3x^3 + x^4)W(x) = x^2$ and hence the result is obvious. □

Theorem 2 *The Binet formula for W_n is*

$$W_n = \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right) - \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

for $n \geq 0$.

Proof Note that the generating function is $W(x) = \frac{x^2}{x^4+3x^3-3x+1}$. It is easily seen that $x^4 + 3x^3 - 3x + 1 = (1 - x - x^2)(1 - 2x - x^2)$. So we can rewrite $W(x)$ as

$$W(x) = \frac{x}{1 - 2x - x^2} - \frac{x}{1 - x - x^2}. \tag{5}$$

From (2), we see that the generating function for Pell numbers is

$$P(x) = \frac{x}{1 - 2x - x^2} \tag{6}$$

and the generating function for the Fibonacci numbers is

$$F(x) = \frac{x}{1 - x - x^2}. \tag{7}$$

From (5), (6), (7), we get $W(x) = P(x) - F(x)$. So $W_n = (\frac{\gamma^n-\delta^n}{\gamma-\delta}) - (\frac{\alpha^n-\beta^n}{\alpha-\beta})$ as we wanted. □

The relationship with Fibonacci, Lucas, and Pell numbers is given below.

Theorem 3 *For the sequences W_n, F_n, L_n, and P_n, we have*:
(1) $W_n = P_n - F_n$ *for* $n \geq 0$.
(2) $W_{n+1} + W_{n-1} = (\gamma^n + \delta^n) - (\alpha^n + \beta^n)$ *for* $n \geq 1$.
(3) $\sqrt{5}F_n + 2\sqrt{2}P_n = (\gamma^n - \delta^n) + (\alpha^n - \beta^n)$ *for* $n \geq 1$.
(4) $L_n + P_{n+1} + P_{n-1} = \alpha^n + \beta^n + \gamma^n + \delta^n$ *for* $n \geq 1$.
(5) $2(W_{n+1} - W_n + F_{n-1}) = \gamma^n + \delta^n$ *for* $n \geq 1$.
(6) $\lim_{n\to\infty} \frac{W_n}{W_{n-1}} = \gamma$.

Proof (1) It is clear from the above theorem, since $W(x) = P(x) - F(x)$.

(2) Since $6W_{n-1} + W_{n+2} = 3W_{n+1} - 3W_{n-1} - W_{n-2} + 6W_{n-1}$, we get

$$\begin{aligned} W_{n+1} + W_{n-1} &= 2W_{n-1} + \frac{1}{3}W_{n-2} + \frac{1}{3}W_{n+2} \\ &= \frac{6}{3}\left(\frac{\gamma^{n-1} - \delta^{n-1}}{\gamma - \delta} - \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) \\ &\quad + \frac{1}{3}\left(\frac{\gamma^{n-2} - \delta^{n-2}}{\gamma - \delta} - \frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta}\right) \end{aligned}$$

$$\begin{aligned}
&+\frac{1}{3}\left(\frac{\gamma^{n+2}-\delta^{n+2}}{\gamma-\delta}-\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right)\\
&=\frac{1}{3(\gamma-\delta)}\left[\gamma^n\left(\frac{6}{\gamma}+\frac{1}{\gamma^2}+\gamma^2\right)+\delta^n\left(\frac{-6}{\delta}-\frac{1}{\delta^2}-\delta^2\right)\right]\\
&\quad+\frac{1}{3(\alpha-\beta)}\left[\alpha^n\left(\frac{-6}{\alpha}-\frac{1}{\alpha^2}-\alpha^2\right)+\beta^n\left(\frac{6}{\beta}+\frac{1}{\beta^2}+\beta^2\right)\right]\\
&=\left(\gamma^n+\delta^n\right)-\left(\alpha^n+\beta^n\right),
\end{aligned}$$

since $\frac{6}{\gamma}+\frac{1}{\gamma^2}+\gamma^2=\frac{-6}{\delta}-\frac{1}{\delta^2}-\delta^2=6\sqrt{2}$ and $\frac{-6}{\alpha}-\frac{1}{\alpha^2}-\alpha^2=\frac{6}{\beta}+\frac{1}{\beta^2}+\beta^2=-3\sqrt{5}$.

