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The almost sure stability of coupled system of stochastic delay differential equations on networks

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Abstract

This paper investigates the coupled systems of stochastic differential equations with variable delays (CSDDEs) on networks. We analyze the existence and uniqueness of solution by combining the method of graph theory with the Lyapunov function analysis. Furthermore, we utilize the graph theory technique and the nonnegative semimartingale convergence theorem to obtain the almost sure stability of sample solutions and the sufficient principles to locate their limit sets, which correlate closely with the topology property of CSDDEs. Finally we illustrate our main results by examples from population dynamics and vibration systems.

MSC: 60H10; 34K50; 34K20

Keywords: nonlinear models and systems; neural networks; almost sure stability; graph theory; nonnegative semimartingale convergence theorem

1 Introduction

Coupled systems of nonlinear differential equations on networks have been applied widely, especially in the mechanical, electronic, and biological fields [1–9]. A network is always described by a directed graph consisting of vertices and directed arcs connecting them. At each vertex, the local dynamics is given by a system of differential equations called a vertex system [2, 5]. The study of mathematical questions on coupled systems (including synchronization, clustering and transitions) has been introduced by [10]. Among various dynamical properties of the coupled systems on networks overall stability based on given vertex systems is very important and interesting from the viewpoint of controlling complex dynamical systems. However, the stability analysis for the coupled systems on networks is generally a complex and formidable task. It is inspiring that for a coupled deterministic system Li *et al.* [2, 5] gave a systematic method to construct an appropriate Lyapunov function making use of the graph-theoretic technique, and then applied this method to epidemic models [2, 3, 5, 6], oscillator models [5], and ecological models [5], obtaining the global stability.

On the other hand, stochastic differential equations (SDEs) have become a powerful tool in the modeling of realistic systems due to various disturbances; refer to [11]. As a matter of fact, the theory of SDEs has been developed very quickly since Itô introduced his stochastic calculus. The stability theory of stochastic delay differential equations (SDDEs) has been studied extensively; see *e.g.* [12–15]. However, so far as we know there are few papers to deal with almost sure asymptotic stability of the sample paths and their limit sets for stochastic systems except for [16–20]. It is worthy to mention that Mao *et al.* [16–19] made an important development by extending the LaSalle-type theorem from ordinary differential equations (ODEs) to stochastic versions after LaSalle discovered the internal relationship between Birkoff's positive limit set and the Lyapunov function [21].

Motivated by the previous works, the purpose of this paper is to investigate the coupled systems of SDEs with variable delays (CSDDEs) on networks described by

$$\begin{aligned} dx_i(t) = {} & \Bigg[f_i\big(x_i(t), x_i\big(t-r_i(t)\big), t\big) \\ & + \sum_{j=1}^{l} P_{ij}\big(x_i(t), x_i\big(t-r_i(t)\big), x_j(t), x_j\big(t-r_j(t)\big), t\big) \Bigg] dt \\ & + g_i\big(x_i(t), x_i\big(t-r_i(t)\big), t\big)\, dB_i(t), \quad t \geq 0, i = 1, 2, \ldots, l, \end{aligned} \tag{1.1}$$

with the initial data

$$x_i(\theta) = \xi_i(\theta) \in \mathbb{R}^{n_i}, \quad -r_i(0) \leq \theta \leq 0, \tag{1.2}$$

where $\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_l(t)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_l} =: \mathbb{R}^N$, $x(t) = (x_1(t), x_2(t), \ldots, x_l(t)) \in \mathbb{R}^N$, and $B(t) = (B_1(t), B_2(t), \ldots, B_l(t))$ is a $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_l} =: \mathbb{R}^M$-valued Brownian motion.

We analyze the existence and uniqueness of a solution to (1.1) by the combined method of graph theory and Lyapunov function analysis. Furthermore, we utilize the graph theory technique and the nonnegative semimartingale convergence theorem to investigate the almost sure stability of sample solutions and give the sufficient principles to locate their limit sets, which is helpful to understand the dynamical behaviors.

The paper is organized as follows: in Section 2, we prepare some notations and lemmas to be used. In Section 3 we discuss the existence of solution and its uniqueness. In Section 4, we give the sufficient conditions for the almost sure asymptotic stability which relate to the topology structure of the network closely. Moreover, we obtain the limit sets of sample paths with probability 1. Finally, we illustrate our main results through some examples.

2 Preliminaries

For convenience we first state some notations. Throughout the paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $B(t)$, $t \geq 0$, be a standard $\mathbb{R}^M$-valued Brownian motion defined on this probability space. For any constant sequence $\{c_i\}$ $(i = 1, 2, \ldots, l)$, define $\check{c} = \max_{1\leq i\leq l} c_i$, $\hat{c} = \min_{1\leq i\leq l} c_i$. For each $x = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^N$, define $\|x\| = [\sum_{i=1}^{l} (x^i)^T \cdot (x^i)]^{\frac{1}{2}}$. For any positive constant r, let $C([-r, 0]; \mathbb{R}^N)$ be the family of all continuous $\mathbb{R}^N$-valued functions ρ on $[-r, 0]$ with a norm $|\rho| = \sup_{-r\leq\theta\leq 0} \|\rho(\theta)\|$. For $h \in C(\mathbb{R}^N; \mathbb{R})$, define $\mathrm{Ker}(h) = \{x \in \mathbb{R}^N | h(x) = 0\}$. Let $C^b_{\mathcal{F}_0}([-r, 0]; \mathbb{R}^N)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-r, 0]; \mathbb{R}^N)$-valued random variables ξ. For each $V_i \in C^{2,1}(\mathbb{R}^{n_i} \times \mathbb{R}_+; \mathbb{R}_+)$, define a function $\mathcal{L}V_i$ from $\mathbb{R}^N \times \mathbb{R}^N \times$

$\mathbb{R}_+$ to $\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V_i(x,y,t) &= \frac{\partial V_i(x_i,t)}{\partial t} + \frac{\partial V_i(x_i,t)}{\partial x_i} f_i(x_i,y_i,t) \\ &\quad + \sum_{j=1}^{l} \frac{\partial V_i(x_i,t)}{\partial x_i} P_{ij}(x_i,y_i,x_j,y_j,t) \\ &\quad + \frac{1}{2}\operatorname{trace}\left[g_i^T(x_i,y_i,t)\frac{\partial^2 V_i(x_i,t)}{\partial x_i^2} g_i(x_i,y_i,t)\right],\end{aligned}$$

where, for $x_i = (x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n_i)})$,

$$\frac{\partial V_i(x_i,t)}{\partial x_i} = \left(\frac{\partial V_i(x_i,t)}{\partial x_i^{(1)}}, \frac{\partial V_i(x_i,t)}{\partial x_i^{(2)}}, \dots, \frac{\partial V_i(x_i,t)}{\partial x_i^{(n_i)}}\right)$$

and

$$\frac{\partial^2 V_i(x_i,t)}{\partial x_i^2} = \left(\frac{\partial^2 V_i(x_i,t)}{\partial x_i^{(j)}\,\partial x_i^{(k)}}\right)_{n_i\times n_i}.$$

A function $V(x,t): \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be radially unbounded, if

$$\lim_{\|x\|\to\infty} \inf_{0\le t<\infty} V(x,t) = \infty.$$

We denote by $\Psi(\mathbb{R}_+;\mathbb{R}_+)$ the family of all continuous functions $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ with the property that for any $\delta > 0$ we have $\liminf_{t\to\infty} \int_t^{t+\delta} \psi(s)\,ds > 0$; see [16] for details.

For self-completeness we cite the following concepts and theorems on graph theory given by [5, 22, 23]. A directed digraph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ contains a vertex set denoted by $\mathcal{V} = \{1,2,\dots,l\}$ and an arc set denoted by $\mathcal{E}$. Denote by (j,i) an arc leading from initial vertex j to terminal vertex i. A directed digraph $\mathcal{G}$ is weighted if each arc (j,i) is assigned a positive weight a_{ij}. The weight $w(\mathcal{G})$ of a subgraph $\mathcal{G}$ is the product of the weights on all its arcs. Given a weighted digraph $\mathcal{G}$ with l vertices, define the weight matrix $A = (a_{ij})$ whose entry a_{ij} equals the weight of arc (j,i) if it exists, and 0 otherwise. A weight matrix A is cogredient to a matrix E if there exists some permutation matrix P such that $PAP^T = E$. A is *reducible* if it is cogredient to $E = \left[\begin{smallmatrix} B & 0 \\ C & D\end{smallmatrix}\right]$, where B and D are square matrices, or if $l = 1$ and $A = 0$. Otherwise, A is *irreducible*. A directed path $\mathcal{H}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, \dots, i_k\}$ such that its set of arcs is $\{(i_m, i_{m+1}) : m = 1,2,\dots,k-1\}$. If $i_k = i_1$, we call $\mathcal{H}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycle, directed or undirected. A tree $\mathcal{T}$ is rooted at vertex i, called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. A digraph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$L = \begin{bmatrix} \sum_{i\neq 1} a_{1i} & -a_{12} & \cdots & -a_{1l} \\ -a_{21} & \sum_{i\neq 2} a_{2i} & \cdots & -a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{l1} & -a_{l2} & \cdots & \sum_{i\neq l} a_{li} \end{bmatrix}. \tag{2.1}$$

Let c_i denote the cofactor of the ith diagonal element of L. We mention the Kirchhoff's matrix tree theorem.

Lemma 2.1 [5, 23] *Assume $l \geq 2$. If $(\mathcal{G}, A)$ is strongly connected, then each $c_i > 0$, $i \in \mathcal{V}$.*

Lemma 2.2 [23] *A weighted digraph $(\mathcal{G}, A)$ is strongly connected, if and only if A is irreducible.*

Lemma 2.3 [5] *Assume $l \geq 2$. Let c_i be given in Lemma 2.1. Then the following identity holds*:

$$\sum_{i,j=1}^{l} c_i a_{ij} U_{ij}(x_i, x_j) = \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(u,v) \in \mathcal{E}(C_{\mathcal{Q}})} U_{vu}(x_v, x_u).$$

Here for any $i, j \in \mathcal{V}$, $U_{ij}(x_i, x_j)$ is an arbitrary function, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $w(\mathcal{Q})$ is the weight of spanning unicycle graph $\mathcal{Q}$, and $C_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$.

Consider the CSDDEs (1.1) on $t \geq 0$ with initial data (1.2) satisfying $\xi_i \in C^b_{\mathcal{F}_0}([-r_i(0), 0]; \mathbb{R}^{n_i})$, $i = 1, 2, \ldots, l$, where, for each $1 \leq i, j \leq l$,

$$\begin{aligned}
&f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \times \mathbb{R}_+ \to \mathbb{R}^{n_i}, \\
&g_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \times \mathbb{R}_+ \to \mathbb{R}^{n_i \times m_i}, \\
&P_{ij} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \times \mathbb{R}^{n_j} \times \mathbb{R}_+ \to \mathbb{R}^{n_i},
\end{aligned}$$

$f = (f_1, f_2, \ldots, f_l)^T$, $g = (g_1, g_2, \ldots, g_l)^T$, $P_j = (P_{1j}, P_{2j}, \ldots, P_{lj})^T$ are all Borel-measurable functions. For CSDDEs (1.1), we propose the following assumptions.

Assumption 1 Time delays $r_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, l$, are differentiable and their derivatives are less than 1. That is to say, there exist constants $d_i < 1$, $i = 1, \ldots, l$, such that

$$\frac{dr_i(t)}{dt} \leq d_i, \quad i = 1, 2, \ldots, l.$$

For each $1 \leq i \leq l$, Assumption 1 implies $t - r_i(t)$ is an increasing function of t, so

$$-r_i(0) \leq t - r_i(t) \leq t, \quad \forall t \geq 0. \tag{2.2}$$

Assumption 2 For each $1 \leq i, j \leq l$, f_i, g_i, and P_{ij} satisfy the local Lipschitz conditions. That is to say, for each $h > 0$ there are positive constants $C_i(h)$ and $\bar{C}_{ij}(h)$, such that

$$\begin{aligned}
&\big\| f_i(x_i, y_i, t) - f_i(\bar{x}_i, \bar{y}_i, t) \big\| \vee \big\| g_i(x_i, y_i, t) - g_i(\bar{x}_i, \bar{y}_i, t) \big\| \\
&\quad \leq C_i(h) \big(\|x_i - \bar{x}_i\| + \|y_i - \bar{y}_i\| \big), \\
&\big\| P_{ij}(x_i, y_i, x_j, y_j, t) - P_{ij}(\bar{x}_i, \bar{y}_i, \bar{x}_j, \bar{y}_j, t) \big\| \\
&\quad \leq \bar{C}_{ij}(h) \big(\|x_i - \bar{x}_i\| + \|y_i - \bar{y}_i\| + \|x_j - \bar{x}_j\| + \|y_j - \bar{y}_j\| \big)
\end{aligned}$$

for all $t \ge 0$, $x_i, \bar{x}_i, y_i, \bar{y}_i \in \mathbb{R}^{n_i}$, $x_j, \bar{x}_j, y_j, \bar{y}_j \in \mathbb{R}^{n_j}$ with $\|x_i\| \vee \|\bar{x}_i\| \vee \|y_i\| \vee \|\bar{y}_i\| \vee \|x_j\| \vee \|\bar{x}_j\| \vee \|y_j\| \vee \|\bar{y}_j\| \le h$.

Assumption 3 $\sup_{0\le t<\infty}(\|f_i(0,0,t)\| \vee \|g_i(0,0,t)\| \vee \|P_{ij}(0,0,0,0,t)\|) < \infty$, $i,j = 1,2,\ldots,l$.

3 Existence and uniqueness of solution

In order to investigate the sophisticated properties of the solution, we need the existence and uniqueness of the global solution to CSDDEs (1.1) firstly, while their coefficients are usually required to satisfy the linear growth condition and local Lipschitz condition (see *e.g.* Friedman [12] and Mao [14]). However, the coefficients of CSDDEs (1.1) may not satisfy the linear growth condition, though they are locally Lipschitz continuous. We therefore wonder what other alternative conditions proposed can avoid the explosion at a finite time. We answer this question in the following theorem.

Theorem 3.1 *Let Assumptions* 1 *and* 2 *hold. Assume for all* $1 \le i,j \le l$, *there exist functions* $V_i(x_i,t) \in C^{2,1}(\mathbb{R}^{n_i} \times \mathbb{R};\mathbb{R}_+)$ (*radially unbounded*), $F_{ij} \in C(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}_+;\mathbb{R})$, $q_i \in C(\mathbb{R}_+;\mathbb{R}_+)$, $p_j \in C(\mathbb{R}^{n_j} \times [-r_j(0),\infty];\mathbb{R}_+)$, *and nonnegative constants* $a_{ij} \ge 0$ ($A = (a_{ij})$ *is irreducible*), *such that*

$$\mathcal{L}V_i(x,y,t) \le q_i(t)\left[1 + \sum_{j=1}^{l} V_j(x_j,t) + \sum_{j=1}^{l} V_j\bigl(y_j, t - r_i(t)\bigr)\right] - \sum_{j=1}^{l} p_j(x_j,t) + \sum_{j=1}^{l}(1-d_j)p_j\bigl(y_j, t - r_j(t)\bigr) + \sum_{j=1}^{l} a_{ij}F_{ij}(x,y,t). \tag{3.1}$$

Assuming that along each directed cycle $\mathcal{C}_Q$ *of weighted digraph* $(\mathcal{G},A)$, *there exist functions* $h^Q \in C(\mathbb{R}_+;\mathbb{R}_+)$, $k_j^Q \in C(\mathbb{R}^{n_j} \times [-r_j(0),\infty];\mathbb{R}_+)$, $j = 1,2,\ldots,l$, *such that*

$$\sum_{(u,v)\in\mathcal{E}(\mathcal{C}_Q)} F_{vu}(x,y,t) \le h^Q(t)\left[1 + \sum_{j=1}^{l} V_j(x_j,t) + \sum_{j=1}^{l} V_j\bigl(y_j, t - r_j(t)\bigr)\right] - \sum_{j=1}^{l} k_j^Q(x_j,t) + \sum_{j=1}^{l}(1-d_j)k_j^Q\bigl(y_j, t - r_j(t)\bigr). \tag{3.2}$$

Then for any given initial data ξ, *CSDDEs* (1.1)-(1.2) *have a unique global solution on* $[0,\infty)$.

Proof Under Assumptions 1 and 2, CSDDEs (1.1)-(1.2) have a unique maximal local solution $x(t)$ on $t \in [[0,\sigma_\infty[[$ for any given initial data ξ, where σ_∞ is the explosion time. We then need only to show that $\sigma_\infty = \infty$ a.s. Therefore, choose sufficiently large N, such that $N \ge |\xi_i|$ ($i = 1,2,\ldots,l$), then for any $n > N$, define the stopping time

$$\tau_n = \inf\bigl\{t \in [0,\sigma_\infty) : \bigl\|x_i(t)\bigr\| \ge n \text{ for some } 1 \le i \le l\bigr\},$$

and set $\inf\emptyset = \infty$. Clearly, the τ_n are increasing so we define the limit $\tau_\infty = \lim_{k\to\infty}\tau_n$. Obviously, $\tau_\infty \le \sigma_\infty$ a.s. Next, we use the stochastic Lyapunov analysis method to prove $\tau_\infty = \infty$, then we will obtain the required equality $\sigma_\infty = \infty$. Let c_i is the cofactor of the ith diagonal element of Laplacian matrix of $(\mathcal{G}, A)$. From Lemma 2.1 and Lemma 2.2, we know each $c_i > 0$. Then we define $V(x,t) = \sum_{i=1}^{l} c_i V_i(x_i,t)$. Utilizing inequalities (3.1), (3.2), and Lemma 2.3, we compute