(3) Notice that $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $P_n=\frac{\gamma^n-\delta^n}{\gamma-\delta}$. So we get $\sqrt{5}F_n=\alpha^n-\beta^n$ and $2\sqrt{2}P_n=\gamma^n-\delta^n$. Thus clearly, $\sqrt{5}F_n+2\sqrt{2}P_n=(\gamma^n-\delta^n)+(\alpha^n-\beta^n)$.

(4) It is easily seen that $P_{n+1}+P_{n-1}=\gamma^n+\delta^n$. Also $L_n=\alpha^n+\beta^n$. So $L_n+P_{n+1}+P_{n-1}=\alpha^n+\beta^n+\gamma^n+\delta^n$.

(5) Since $W_{n+1}=3W_n-3W_{n-2}-W_{n-3}$, we easily get

$$\begin{aligned}
W_{n+1}-W_n&=2W_n-3W_{n-2}-W_{n-3}\\
&=2\left(\frac{\gamma^n-\delta^n}{\gamma-\delta}-\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)-3\left(\frac{\gamma^{n-2}-\delta^{n-2}}{\gamma-\delta}-\frac{\alpha^{n-2}-\beta^{n-2}}{\alpha-\beta}\right)\\
&\quad-\left(\frac{\gamma^{n-3}-\delta^{n-3}}{\gamma-\delta}-\frac{\alpha^{n-3}-\beta^{n-3}}{\alpha-\beta}\right)\\
&=\frac{1}{\gamma-\delta}\left[\gamma^n\left(2-\frac{3}{\gamma^2}-\frac{1}{\gamma^3}\right)+\delta^n\left(-2+\frac{3}{\delta^2}+\frac{1}{\delta^3}\right)\right]\\
&\quad+\frac{1}{\alpha-\beta}\left[\alpha^{n-1}\left(2\alpha-\frac{3}{\alpha}-\frac{1}{\alpha^2}\right)-\beta^{n-1}\left(2\beta-\frac{3}{\beta}-\frac{1}{\beta^2}\right)\right]
\end{aligned}$$

and hence

$$\begin{aligned}
2W_{n+1}-2W_n&=\frac{2}{2\sqrt{2}}\left[\gamma^n\left(\frac{2\gamma^3-3\gamma-1}{\gamma^3}\right)+\delta^n\left(\frac{-2\delta^2+3\delta+1}{\delta^3}\right)\right]\\
&\quad-\frac{2}{\alpha-\beta}\left[\alpha^{n-1}\left(\frac{2\alpha^3-3\alpha-1}{\alpha^2}\right)-\beta^{n-1}\left(\frac{2\beta^3-3\beta-1}{\beta^2}\right)\right]\\
\Leftrightarrow\quad 2W_{n+1}-2W_n&+\frac{2}{\alpha-\beta}\left[\alpha^{n-1}\left(\frac{2\alpha^3-3\alpha-1}{\alpha^2}\right)-\beta^{n-1}\left(\frac{2\beta^3-3\beta-1}{\beta^2}\right)\right]\\
&=\frac{1}{\sqrt{2}}\left[\gamma^n\left(\frac{2\gamma^3-3\gamma-1}{\gamma^3}\right)+\delta^n\left(\frac{-2\delta^3+3\delta+1}{\delta^3}\right)\right]\\
\Leftrightarrow\quad 2(W_{n+1}-W_n+F_{n-1})&=\gamma^n+\delta^n,
\end{aligned}$$

since $\frac{2\gamma^3-3\gamma-1}{\gamma^3}=\frac{-2\delta^3+3\delta+1}{\delta^3}=\sqrt{2}$ and $\frac{2\alpha^3-3\alpha-1}{\alpha^2}=\frac{2\beta^3-3\beta-1}{\beta^2}=1$.

(6) It is just an algebraic computation, since $W_n=(\frac{\gamma^n-\delta^n}{\gamma-\delta})-(\frac{\alpha^n-\beta^n}{\alpha-\beta})$. □

Theorem 4 *The sum of the first n terms of W_n is*

$$\sum_{i=1}^{n}W_i=\frac{W_n+4W_{n-1}+4W_{n-2}+W_{n-3}+1}{2} \tag{8}$$

for $n\geq 3$.