$$
\begin{aligned}
&\mathcal{L}V(x,y,t)\\
&\quad= \sum_{i=1}^{l} c_i \mathcal{L}V_i(x,y,t)\\
&\quad\le \sum_{i=1}^{l} c_i \Biggl\{ q_i(t)\Biggl[1 + \sum_{j=1}^{l} V_j(x_j,t) + \sum_{j=1}^{l} V_j\bigl(y_j, t - r_j(t)\bigr)\Biggr] - \sum_{j=1}^{l} p_j(x_j,t)\\
&\qquad + \sum_{j=1}^{l} (1-d_j) p_j\bigl(y_j, t - r_j(t)\bigr)\Biggr\} + \sum_{i,j=1}^{l} c_i a_{ij} F_{ij}(x,y,t)\\
&\quad= \sum_{i=1}^{l} c_i q_i(t) + \Biggl(\sum_{i=1}^{l} c_i\Biggr) \sum_{i=1}^{l} V_i(x_i,t)\\
&\qquad + \Biggl(\sum_{i=1}^{l} c_i\Biggr) \sum_{i=1}^{l} V_i\bigl(y_i, t - r_i(t)\bigr) - \sum_{i,j=1}^{l} c_i p_j(x_j,t)\\
&\qquad + \sum_{i,j=1}^{l} c_i (1-d_j) p_j\bigl(y_j, t - r_j(t)\bigr) + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(u,v)\in\mathcal{E}(\mathcal{C}_{\mathcal{Q}})} F_{vu}(x,y,t)\\
&\quad\le \Biggl[\sum_{i=1}^{l} c_i q_i(t) + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) h^{\mathcal{Q}}(t)\Biggr] + \Biggl[\sum_{i=1}^{l} c_i + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) h^{\mathcal{Q}}(t)\Biggr] \sum_{i=1}^{l} V_i(x_i,t)\\
&\qquad + \Biggl[\sum_{i=1}^{l} c_i + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) h^{\mathcal{Q}}(t)\Biggr] \sum_{i=1}^{l} V_i\bigl(y_i, t - r_i(t)\bigr)\\
&\qquad - \Biggl(\sum_{i=1}^{l} c_i\Biggr) \sum_{i=1}^{l} p_i(x_i,t) - \sum_{i=1}^{l} \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) k_i^{\mathcal{Q}}(x_i,t)\\
&\qquad + \Biggl(\sum_{i=1}^{l} c_i\Biggr) \sum_{i=1}^{l} (1-d_i) p_i\bigl(y_i, t - r_i(t)\bigr) + \sum_{i=1}^{l} (1-d_i) \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) k_i^{\mathcal{Q}}\bigl(y_i, t - r_i(t)\bigr)\\
&\quad\le \eta(t)\Biggl[\hat{c} + \sum_{i=1}^{l} c_i V_i(x_i,t) + \sum_{i=1}^{l} c_i V_i\bigl(y_i, t - r_i(t)\bigr)\Biggr]\\
&\qquad - \sum_{i=1}^{l} \phi_i(x_i,t) + \sum_{i=1}^{l} (1-d_i) \phi_i\bigl(y_i, t - r_i(t)\bigr),
\end{aligned}
\tag{3.3}
$$

where

$$
\eta(t) = \frac{1}{\hat{c}}\Biggl[\sum_{i=1}^{l} c_i\bigl(1 + q_i(t)\bigr) + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) h^{\mathcal{Q}}(t)\Biggr],
$$

$$
\phi_i(x_i,t) = p_i(x_i,t) \sum_{j=1}^{l} c_j + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) k_i^{\mathcal{Q}}(x_i,t).
$$

For any $n > N$ and $t \geq 0$, utilizing Itô's formula, we get

$$\begin{aligned}
&\mathbb{E}\bigl[V\bigl(x(t\wedge\tau_n),t\wedge\tau_n\bigr)\bigr] \\
&\quad= \mathbb{E}\Biggl[\sum_{i=1}^{l} c_i V_i\bigl(x_i(t\wedge\tau_n),t\wedge\tau_n\bigr)\Biggr] \\
&\quad= \mathbb{E}\Biggl[\sum_{i=1}^{l} c_i V_i\bigl(x_i(0),0\bigr)\Biggr] + \mathbb{E}\Biggl[\int_0^{t\wedge\tau_n}\sum_{i=1}^{l} c_i \mathcal{L}V_i\bigl(x_i(s),y_i(s),s\bigr)\,ds\Biggr].
\end{aligned}$$

From Assumption 1, we know for each $1 \leq i \leq l$, $t - r_i(t)$ is an increasing function and

$$0 < \frac{1}{1-r_i'(s)} \leq \frac{1}{1-d_i} \leq \frac{1}{1-\check{d}}.$$

Then inequality (3.3) implies

$$\begin{aligned}
&\mathbb{E}\bigl[V\bigl(x(t\wedge\tau_n),t\wedge\tau_n\bigr)\bigr] \\
&\quad\leq \sum_{i=1}^{l} c_i \mathbb{E}V_i\bigl(x_i(0),0\bigr) + \mathbb{E}\int_0^{t\wedge\tau_n}\hat{c}\eta(s)\,ds + \mathbb{E}\int_0^{t\wedge\tau_n}\eta(s)\sum_{i=1}^{l} c_i V_i\bigl(x_i(s),s\bigr)\,ds \\
&\qquad+ \mathbb{E}\int_0^{t\wedge\tau_n}\eta(s)\sum_{i=1}^{l} c_i V_i\bigl(x_i\bigl(s-r_i(s)\bigr),s-r_i(s)\bigr)\,ds \\
&\qquad- \mathbb{E}\int_0^{t\wedge\tau_n}\sum_{i=1}^{l}\phi_i\bigl(x_i(s),s\bigr)\,ds + \mathbb{E}\int_0^{t\wedge\tau_n}\sum_{i=1}^{l}(1-d_i)\phi_i\bigl(x_i\bigl(s-r_i(s)\bigr),s-r_i(s)\bigr)\,ds \\
&\quad\leq \sum_{i=1}^{l} c_i \mathbb{E}V_i\bigl(\xi_i(0),0\bigr) + \mathbb{E}\int_0^{t\wedge\tau_n}\hat{c}\eta(s)\,ds + \mathbb{E}\sum_{i=1}^{l}\int_0^{t\wedge\tau_n} c_i\eta(s)V_i\bigl(x_i(s),s\bigr)\,ds \\
&\qquad+ \sum_{i=1}^{l}\mathbb{E}\int_{-r_i(0)}^{t\wedge\tau_n-r_i(t\wedge\tau_n)}\frac{c_i}{1-r_i'(s)}\eta\bigl(s+r_i(s)\bigr)V_i\bigl(x_i(s),s\bigr)\,ds \\
&\qquad- \sum_{i=1}^{l}\mathbb{E}\int_0^{t\wedge\tau_n}\phi_i\bigl(x_i(s),s\bigr)\,ds + \sum_{i=1}^{l}\mathbb{E}\int_{-r_i(0)}^{t\wedge\tau_n-r_i(t\wedge\tau_n)}\frac{1-d_i}{1-r_i'(s)}\phi_i\bigl(x_i(s),s\bigr)\,ds \\
&\quad\leq \sum_{i=1}^{l} c_i \mathbb{E}V_i\bigl(\xi_i(0),0\bigr) + \mathbb{E}\int_0^{t\wedge\tau_n}\hat{c}\eta(s)\,ds \\
&\qquad+ \sum_{i=1}^{l}\int_{-r_i(0)}^{0}\frac{c_i}{1-d_i}\mathbb{E}\eta\bigl(s+r_i(s)\bigr)V_i\bigl(\xi_i(s),s\bigr)\,ds \\
&\qquad+ \sum_{i=1}^{l}\int_{-r_i(0)}^{0}\mathbb{E}\phi_i\bigl(\xi_i(s),s\bigr)\,ds + \mathbb{E}\int_0^{t\wedge\tau_n}\sum_{i=1}^{l} c_i\eta(s)V_i\bigl(x_i(s),s\bigr)\,ds \\
&\qquad+ \frac{1}{1-\check{d}}\mathbb{E}\int_0^{t\wedge\tau_n}\sum_{i=1}^{l} c_i\eta\bigl(s+r_i(s)\bigr)V_i\bigl(x_i(s),s\bigr)\,ds \\
&\quad\leq C(t) + \frac{2-\check{d}}{1-\check{d}}\Bigl[\sup_{0\leq r\leq t}\eta(r)\Bigr]\int_0^t \mathbb{E}V\bigl(x(s\wedge\tau_n),s\wedge\tau_n\bigr)\,ds,
\end{aligned} \tag{3.4}$$

where

$$C(t)=\sum_{i=1}^{l}c_i\mathbb{E}V_i\big(\xi_i(0),0\big)+\int_0^t\hat{c}\eta(s)\,ds+\sum_{i=1}^{l}\int_{-r_i(0)}^{0}c_i\eta\big(s+r_i(s)\big)\mathbb{E}V_i\big(\xi_i(s),s\big)\,ds$$
$$+\sum_{i=1}^{l}\int_{-r_i(0)}^{0}\mathbb{E}\phi_i\big(\xi_i(s),s\big)\,ds.$$

The Gronwall inequality yields

$$\mathbb{E}\big[V\big(x(r\wedge\tau_n),r\wedge\tau_n\big)\big]\le C(t)e^{\frac{2-\check{d}}{1-\check{d}}t(\sup_{0\le r\le t}\eta(r))},\quad\forall t\ge 0.\tag{3.5}$$

On the other hand, define $\mu:R_+\to R_+$ by

$$\mu(a)=\inf_{\|x\|\ge a,0\le t<\infty}V(x,t)\quad\text{for } a\ge 0.\tag{3.6}$$