Proof Recall that $W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4}$. So

$$W_{n-3} + W_{n-4} = 3W_{n-1} - 2W_{n-3} - W_n. \tag{9}$$

Applying (9), we deduce that

$$\begin{aligned} &W_1 + W_0 = 3W_3 - 2W_1 - W_4, \\ &W_2 + W_1 = 3W_4 - 2W_2 - W_5, \\ &W_3 + W_2 = 3W_5 - 2W_3 - W_6, \\ &\ldots, \\ &W_{n-4} + W_{n-5} = 3W_{n-2} - 2W_{n-4} - W_{n-1}, \\ &W_{n-3} + W_{n-4} = 3W_{n-1} - 2W_{n-3} - W_n. \end{aligned} \tag{10}$$

If we sum of both sides of (10), then we obtain $W_{n-3} + W_0 + 2(W_1 + \cdots + W_{n-4}) = 3(W_3 + W_4 + \cdots + W_{n-1}) - 2(W_1 + W_2 + \cdots + W_{n-3}) - (W_4 + W_5 + \cdots + W_n)$. So we get $W_{n-3} + 2(W_1 + W_2 + \cdots + W_{n-4}) = 1 - W_{n-2} - W_{n-1} - W_n + 3W_{n-2} + 3W_{n-1}$ and hence we get the desired result. □

Theorem 5 *The recurrence relations are*

$$\begin{aligned} W_{2n} &= 9W_{2n-2} - 20W_{2n-4} + 9W_{2n-6} - W_{2n-8}, \\ W_{2n+1} &= 9W_{2n-1} - 20W_{2n-3} + 9W_{2n-5} - W_{2n-7} \end{aligned}$$

for $n \geq 4$.

Proof Recall that $W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4}$. So $W_{2n} = 3W_{2n-1} - 3W_{2n-3} - W_{2n-4}$ and hence

$$\begin{aligned} W_{2n} &= 3W_{2n-1} - 3W_{2n-3} - W_{2n-4} \\ &= 9W_{2n-2} - 9W_{2n-4} - 3W_{2n-5} - 9W_{2n-4} + 9W_{2n-6} + 3W_{2n-7} \\ &\quad + W_{2n-8} - W_{2n-8} - W_{2n-4} \\ &= -(3W_{2n-5} - 3W_{2n-7} - W_{2n-8}) + 9W_{2n-2} - 18W_{2n-4} + 9W_{2n-6} \\ &\quad - W_{2n-8} - W_{2n-4} \\ &= -W_{2n-4} + 9W_{2n-2} - 9W_{2n-4} - 9W_{2n-4} + 9W_{2n-6} - W_{2n-8} - W_{2n-4} \\ &= 9W_{2n-2} - 20W_{2n-4} + 9W_{2n-6} - W_{2n-8}. \end{aligned}$$

The other assertion can be proved similarly. □

The rank of an integer N is defined to be

$$\rho(N) = \begin{cases} p & \text{if } p \text{ is the smallest prime with } p|N, \\ \infty & \text{if } N \text{ is prime.} \end{cases}$$

Thus we can give the following theorem.

Theorem 6 *The rank of W_n is*

$$\rho(W_n)=\begin{cases}2 & \textit{if } n=5+6k,6+6k,7+6k,\\ 3 & \textit{if } n=8+12k,9+12k,15+12k,16+12k,\\ 5 & \textit{if } n=14+60k,46+60k\end{cases}$$

for an integer $k\geq 0$.

Proof Let $n=5+6k$. We prove it by induction on k. Let $k=0$. Then we get $W_5=24=2^3\cdot 3$. So $\rho(W_5)=2$. Let us assume that the rank of W_n is 2 for $n=k-1$, that is, $\rho(W_{6k-1})=2$, so $W_{5+6(k-1)}=W_{6k-1}=2^a\cdot B$ for some integers $a\geq 1$ and $B>0$. For $n=k$, we get

$$\begin{aligned}
W_{6k+5} &= 3W_{6k+4}-3W_{6k+2}-W_{6k+1}\\
&= 3(3W_{6k+3}-3W_{6k+1}-W_{6k})-3W_{6k+2}-W_{6k+1}\\
&= 9W_{6k+3}-9W_{6k+1}-3W_{6k}-3W_{6k+2}-W_{6k+1}\\
&= 9(3W_{6k+2}-3W_{6k}-W_{6k-1})-9W_{6k+1}-3W_{6k}-3W_{6k+2}-W_{6k+1}\\
&= 27W_{6k+2}-27W_{6k}-9W_{6k-1}-9W_{6k+1}-3W_{6k}-3W_{6k+2}-W_{6k+1}\\
&= 24W_{6k+2}-30W_{6k}-10W_{6k+1}-9W_{6k-1}\\
&= 24W_{6k+2}-30W_{6k}-10W_{6k+1}-9\cdot 2^aB\\
&= 2\bigl[12W_{6k+2}-15W_{6k}-5W_{6k+1}-9\cdot 2^{a-1}B\bigr].
\end{aligned}$$

Therefore $\rho(W_{5+6k})=2$. Similarly it can be shown that $\rho(W_{6+6k})=\rho(W_{7+6k})=2$.