Clearly, $\mu(\|x(t)\|)\le V(x(t),t)$. Since $V_i(x_i,t)$, $i=1,2,\dots,l$ is radially unbounded,

$$\lim_{a\to\infty}\mu(a)=\infty.$$

It therefore follows from (3.5) that

$$C(t)e^{\frac{2-\check{d}}{1-\check{d}}t(\sup_{0\le r\le t}\eta(r))}\ge\mathbb{E}\mu\big(\big\|x(t\wedge\tau_n)\big\|\big)\ge\mu(n)\mathbb{P}(\tau_n\le t).$$

Letting $n\to\infty$, we have

$$\mathbb{P}(\tau_\infty\le t)=0,\quad\forall t>0.$$

Then $t\to\infty$, and we obtain

$$\mathbb{P}(\tau_\infty<\infty)=0.$$

That is, $\tau_\infty=\infty$ a.s. We, therefore, must have $\sigma_\infty=\infty$ a.s. This completes the proof. □

Corollary 3.1 *Suppose that the assumptions of Theorem* 3.1 *are satisfied except that inequalities* (3.1) *and* (3.2) *are replaced by*

$$\mathcal{L}V_i(x_i,y_i,t)\le\gamma_i(t)+\sum_{j=1}^{l}\alpha_{ij}(t)V_j(x_j,t)+\sum_{j=1}^{l}\beta_{ij}(t)V_j\big(y_j,t-r_j(t)\big)+\sum_{j=1}^{l}a_{ij}F_{ij}(x,y,t)$$

and

$$\sum_{(u,v)\in\mathcal{E}(\mathcal{C}_{\mathcal{Q}})}F_{vu}(x,y,t)\le\delta^{\mathcal{Q}}(t)+\sum_{j=1}^{l}\theta_j^{\mathcal{Q}}(t)V_j(x_j,t)+\sum_{j=1}^{l}\vartheta_j^{\mathcal{Q}}(t)V_j\big(y_j,t-r_j(t)\big),$$

where $\gamma_i(t),\alpha_{ij}(t),\beta_{ij}(t),\delta^{\mathcal{Q}}(t),\theta_j^{\mathcal{Q}}(t),\vartheta_j^{\mathcal{Q}}(t)\in C(\mathbb{R}_+;\mathbb{R}_+)$. *Then the result of Theorem* 3.1 *still holds*.

Proof Define $q_i(t) = \max_{1\le j\le l}\{\gamma_i(t),\alpha_{ij}(t),\beta_{ij}(t)\}$, $h^{\mathcal{Q}}(t) = \max_{1\le j\le l}\{\delta^{\mathcal{Q}}(t),\theta_j^{\mathcal{Q}}(t),\vartheta_j^{\mathcal{Q}}(t)\}$, $p_j(x_j,t) = k_j^{\mathcal{Q}}(x_j,t) \equiv 0$, then the required assertion follows from Theorem 3.1. □

Theorem 3.2 *Suppose that all of the assumptions of Theorem* 3.1 *are satisfied, moreover, for each* $1 \le i \le l$, $V_i(x_i,t) \in C^{2,1}(\mathbb{R}_+^{n_i} \times \mathbb{R};\mathbb{R}_+)$ (*radially unbounded*) *satisfies*

$$\lim_{x_i^{(j)}\to 0^+} \inf_{0\le t<\infty} V_i(x_i,t) = \infty, \quad j = 1,2,\dots,n_i, \tag{3.7}$$

where $x_i = (x_i^{(1)},x_i^{(2)},\dots,x_i^{(n_i)})$. *Then for any given initial data* ξ ($\xi(t) \in \mathbb{R}_+^N$), *CSDDEs* (1.1)-(1.2) *have a unique global solution* $x(t)$ *on* $[0,\infty)$ *and* $x(t) \in \mathbb{R}_+^N$ *for all* $t \in [0,\infty)$ *a.s.*

Proof Let $N > 0$ be sufficiently large for

$$\frac{1}{N} < \min_{1\le i\le l,1\le j\le n_i,-r_i(0)\le t\le 0} \xi_i^{(j)}(t) \le \max_{1\le i\le l,1\le j\le n_i,-r_i(0)\le t\le 0} \xi_i^{(j)}(t) < N.$$

For each integer $n \ge N$, define the stopping time

$$\tau_n = \inf\bigl\{t \in [0,\sigma_\infty) : x_i^{(j)}(t) \in (1/n,n) \text{ for some } 1 \le i \le l \text{ and some } 1 \le j \le n_i\bigr\}. \tag{3.8}$$

The left proof is a modification of that of Theorem 3.1 directly, we omit it for avoiding duplication. □

Example 3.1 (n-Dimensional stochastic diffusion population model) Consider the environmentally perturbed n-dimensional diffusion population system given by the SDEs

$$\begin{aligned} dx_i(t) = {}& \Biggl[x_i(t)\bigl(a_i(t) - b_i(t)x_i\bigl(t - r_i(t)\bigr)\bigr) + \sum_{j=1,j\ne i}^{n} d_{ij}\bigl(x_j^2(t) - x_i^2(t)\bigr)\Biggr]dt \\ & + \sum_{j=1}^{l} \sigma_{ij}(t)x_i(t)\,dB_j(t), \quad i = 1,2,\dots,n, \end{aligned} \tag{3.9}$$

where $a_i(t),b_i(t),\sigma_{ij}(t) \in C(\mathbb{R}_+;\mathbb{R}_+)$, $d_{ij} \ge 0$ are the diffusion coefficients and $r_i(t)$ satisfy Assumption 1. The above system describes the phenomenon that species always migrate from high concentration regions to low concentration ones, *i.e.*, the movement is a function of species density [24]. Let $d_{ii} = 0$. In order to yield the existence of the global positive solution we define C^2-functions $V_i : R_+ \to R_+$, $i = 1,2,\dots,n$, by

$$V_i(x_i) = x_i - 1 - \log x_i.$$

Obviously, V_i is radially unbounded and $\lim_{x_i\to 0^+} \inf_{0\le t<\infty} V_i(x_i) = \infty$. By Itô's formula, compute

$$\begin{aligned} \mathcal{L}V_i(x,y,t) = {}& x_i(a_i - b_iy_i) + \sum_{j=1}^{n} d_{ij}\bigl(x_j^2 - x_i^2\bigr) \\ & - \Biggl[a_i - b_iy_i + \sum_{j=1}^{n} d_{ij}\biggl(\frac{x_j^2}{x_i} - x_i\biggr)\Biggr] + \frac{1}{2}\sum_{j=1}^{l} \sigma_{ij}^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} d_{ij}\left(x_j^2 - \frac{x_j^2}{x_i}\right) - \sum_{j=1}^{n} d_{ij}\left(x_i^2 - x_i\right) + a_i x_i \\
&\quad - \left(a_i - \frac{1}{2}\sum_{j=1}^{l}\sigma_{ij}^2\right) - b_i x_i y_j + b_i y_i \\
&= \sum_{j=1}^{n} d_{ij}\left(x_j^2 - \frac{x_j^2}{x_i} - x_i^2 + \frac{x_i^2}{x_j}\right) - \sum_{j=1}^{n} d_{ij}\frac{x_i^2}{x_j} \\
&\quad + \left(a_i + \sum_{j=1}^{n} d_{ij}\right)x_i - \left(a_i - \frac{1}{2}\sum_{j=1}^{l}\sigma_{ij}^2\right) - b_i x_i y_j + b_i y_i \\
&\leq \sum_{j=1}^{n} d_{ij}\left(x_j^2 - \frac{x_j^2}{x_i} - x_i^2 + \frac{x_i^2}{x_j}\right) + \left(a_i + \sum_{j=1}^{n} d_{ij}\right)x_i + \frac{1}{2}\sum_{j=1}^{l}\sigma_{ij}^2 + b_i y_i \\
&=: \sum_{j=1}^{l} u_{ij}F_{ij}(x) + \left(a_i + \sum_{j=1}^{n} d_{ij}\right)x_i + \frac{1}{2}\sum_{j=1}^{l}\sigma_{ij}^2 + b_i y_i, \qquad (3.10)
\end{aligned}
$$

dropping (t) from a, b, σ, where

$$
u_{ij} = \begin{cases} d_{ij}, & j \neq i, \\ \delta_i, & j = i, \end{cases} \qquad F_{ij}(x) = x_j^2 - \frac{x_j^2}{x_i} - x_i^2 + \frac{x_i^2}{x_j},
$$

here δ_i may be any nonnegative constant for $i = 1, 2, \ldots, n$, because $F_{ii}(x) \equiv 0$.

Utilizing the inequality for $x > 0$, $2(x - 1 - \log x) > x$, we have

$$
\left(a_i(t) + \sum_{j=1}^{n} d_{ij}\right)x_i + \frac{1}{2}\sum_{j=1}^{l}\sigma_{ij}^2(t) \leq K(t)\bigl(1 + V_i(x_i)\bigr), \qquad b_i(t)y_i \leq K(t)V_j(y_j),
$$

where $K(t) \in C(\mathbb{R}_+; \mathbb{R}_+)$. Obviously,

$$
\sum_{(u,v)\in\mathcal{E}(\mathcal{C}_{\mathcal{Q}})} F_{vu}(x) = 0.
$$

Then by Theorem 3.2, we obtain the following result.