Now let $n=8+12k$. For $k=0$, we get $W_8=387=3^2\cdot 43$. So $\rho(W_8)=3$. Let us assume that for $n=k-1$ the rank of W_n is 3, that is, $\rho(W_{8+12(k-1)})=\rho(W_{12k-4})=3^b\cdot H$ for some integers $b\geq 1$ and $H>0$ which is not even integer. For $n=k$, we get

$$\begin{aligned}
W_{12k+8} &= 3W_{12k+7}-3W_{12k+5}-W_{12k+4}\\
&= 3W_{12k+7}-3W_{12k+5}-(3W_{12k+3}-3W_{12k+1}-W_{12k})\\
&= 3W_{12k+7}-3W_{12k+5}-3W_{12k+3}+3W_{12k+1}+W_{12k}\\
&= 3W_{12k+7}-3W_{12k+5}-3W_{12k+3}+3W_{12k+1}\\
&\quad +(3W_{12k-1}-3W_{12k-3}-W_{12k-4})\\
&= 3W_{12k+7}-3W_{12k+5}-3W_{12k+3}+3W_{12k+1}+3W_{12k-1}\\
&\quad -3W_{12k-3}-W_{12k-4}\\
&= 3W_{12k+7}-3W_{12k+5}-3W_{12k+3}+3W_{12k+1}+3W_{12k-1}\\
&\quad -3W_{12k-3}-3^b\cdot H\\
&= 3\bigl(W_{12k+7}-W_{12k+5}-W_{12k+3}+W_{12k+1}+W_{12k-1}\\
&\quad -W_{12k-3}-3^{b-1}\cdot H\bigr).
\end{aligned}$$

So $\rho(W_{12k+8})=3$. The others can be proved similarly. □

Remark 1 Apart from the above theorem, we see that $\rho(W_{22}) = \rho(W_{26}) = \infty$, while $\rho(W_{70}) = \rho(W_{98}) = 13$ and $\rho(W_{10}) = \rho(W_{34}) = \rho(W_{50}) = 23$. But there is no general formula.

The companion matrix for W_n is

$$M = \begin{bmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Set

$$N = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$R = [3 \quad 1 \quad 0 \quad 0].$$

Then we can give the following theorem, which can be proved by induction on n.

Theorem 7 *For the sequence W_n, we have*:
(1) $RM^nN = W_{n+3} + P_n + 2(W_{n+1} - F_n)$ *for* $n \geq 1$.
(2) $R(M^T)^{n-3}N = W_n$ *for* $n \geq 3$.
(3) *If* $n \geq 7$ *is odd, then*

$$M^n = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix},$$

where

$$m_{11} = W_{n+2}, \qquad m_{21} = W_{n+1}, \qquad m_{31} = W_n, \qquad m_{41} = W_{n-1},$$

$$m_{14} = -W_{n+1}, \qquad m_{24} = -W_n, \qquad m_{34} = -W_{n-1}, \qquad m_{44} = -W_{n-2},$$

$$m_{12} = -1 - W_{n+1} - 2\sum_{i=0}^{\frac{n-5}{2}} W_{n-1-2i}, \qquad m_{13} = -W_{n+2} - 2\sum_{i=0}^{\frac{n-3}{2}} W_{n-2i},$$

$$m_{22} = -W_n - 2\sum_{i=0}^{\frac{n-5}{2}} W_{n-2-2i}, \qquad m_{23} = -1 - W_{n+1} - 2\sum_{i=0}^{\frac{n-5}{2}} W_{n-1-2i},$$

$$m_{32} = -1 - W_{n-1} - 2\sum_{i=0}^{\frac{n-7}{2}} W_{n-3-2i}, \qquad m_{33} = -W_n - 2\sum_{i=0}^{\frac{n-5}{2}} W_{n-2-2i},$$