Corollary 3.2 *If Assumption* 1 *hold, moreover, there is a series of nonnegative constants* $(\delta_1, \delta_2, \ldots, \delta_n)$ *such that*

$$
G = \begin{pmatrix} \delta_1 & d_{12} & \cdots & d_{1n} \\ d_{21} & \delta_2 & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & \delta_n \end{pmatrix} \qquad (3.11)
$$

is irreducible. Then SDE (3.9) *with initial data* (1.2) *have a unique global solution* $x(t)$ *on* $[0, \infty)$ *and* $x(t) \in \mathbb{R}_+^n$ *for all* $t \in [0, \infty)$ *a.s.*

4 The almost sure stability

One of the important issues in the study of coupled systems is the automatic control with consequent emphasis being placed on the analysis of stability. In this paper, we will mainly discuss the almost surely asymptotic stability of the CSDDEs. The main result is as follows.

Theorem 4.1 *Let Assumptions* 1-3 *hold. For all* $1 \leq i,j \leq l$, *there exists a radially unbounded function* $V_i(x_i,t) \in C^{2,1}(\mathbb{R}^{n_i} \times \mathbb{R};\mathbb{R}_+)$, $\beta_i \in L(\mathbb{R}_+;\mathbb{R}_+)$, $w_i \in C(\mathbb{R}^N \times \mathbb{R};\mathbb{R}_+)$, $\bar{w}_{ij} \in C(\mathbb{R}^{n_j} \times [-r_j(0),\infty];\mathbb{R}_+)$, $\alpha_i \in \Psi(\mathbb{R}_+;\mathbb{R}_+)$, $h_i \in C(\mathbb{R}^N;\mathbb{R}_+)$, $F_{ij} \in C(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}_+;\mathbb{R})$ *and constants* $a_{ij} \geq 0$ ($A = (a_{ij})$ *is irreducible*) *such that*

$$\mathcal{L}V_i(x,y,t) \leq \beta_i(t) - w_i(x,t) + \sum_{j=1}^{l}(1-d_j)\bar{w}_{ij}\big(y_j,t-r_j(t)\big) + \sum_{j=1}^{l} a_{ij}F_{ij}(x,y,t), \tag{4.1}$$

where

$$w_i(x,t) - \sum_{j=1}^{l}\bar{w}_{ij}(x_j,t) \geq \alpha_i(t)h_i(x). \tag{4.2}$$

Assume along each directed cycle $\mathcal{C}_{\mathcal{Q}}$ *of a weighted digraph* $(\mathcal{G},A)$ *that there are functions* $\beta^{\mathcal{Q}} \in L(\mathbb{R}_+;\mathbb{R}_+)$, $w^{\mathcal{Q}} \in C(\mathbb{R}^N \times \mathbb{R};\mathbb{R}_+)$, $\bar{w}_j^{\mathcal{Q}} \in C(\mathbb{R}^{n_j} \times [-r_j(0),\infty];\mathbb{R}_+)$, $\alpha^{\mathcal{Q}} \in \Psi(\mathbb{R}_+;\mathbb{R}_+)$, $h^{\mathcal{Q}} \in C(\mathbb{R}^N;\mathbb{R}_+)$ *such that*

$$\sum_{(u,v)\in\mathcal{E}(\mathcal{C}_{\mathcal{Q}})} F_{vu}(x,y,t) \leq \beta^{\mathcal{Q}}(t) - w^{\mathcal{Q}}(x,t) + \sum_{j=1}^{l}(1-d_j)\bar{w}_j^{\mathcal{Q}}\big(y_j,t-r_j(t)\big), \tag{4.3}$$

where

$$w^{\mathcal{Q}}(x,t) - \sum_{j=1}^{l}\bar{w}_j^{\mathcal{Q}}(x_j,t) \geq \alpha^{\mathcal{Q}}(t)h^{\mathcal{Q}}(x). \tag{4.4}$$

Then $S = (\bigcap_{i=1}^{l}\operatorname{Ker}(h_i)) \cap (\bigcap_{\mathcal{Q}\in\mathbb{Q}}\operatorname{Ker}(h^{\mathcal{Q}})) \neq \emptyset$ *and for any initial data* ξ, *the CSDDEs* (1.1)-(1.2) *has a unique global solution on* $[0,\infty)$ *denoted by* $x(t;\xi)$ *satisfying*

$$\lim_{t\to\infty} d\big(x(t,\xi),S\big) = 0 \quad a.s. \tag{4.5}$$

Proof For any given initial data, the global existence of the unique solution on $t \geq 0$ is a direct application of Theorem 3.1. Next, we borrow the technique in [16] and combine it with the graph method from [5] to get the limit set. Because the proof is rather technical, we divide it into three steps.

Step 1. Write the solution $x_i(t;\xi) = x_i(t)$ for simplicity. Let c_i be the cofactor of the ith diagonal element of Laplacian matrix of $(\mathcal{G},A)$, and compute

$$\begin{aligned}
&\sum_{i=1}^{l} c_i\mathcal{L}V_i(x,y,t) \\
&\quad\leq \sum_{i=1}^{l} c_i\beta_i(t) + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(u,v)\in\mathcal{E}(\mathcal{C}_{\mathcal{Q}})} F_{vu}(x,y,t)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{l} c_i \left[w_i(x,t) - \sum_{j=1}^{l} (1-d_j)\bar{w}_{ij}\big(y_j, t - r_j(t)\big) \right] \\
&\leq \zeta(t) - \sum_{i=1}^{l} c_i \left[w_i(x,t) - \sum_{j=1}^{l} (1-d_j)\bar{w}_{ij}\big(y_j, t - r_j(t)\big) \right] \\
&\quad - \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \left[w^{\mathcal{Q}}(x,t) - \sum_{j=1}^{l} (1-d_j)\bar{w}_j^{\mathcal{Q}}\big(y_j, t - r_j(t)\big) \right],
\end{aligned}
$$

dropping (t) from x and y, where

$$
\zeta(t) = \sum_{i=1}^{l} c_i \beta_i(t) + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q})\beta^{\mathcal{Q}}(t) \in L(\mathbb{R}_+;\mathbb{R}_+).
$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{l} c_i V_i\big(x_i(t), t\big) \\
&\quad \leq \sum_{i=1}^{l} c_i V_i\big(\xi_i(0), 0\big) + \int_0^t \zeta(s)\,ds + \sum_{i,j=1}^{l} c_i \int_{-r_j(0)}^{0} \bar{w}_{ij}\big(\xi_j(s), s\big)\,ds \\
&\qquad + \sum_{j=1}^{l} \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \int_{-r_j(0)}^{0} \bar{w}_j^{\mathcal{Q}}\big(\xi_j(s), s\big)\,ds - \sum_{i=1}^{l} c_i \int_0^t \alpha_i(s) h_i\big(x(s)\big)\,ds \\
&\qquad - \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \int_0^t \alpha^{\mathcal{Q}}(s) h^{\mathcal{Q}}\big(x(s)\big)\,ds + \sum_{i=1}^{l} M_i(t),
\end{aligned}
\tag{4.6}
$$

where

$$
M_i(t) = \int_0^t c_i \frac{\partial V_i}{\partial x_i}\big(x_i(s), s\big) g_i\big(x_i(s), x_i\big(s - r_i(s)\big), s\big)\,dB_i(s).
$$

Next, we prove $\sum_{i=1}^{l} M_i(t)$ is a local martingale. Choose sufficiently large N, such that $N \geq |\xi_i|$ $(i = 1, 2, \ldots, l)$, then for any $n > N$, define the stopping time

$$
\tau_n = \inf\big\{t \geq 0 : \|x_i(t)\| \geq n \text{ for some } 1 \leq i \leq l\big\}.
$$

So this stopping sequence $\tau_n \uparrow \infty$ a.s. From Assumptions 2 and 3, we know that, for each i, g_i is local bounded. Notice that $\frac{\partial V_i}{\partial x_i}(x_i(s), s)$ is continuous in s and $g_i(x_i(s), x_i(s - r_i(s)), s)I_{\{0\leq s\leq\tau_n\}}$ is uniformly bounded in $s \in [0, t]$, where I_A is the indicator function of set A. So $M_i(t \wedge \tau_n) = \int_0^t c_i \frac{\partial V_i}{\partial x_i}(x_i(s), s) g_i(x_i(s), x_i(s - r_i(s)), s) I_{\{0\leq s\leq\tau_n\}}\,dB_i(s)$ is a martingale (see [14] for the details of the martingale proof). Thus each $M_i(t)$ is a local martingale. Then the required assertion follows.

Using the nonnegative semimartingale convergence theorem (*cf.* Liptser and Shiryayev [25] or Mao [14]) we get

$$
\limsup_{t\to\infty} \sum_{i=1}^{l} c_i V_i\big(x_i(t), t\big) < \infty \quad \text{a.s.}
\tag{4.7}
$$

On the other hand, the left side of (4.6) is positive, taking the expectations on the right hand and letting $t \to \infty$, we obtain

$$\sum_{i=1}^{l} c_i \mathbb{E}\int_0^\infty \alpha_i(s)h_i\bigl(x(s)\bigr)\,ds + \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q})\mathbb{E}\int_0^\infty \alpha^{\mathcal{Q}}(s)h^{\mathcal{Q}}\bigl(x(s)\bigr)\,ds < \infty. \tag{4.8}$$

This implies

$$\int_0^\infty \alpha_i(s)h_i\bigl(x(s)\bigr)\,ds < \infty, \quad i = 1,2,\ldots,l \text{ a.s.}, \tag{4.9}$$

$$\int_0^t \alpha^{\mathcal{Q}}(s)h^{\mathcal{Q}}\bigl(x(s)\bigr)\,ds < \infty, \quad \mathcal{Q} \in \mathbb{Q} \text{ a.s.} \tag{4.10}$$

Step 2. From (4.8) and $\alpha \in \Psi(\mathbb{R}_+,\mathbb{R}_+)$, it is not difficult to prove that, for each $1 \le i \le l$ and each $\mathcal{Q} \in \mathbb{Q}$,

$$\liminf_{t\to\infty} h_i\bigl(x(t)\bigr) = 0, \qquad \liminf_{t\to\infty} h^{\mathcal{Q}}\bigl(x(t)\bigr) = 0 \quad \text{a.s.} \tag{4.11}$$

We now claim that, for each $1 \le i \le l$ and each $\mathcal{Q} \in \mathbb{Q}$,

$$\limsup_{t\to\infty} h_i\bigl(x(t)\bigr) = 0, \qquad \limsup_{t\to\infty} h^{\mathcal{Q}}\bigl(x(t)\bigr) = 0 \quad \text{a.s.} \tag{4.12}$$

If this is false, then there is some $i_1 \in \mathbb{L}$ such that

$$\mathbb{P}\Bigl\{\limsup_{t\to\infty} h_{i_1}\bigl(x(t)\bigr) > 0\Bigr\} > 0, \tag{4.13}$$

or there is some $\mathcal{Q}_1 \in \mathbb{Q}$ such that

$$\mathbb{P}\Bigl\{\limsup_{t\to\infty} h^{\mathcal{Q}_1}\bigl(x(t)\bigr) > 0\Bigr\} > 0. \tag{4.14}$$

Without loss of generality we suppose (4.13) holds. Hence there is a small constant $0 < \epsilon_1 < \frac{1}{3}$ such that

$$\mathbb{P}\bigl(A_{i_1}^1\bigr) \ge 3\epsilon_1, \tag{4.15}$$

where

$$A_{i_1}^1 = \Bigl\{\limsup_{t\to\infty} h_{i_1}\bigl(x(t)\bigr) > 2\epsilon_1\Bigr\}.$$

For each $i \in \mathbb{L}$, define $\mu_i : R_+ \to R_+$ by

$$\mu_i(a) = \inf_{\|x_i\|\ge a, 0\le t<\infty} V_i(x_i,t) \quad \text{for } a \ge 0. \tag{4.16}$$

So by (4.7) and the continuity of both $x_i(t)$ and $V_i(x_i,t)$, we have

$$\sup_{0\le t<\infty} \mu_i\bigl(\|x_i(t)\|\bigr) \le \sup_{0\le t<\infty} V_i\bigl(x_i(t),t\bigr) < \infty \quad \text{a.s.} \tag{4.17}$$

On the other side, the radial unboundedness of each $V_i(x_i, t)$ implies

$$\sup_{0\le t<\infty} \|x(t)\| < \infty \quad \text{a.s.} \tag{4.18}$$

Recalling the boundedness of the initial data we can then find a positive number χ, which depends on ϵ_1, sufficiently large such that $|\xi| < \chi$, and

$$\mathbb{P}(A^2) \ge 1 - \epsilon_1, \tag{4.19}$$

where

$$A^2 = \Big\{ \sup_{0\le t<\infty} \|x(t)\| < \chi \Big\}.$$

It is easy to see from (4.15) and (4.19) that

$$\mathbb{P}(A^1_{i_1} \cap A^2) \ge 2\epsilon_1. \tag{4.20}$$

For any fixed number $\eta > 0$, let us now define a sequence of stopping times

$$\begin{aligned}
&\tau = \inf\{t \ge 0 : \|x(t)\| \ge \chi\}, \\
&\sigma^1_{i_1} = \inf\{t \ge 0 : h_{i_1}(x(t)) \ge 2\epsilon_1\}, \\
&\sigma^2_{i_1} = \inf\{t \ge \sigma^1_{i_1} + \eta : h_{i_1}(x(t)) \le \epsilon_1\}, \\
&\sigma^{2k+1}_{i_1} = \inf\{t \ge \sigma^{2k}_{i_1} : h_{i_1}(x(t)) \ge 2\epsilon_1\}, \quad k = 1, 2, \ldots, \\
&\sigma^{2k+2}_{i_1} = \inf\{t \ge \sigma^{2k+1}_{i_1} + \eta : h_{i_1}(x(t)) \le \epsilon_1\}, \quad k = 1, 2, \ldots.
\end{aligned}$$

Throughout this paper we set $\inf\emptyset = \infty$. From (4.11), for each $\omega \in A^1_{i_1} \cap A^2$, we have

$$\tau(\omega) = \infty \quad \text{and} \quad \sigma^k_i(\omega) < \infty, \quad k = 1, 2, \ldots. \tag{4.21}$$

By Hölder's inequality and Doob's martingale inequality (see [14]), from the CSDDEs (1.1), for any $T > 0$, we have

$$\begin{aligned}
&\mathbb{E}\Big[I_{A^{3,k}_{i_1}} \sup_{0\le t\le T} \big\| x_{i_1}\big(\tau \wedge (\sigma^{2k-1}_{i_1} + t)\big) - x_{i_1}\big(\tau \wedge \sigma^{2k-1}_{i_1}\big) \big\|^2 \Big] \\
&\quad \le 6(T+2)TK_\chi,
\end{aligned} \tag{4.22}$$

where $A^{3,k}_{i_1} = \{\tau \wedge \sigma^{2k-1}_{i_1} < \infty\}$, $K_\chi > 0$ is a constant only dependent on χ. Since $h_{i_1}(\cdot)$ is uniformly continuous in the closed ball $\bar{S}^\chi_{i_1} = \{x \in \mathbb{R}^{n_{i_1}} : \|x\| \le \chi\}$, we can therefore choose $\delta = \delta(\epsilon_3) > 0$ such that, for any pair $x_i, \bar{x}_i \in \bar{S}^\chi_{i_1}$ with $\|x_i - \bar{x}_i\| < \delta$, we have

$$\big\| h_{i_1}(x_i) - h_{i_1}(\bar{x}_i) \big\| < \epsilon_1. \tag{4.23}$$

We furthermore choose a positive constant T_1 ($< \eta$) sufficiently small such that

$$\frac{6(T_1+2)T_1K_\chi}{\delta^2} < \epsilon_1,$$

then from (4.21) and (4.22), we have

$$\mathbb{P}\big(A_{i_1}^1 \cap A^2 \cap A_{i_1}^{4,k}\big) \le \mathbb{P}\big(A_{i_1}^{3,k} \cap A_{i_1}^{4,k}\big) < \epsilon_1, \tag{4.24}$$

where

$$A_{i_1}^{4,k} = \Big\{\sup_{0\le t\le T_1} \big\| x_{i_1}\big(\tau \wedge \big(\sigma_{i_1}^{2k-1} + t\big)\big) - x_{i_1}\big(\tau \wedge \sigma_{i_1}^{2k-1}\big)\big\| \ge \delta\Big\}.$$

Recalling (4.20) and (4.24), we further compute

$$\mathbb{P}\big(A_{i_1}^1 \cap A^2 \cap A_{i_1}^{5,k}\big) \ge \mathbb{P}\big(A_{i_1}^1 \cap A^2 \cap \big(A_{i_1}^{4,k}\big)^c\big) \ge \epsilon_1, \tag{4.25}$$

where

$$A_{i_1}^{5,k} = \Big\{\sup_{0\le t\le T_1} \big\| h_{i_1}\big(x_{i_1}\big(\sigma_{i_1}^{2k-1} + t\big)\big) - h_{i_1}\big(x_{i_1}\big(\sigma_{i_1}^{2k-1}\big)\big)\big\| < \epsilon_1\Big\}.$$

On the other side, it follows from $\alpha_{i_1} \in \Psi(R_+, R_+)$ that for $T_1 > 0$ there exist two constants $\epsilon_2 = \epsilon_2(T_1) > 0$ and $t_3 = t_3(\epsilon_2) > 0$ such that

$$\int_t^{t+T_1} \alpha_{i_1}(s)\,ds \ge \epsilon_2, \quad \text{whenever } t \ge t_3.$$

By the definition of $\sigma_{i_1}^k$, there exists a positive integer N such that, for any $\omega \in A_{i_1}^1 \cap A^2$,

$$\sigma_{i_1}^{2k-1} \ge t_3, \quad \text{whenever } k \ge N+1.$$

Therefore, by (4.8), we compute

$$\begin{aligned}
\infty &> \mathbb{E}\int_0^\infty \alpha_{i_1}(s) h_{i_1}\big(x(s)\big)\,ds \\
&\ge \epsilon_1 \mathbb{E}\Bigg[\sum_{k=1}^\infty I_{A_{i_1}^1 \cap A^2} \int_{\sigma_{i_1}^{2k-1}}^{\sigma_{i_1}^{2k}} \alpha_{i_1}(s)\,ds\Bigg] \\
&\ge \epsilon_1 \mathbb{E}\Bigg[\sum_{k=N+1}^\infty I_{A_{i_1}^1 \cap A^2 \cap A_{i_1}^{5,k}} \int_{\sigma_{i_1}^{2k-1}}^{\sigma_{i_1}^{2k-1}+T_1} \alpha_{i_1}(s)\,ds\Bigg] \\
&\ge \epsilon_1\epsilon_2 \sum_{k=N+1}^\infty \epsilon_1 = \infty,
\end{aligned} \tag{4.26}$$

which is a contradiction. Thus (4.12) must hold, implying for each i and $\mathcal{Q}$,

$$\lim_{t\to\infty} h_i\big(x(t)\big) = 0, \qquad \lim_{t\to\infty} h^{\mathcal{Q}}\big(x(t)\big) = 0 \quad \text{a.s.} \tag{4.27}$$