$$m_{42} = -W_{n-2} - 2\sum_{i=0}^{\frac{n-7}{2}} W_{n-4-2i}, \qquad m_{43} = -1 - W_{n-1} - 2\sum_{i=0}^{\frac{n-7}{2}} W_{n-3-2i},$$

and if $n \geq 8$ *is even, then*

$$M^n = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix},$$

where

$$m_{11} = W_{n+2}, \qquad m_{21} = W_{n+1}, \qquad m_{31} = W_n, \qquad m_{41} = W_{n-1},$$

$$m_{14} = -W_{n+1}, \qquad m_{24} = -W_n, \qquad m_{34} = -W_{n-1}, \qquad m_{44} = -W_{n-2},$$

$$m_{12} = -W_{n+1} - 2\sum_{i=0}^{\frac{n-4}{2}} W_{n-1-2i}, \qquad m_{13} = -1 - W_{n+2} - 2\sum_{i=0}^{\frac{n-4}{2}} W_{n-2i},$$

$$m_{22} = -1 - W_n - 2\sum_{i=0}^{\frac{n-6}{2}} W_{n-2-2i}, \qquad m_{23} = -W_{n+1} - 2\sum_{i=0}^{\frac{n-4}{2}} W_{n-1-2i},$$

$$m_{32} = -W_{n-1} - 2\sum_{i=0}^{\frac{n-6}{2}} W_{n-3-2i}, \qquad m_{33} = -1 - W_n - 2\sum_{i=0}^{\frac{n-6}{2}} W_{n-2-2i},$$

$$m_{42} = -1 - W_{n-2} - 2\sum_{i=0}^{\frac{n-8}{2}} W_{n-4-2i}, \qquad m_{43} = -W_{n-1} - 2\sum_{i=0}^{\frac{n-6}{2}} W_{n-3-2i}.$$

A circulant matrix is a matrix $A = [a_{ij}]_{n\times n}$ defined to be

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix},$$

where a_i are constants. The eigenvalues of A are

$$\lambda_j(A) = \sum_{k=0}^{n-1} a_k w^{-jk}, \tag{11}$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, and $j = 0, 1, \ldots, n-1$. The spectral norm for a matrix $B = [b_{ij}]_{n\times m}$ is defined to be $\|B\|_{\text{spec}} = \max\{\sqrt{\lambda_i}\}$, where λ_i are the eigenvalues of $B^H B$ for $0 \leq j \leq n-1$ and B^H denotes the conjugate transpose of B.

For the circulant matrix

$$W = W(W_n) = \begin{bmatrix} W_0 & W_1 & W_2 & \cdots & W_{n-1} \\ W_{n-1} & W_0 & W_1 & \cdots & W_{n-2} \\ W_{n-2} & W_{n-1} & W_0 & \cdots & W_{n-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ W_1 & W_2 & W_3 & \cdots & W_0 \end{bmatrix}$$

for W_n, we can give the following theorem.

Theorem 8 *The eigenvalues of W are*

$$\lambda_j(W) = \frac{\left\{\begin{array}{l} W_{n-1}w^{-3j} + (W_n + P_{n-1} - 2F_{n-1} + 1)w^{-2j} \\ \quad + (P_n - 2F_n - W_{n-1})w^{-j} - W_n \end{array}\right\}}{w^{-4j} + 3w^{-3j} - 3w^{-j} + 1}$$

for $j = 0, 1, 2, \ldots, n-1$.