Step 3. Observe from (4.27) that there exists an $A^0 \subset \Omega$ with $\mathbb{P}(A^0) = 1$ such that, for each $1 \le i \le l$ and each $\mathcal{Q} \in \mathbb{Q}$,

$$\begin{aligned}
&\lim_{t\to\infty} h_i\big(x(t,\omega)\big) = 0, \qquad \lim_{t\to\infty} h^{\mathcal{Q}}\big(x(t,\omega)\big) = 0 \quad \text{and} \\
&\sup_{0\le t<\infty} \big\|x(t,\omega)\big\| < \infty, \quad \forall \omega \in A^0.
\end{aligned} \tag{4.28}$$

Fix an $\omega \in A^0$, then $\{x(t,\omega)\}_{t\geq 0}$ is bounded in $\mathbb{R}^N$. So there must be an increasing sequence $\{t_j\}_{j\geq 1}$ such that $\{x(t_j,\omega)\}_{j\geq 1}$ converges to some $x^* \in \mathbb{R}^N$. Hence

$$h_i(x^*) = \lim_{j\to\infty} h_i\big(x(t_j,\omega)\big) = 0, \quad i = 1,2,\ldots,l,$$
$$h^{\mathcal{Q}}(x^*) = \lim_{t\to\infty} h^{\mathcal{Q}}\big(x(t_j,\omega)\big) = 0, \quad \mathcal{Q} \in \mathbb{Q},$$

which implies $x^* \in (\bigcap_{i=1}^{l} \operatorname{Ker}(h_i)) \cap (\bigcap_{\mathcal{Q}\in\mathbb{Q}} \operatorname{Ker}(h^{\mathcal{Q}}))$, so $S \neq \emptyset$. We shall now show that

$$\lim_{t\to\infty} d\big(x(t,\omega), S\big) = 0 \quad \text{for all } \omega \in A^0. \tag{4.29}$$

If this is false, then there is some $\bar{\omega} \in A^0$, such that

$$\lim_{t\to\infty} d\big(x(t,\bar{\omega}), S\big) = 0 \quad \text{for all } \omega \in A^0.$$

Then there exists either an i_2, $1 \leq i_2 \leq l$ such that

$$\limsup_{t\to\infty} d\big(x(t,\bar{\omega}), \operatorname{Ker}(h_{i_2})\big) > 0, \tag{4.30}$$

or a $\mathcal{Q}_2 \in \mathbb{Q}$ such that

$$\limsup_{t\to\infty} d\big(x(t,\bar{\omega}), \operatorname{Ker}\big(h^{\mathcal{Q}_2}\big)\big) > 0. \tag{4.31}$$

Without loss of generality we suppose inequality (4.30) holds. Then there is a constant $\epsilon_3 > 0$ and a sequence $\{x(t_j,\bar{\omega})\}_{j\geq 1}$ of $\{x(t,\bar{\omega})\}_{t\geq 0}$ such that

$$d\big(x(t_j,\bar{\omega}), \operatorname{Ker}(h_{i_2})\big) \geq \epsilon_3, \quad \forall j \geq 1.$$

Since $\{x(t_j,\bar{\omega})\}_{j\geq 1}$ is bounded, we can find a subsequence $\{x(t_{j_k},\bar{\omega})\}_{k\geq 1}$ which converges to z. Clearly, $z \notin \operatorname{Ker}(h_{i_2})$, so $h_{i_2}(z) > 0$. However, by (4.28),

$$h_{i_2}(z) = \lim_{k\to\infty} h_{i_2}\big(x(t_{j_k},\bar{\omega})\big) = 0,$$

which contradicts $h_{i_2}(z) > 0$. Hence (4.29) must hold and the required assertion (4.5) follows from $\mathbb{P}(A^0) = 1$. The proof is therefore complete. □

Theorem 4.1 implies the solutions to the CSFDEs (1.1) will asymptotically approach the set S almost surely. Then we deduce some useful corollaries.

Corollary 4.1 *Let Assumptions* 1-3 *hold. For all* $1 \leq i,j \leq l$, *there exists a radially unbounded function* $V_i(x_i,t) \in C^{2,1}(\mathbb{R}^{n_i} \times \mathbb{R}; \mathbb{R}_+)$, $\beta_i \in L(\mathbb{R}_+; \mathbb{R}_+)$, $\alpha_i \in \Psi(\mathbb{R}_+; \mathbb{R}_+)$, $\bar{\alpha}_{ij} \in C([-r_j(0),\infty), \mathbb{R}_+)$, $h_i \in C(\mathbb{R}^N; \mathbb{R}_+)$, $\bar{h}_{ij} \in C(\mathbb{R}^{n_j}; \mathbb{R}_+)$, $F_{ij} \in C(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}_+; \mathbb{R})$ *and constants* $a_{ij} \geq 0$ ($A = (a_{ij})$ *is irreducible*) *such that*

$$\mathcal{L}V_i(x,y,t) \leq \beta_i(t) - \alpha_i(t)h_i(x) + \sum_{j=1}^{l}(1-d_j)\bar{\alpha}_{ij}\big(t - r_j(t)\big)\bar{h}_{ij}(y_j) + \sum_{j=1}^{l} a_{ij}F_{ij}(x,y,t),$$

where

$$\alpha_i(t) \ge \bar{\alpha}_{ij}(t), \qquad h_i(x) \ge \bar{h}_i(x) := \sum_{j=1}^{l} \bar{h}_{ij}(x_j), \quad \forall i \in \mathbb{L}.$$

Assume along each directed cycle $\mathcal{C}_{\mathcal{Q}}$ of weighted digraph $(\mathcal{G}, A)$, there are functions $\beta^{\mathcal{Q}} \in L(\mathbb{R}_+;\mathbb{R}_+)$, $\alpha^{\mathcal{Q}} \in \Psi(\mathbb{R}_+;\mathbb{R}_+)$, $\bar{\alpha}_j^{\mathcal{Q}} \in C([-r_j(0),\infty),\mathbb{R}_+)$, $h^{\mathcal{Q}} \in C(\mathbb{R}^N;\mathbb{R}_+)$, $\bar{h}_j^{\mathcal{Q}} \in C(\mathbb{R}^{n_j};\mathbb{R}_+)$ such that

$$\sum_{(u,v)\in\mathcal{E}(\mathcal{C}_{\mathcal{Q}})} F_{vu}(x,y,t) \le \beta^{\mathcal{Q}}(t) - \alpha^{\mathcal{Q}}(t)h^{\mathcal{Q}}(x) + \sum_{j=1}^{l}(1-d_j)\bar{\alpha}_j^{\mathcal{Q}}\bigl(t-r_j(t)\bigr)\bar{h}_j^{\mathcal{Q}}(y_j),$$

where

$$\alpha^{\mathcal{Q}}(t) \ge \bar{\alpha}_j^{\mathcal{Q}}(t), \qquad h^{\mathcal{Q}}(x) \ge \bar{h}^{\mathcal{Q}}(x) := \sum_{j=1}^{l} \bar{h}_j^{\mathcal{Q}}(x_j), \quad \forall \mathcal{Q} \in \mathbb{Q}.$$

Then

$$\bar{S} := \left(\bigcap_{i=1}^{l} \operatorname{Ker}(h_i - \bar{h}_i)\right) \cap \left(\bigcap_{\mathcal{Q}\in\mathbb{Q}} \operatorname{Ker}\bigl(h^{\mathcal{Q}} - \bar{h}^{\mathcal{Q}}\bigr)\right) \neq \emptyset \tag{4.32}$$

and for any initial data ξ, the CSDDEs (1.1)-(1.2) *have a unique global solution on* $[0,\infty)$ *denoted by $x(t,\xi)$ satisfying*

$$\lim_{t\to\infty} d\bigl(x(t,\xi), S\bigr) = 0 \quad a.s.$$

Proof Define $w_i(x,t) = \alpha_i(t)h_i(x)$, $\bar{w}_{ij}(x_j,t) = \bar{\alpha}_{ij}(t)\bar{h}_{ij}(x_j)$, $w^{\mathcal{Q}}(x,t) = \alpha^{\mathcal{Q}}(t)h^{\mathcal{Q}}(x)$, and $\bar{w}_j^{\mathcal{Q}}(x_j, t) = \bar{\alpha}_j^{\mathcal{Q}}(t)\bar{h}_j^{\mathcal{Q}}(x_j)$, then we have