Proof Applying (11) we easily get

$$\begin{aligned}
\lambda_j(W) &= \sum_{k=0}^{n-1} W_k w^{-jk} = \sum_{k=0}^{n-1} \left(\frac{\gamma^k - \delta^k}{\gamma - \delta} - \frac{\alpha^k - \beta^k}{\alpha - \beta} \right) w^{-jk} \\
&= \frac{1}{\gamma - \delta} \left[\frac{\gamma^n - 1}{\gamma w^{-j} - 1} - \frac{\delta^n - 1}{\delta w^{-j} - 1} \right] - \frac{1}{\alpha - \beta} \left[\frac{\alpha^n - 1}{\alpha w^{-j} - 1} - \frac{\beta^n - 1}{\beta w^{-j} - 1} \right] \\
&= \frac{1}{\gamma - \delta} \left[\frac{(\gamma^n - 1)(\delta w^{-j} - 1) - (\delta^n - 1)(\gamma w^{-j} - 1)}{(\gamma w^{-j} - 1)(\delta w^{-j} - 1)} \right] \\
&\quad - \frac{1}{\alpha - \beta} \left[\frac{(\alpha^n - 1)(\beta w^{-j} - 1) - (\beta^n - 1)(\alpha w^{-j} - 1)}{(\alpha w^{-j} - 1)(\beta w^{-j} - 1)} \right] \\
&= \frac{1}{\gamma - \delta} \left[\frac{w^{-j}(\gamma^n \delta - \delta^n \gamma + \gamma - \delta) + \delta^n - \gamma^n}{\delta \gamma w^{-2j} - w^{-j}(\delta + \gamma) + 1} \right] \\
&\quad - \frac{1}{\alpha - \beta} \left[\frac{w^{-j}(\alpha^n \beta - \beta^n \alpha + \alpha - \beta) + \beta^n - \alpha^n}{\beta \alpha w^{-2j} - w^{-j}(\beta + \alpha) + 1} \right] \\
&= \frac{\left\{\begin{array}{l} w^{-3j}[\sqrt{5}(\delta - \gamma + \gamma\delta^n - \delta\gamma^n) + 2\sqrt{2}(\alpha - \beta + \alpha^n\beta - \alpha\beta^n)] \\ \quad + w^{-2j}[\sqrt{5}(\gamma^n - \delta^n + \delta - \gamma + \gamma\delta^n - \gamma^n\delta) + 2\sqrt{2}(\beta^n - \alpha^n) \\ \quad + 4\sqrt{2}(\alpha - \beta + \alpha^n\beta - \alpha\beta^n)] + w^{-j}[\sqrt{5}(\gamma^n - \delta^n + \gamma - \delta \\ \quad + \gamma^n\delta - \gamma\delta^n) + 2\sqrt{2}(\beta - \alpha + \beta^n\alpha - \alpha^n\beta) + 4\sqrt{2}(\beta^n - \alpha^n)] \\ \quad + [\sqrt{5}(\delta^n - \gamma^n) + 2\sqrt{2}(\alpha^n - \beta^n)] \end{array}\right\}}{2\sqrt{10}(w^{-4j} + 3w^{-3j} - 3w^{-j} + 1)} \\
&= \frac{\left\{\begin{array}{l} W_{n-1}w^{-3j} + (W_n + P_{n-1} - 2F_{n-1} + 1)w^{-2j} \\ \quad + (P_n - 2F_n - W_{n-1})w^{-j} - W_n \end{array}\right\}}{w^{-4j} + 3w^{-3j} - 3w^{-j} + 1},
\end{aligned}$$

since $\alpha\beta = -1$, $\gamma\delta = -1$, $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, $\gamma + \delta = 2$, and $\gamma - \delta = 2\sqrt{2}$. □

After all, we consider the spectral norm of W. Let $n = 2$. Then $W_2 = [0]_{2\times 2}$. So $\| W_2 \|_{\text{spec}} = 0$. Similarly for $n = 3$, we get

$$W_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and hence $W_3^H W_3 = I_3$. So $\| W_3 \|_{\text{spec}} = 1$. For $n \geq 4$, the spectral norm of W_n is given by the following theorem, which can be proved by induction on n.

Theorem 9 *The spectral norm of W_n is*

$$\| W_n \|_{\text{spec}} = \frac{W_{n-1} + 4W_{n-2} + 4W_{n-3} + W_{n-4} + 1}{2}$$

for $n \geq 4$.

For example, let $n = 6$. Then the eigenvalues of $W_6^H W_6$ are

$$\lambda_0 = 1{,}369, \qquad \lambda_1 = 289, \qquad \lambda_2 = \lambda_4 = 784 \quad \text{and} \quad \lambda_3 = \lambda_5 = 388.$$

So the spectral norm is $\| W_6 \|_{\text{spec}} = \sqrt{\lambda_0} = 37$. Also $\frac{W_5+4W_4+4W_3+W_2+1}{2} = 37$. Consequently,

$$\| W_6 \|_{\text{spec}} = \frac{W_5 + 4W_4 + 4W_3 + W_2 + 1}{2} = 37$$

as we claimed.

Competing interests
The author declares that they have no competing interests.

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