$$\begin{aligned}
&w_i(x,t) - \sum_{j=1}^{l} \bar{w}_{ij}(x_j,t) \\
&\quad = \alpha_i(t)\left(h_i(x) - \sum_{j=1}^{l}\bar{h}_{ij}(x_j)\right) + \sum_{j=1}^{l}\bigl(\alpha_i(t) - \bar{\alpha}_{ij}(t)\bigr)\bar{h}_{ij}(x_j) \\
&\quad \ge \alpha_i(t)\left(h_i(x) - \sum_{j=1}^{l}\bar{h}_{ij}(x_j)\right) = \alpha_i(t)\bigl(h_i(x) - \bar{h}_i(x)\bigr),
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
&w^{\mathcal{Q}}(x,t) - \sum_{j=1}^{l} \bar{w}_j^{\mathcal{Q}}(x_j,t) \\
&\quad = \alpha^{\mathcal{Q}}(t)\left(h^{\mathcal{Q}}(x) - \sum_{j=1}^{l}\bar{h}_j^{\mathcal{Q}}(x_j)\right) + \sum_{j=1}^{l}\bigl(\alpha^{\mathcal{Q}}(t) - \bar{\alpha}_j^{\mathcal{Q}}(t)\bigr)\bar{h}_j^{\mathcal{Q}}(x_j) \\
&\quad \ge \alpha^{\mathcal{Q}}(t)\left(h^{\mathcal{Q}}(x) - \sum_{j=1}^{l}\bar{h}_j^{\mathcal{Q}}(x_j)\right) = \alpha^{\mathcal{Q}}(t)\bigl(h^{\mathcal{Q}}(x) - \bar{h}^{\mathcal{Q}}(x)\bigr).
\end{aligned} \tag{4.34}$$

Hence, the result is obtained by Theorem 4.1 directly. □

Notice that if $h_i(x) \equiv \bar{h}_i(x)$ and $h^{\mathcal{Q}}(x) \equiv \bar{h}^{\mathcal{Q}}(x)$, the result of above corollary is useless. Therefore, the stronger conditions are needed to locate the limit set of the CSDDEs (1.1).

Corollary 4.2 *Let all the assumptions of Corollary* 4.1 *hold, moreover, assume there are subsets* $\mathbb{I}, \mathbb{J} \subseteq \mathbb{L}$ *and* $\mathbb{P} \subseteq \mathbb{Q}$ *such that, for each* $i \in \mathbb{I}$, $j \in \mathbb{J}$, $\mathcal{Q} \in \mathbb{P}$, $(\alpha_i(t) - \bar{\alpha}_{ij}(t)), (\alpha^{\mathcal{Q}}(t) - \bar{\alpha}_j^{\mathcal{Q}}(t)) \in \Psi(\mathbb{R}_+; \mathbb{R}_+)$. *Then*

$$\tilde{S} := \bar{S} \cap \left(\bigcap_{i\in\mathbb{I}, j\in\mathbb{J}} \operatorname{Ker}(\bar{h}_{ij}) \right) \cap \left(\bigcap_{j\in\mathbb{J}, \mathcal{Q}\in\mathbb{P}} \operatorname{Ker}\bigl(\bar{h}_j^{\mathcal{Q}}\bigr) \right) \neq \emptyset, \tag{4.35}$$

where $\bar{S}$ *is defined as equality* (4.32) *and for any initial data* ξ, *the CSDDEs* (1.1)-(1.2) *have a unique global solution on* $[0, \infty)$ *denoted by* $x(t, \xi)$ *satisfying*

$$\lim_{t\to\infty} d\bigl(x(t,\xi), \tilde{S}\bigr) = 0 \quad a.s.$$

Proof Obviously, the conditions of Corollary 4.1 are satisfied. So the conclusions of Corollary 4.1 still hold. By inequalities (4.33) and (4.34), we get

$$w_i(x,t) - \sum_{j=1}^{l} \bar{w}_{ij}(x_j, t) \geq \sum_{j=1}^{l} \bigl(\alpha_i(t) - \bar{\alpha}_{ij}(t)\bigr) \bar{h}_{ij}(x_j),$$

$$w^{\mathcal{Q}}(x,t) - \sum_{j=1}^{l} \bar{w}_j^{\mathcal{Q}}(x_j, t) \geq \sum_{j=1}^{l} \bigl(\alpha^{\mathcal{Q}}(t) - \bar{\alpha}_j^{\mathcal{Q}}(t)\bigr) \bar{h}_j^{\mathcal{Q}}(x_j).$$

By Theorem 4.1, the required assertions are obtained. □

From Theorem 4.1, S is the limit set of solutions to the CSDDEs (1.1). Therefore, the asymptotic properties of the solutions, such as the asymptotic boundedness, can be obtained more precisely from the information of S.

Corollary 4.3 *Let all the assumptions of Theorem* 4.1 *hold. If* S *is bounded, then with the initial data* ξ, *the solution has the property*

$$\lim_{t\to\infty} \bigl\| x(t,\xi) \bigr\| \leq C \quad a.s.,$$

where $C = \sup_{x\in S} \|x\|$.

If S only contains the origin of $\mathbb{R}^N$, we can obtain the following result on the globally asymptotic stability with probability 1.

Corollary 4.4 *Let all the assumptions of Theorem* 4.1 *hold. If* $h_i(x) = 0$, $h^{\mathcal{Q}}(x) = 0$, $i \in \mathbb{L}$, $\mathcal{Q} \in \mathbb{Q}$ *iff* $x = 0$, *then the solution with the initial data* ξ *has the property*

$$\lim_{t\to\infty} x(t,\xi) = 0 \quad a.s.$$

Furthermore, if the rate of $V_i(x_i)$ tending to infinity is known, the rate of the solutions converging to $\operatorname{Ker}(h)$ is obtained. Exponential stability is a simple application of Theorem 4.1, which is arranged in the following corollary.

Corollary 4.5 *Let all the assumptions of Theorem* 4.1 *hold. Assume for each $i \in \mathbb{L}$, there are pairs of positive constants λ_i and p_i, such that*

$$V_i(x_i, t) \geq e^{\lambda_i t} \|x_i\|^{p_i}. \tag{4.36}$$

Then the solution with the initial data ξ has the property

$$\lim_{t\to\infty} \frac{1}{t} \ln\|x(t,\xi)\| \leq -\frac{\hat{\lambda}}{\hat{p}} \quad a.s.$$

Proof From (4.7) in the proof of Theorem 4.1, for almost all $\omega \in \Omega$, the solution $x(t,\xi)$ satisfies

$$\limsup_{t\to\infty} \sum_{i=1}^{l} c_i V_i\bigl(x_i(t,\xi), t\bigr) < \infty \quad \text{a.s.}$$

Equation (4.36) implies

$$Ke^{\hat{\lambda} t}\|x(t,\xi)\|^{\hat{p}} \leq \sum_{i=1}^{l} c_i e^{\lambda_i t}\|x_i(t,\xi)\|^{p_i} \leq \sum_{i=1}^{l} c_i V_i\bigl(x_i(t,\xi), t\bigr) \quad \text{a.s.},$$

where K is a positive constant. Then the required assertion follows directly. □

Example 4.1 (Coupled stochastic oscillators with unbounded delays) Given a network represented by digraph $\mathcal{Q}$ with l vertices, a couple system can be built by assigning the ith vertex ($i = 1, 2, \ldots, l$) its own oscillator satisfying the Itô equation

$$\ddot{z}_i(t) + p_i(t)\dot{z}_i(t) + f_i\bigl(z_i(t)\bigr) = -q_i(t)\dot{z}_i(t - \theta_i t)\dot{B}_i(t), \tag{4.37}$$

and then coupling these oscillators based on directed arcs in the digraph. For each i, define $x_i = (x_i^{(1)}, x_i^{(2)})^T = (z_i, \dot{z}_i)^T$, then we obtain the following self-excited coupled system (SCS) on $\mathcal{Q}$:

$$\begin{aligned} dx_i(t) = {}& \begin{pmatrix} x_i^{(2)}(t) \\ -p_i(t)x_i^{(2)}(t) - f_i(x_i^{(1)}(t)) - \sum_{j=1}^{l} a_{ij}(x_i^{(2)}(t) - x_j^{(2)}(t)) \end{pmatrix} dt \\ & + \begin{pmatrix} 0 \\ -q_i(t)x_i^{(2)}(t - \theta_i t) \end{pmatrix} dB_i(t), \quad 1 \leq i \leq l, \end{aligned} \tag{4.38}$$

where, for each $1 \leq i,j \leq l$, $\theta_i \in (0,1)$, $p_i(\cdot), q_i(\cdot), f_i(\cdot) \in C(\mathbb{R};\mathbb{R})$, $x_i^{(1)} f_i(x_i^{(1)}) > 0$ for $x_i^{(1)} \neq 0$, $\int_0^\infty f_i(u)\,du = \infty$, $a_{ij} \geq 0$, and $A = (a_{ij})$ is irreducible. Then the total energy function $V_i(x_i, t) = \frac{(x_i^{(2)})^2}{2} + \int_0^{x_i^{(1)}} f_i(u)\,du$ is radially unbounded. Compute

$$\mathcal{L}V_i(x,y,t) =: -w_i(x,t) + \sum_{j=1}^{l}(1-\theta_j)\bar{w}_{ij}(y_j,t) + \sum_{j=1}^{l} a_{ij}F_{ij}(x,y,t), \tag{4.39}$$

where $w_i(x,t) = p_i(t)(x_i^{(2)})^2$,

$$\bar{w}_{ij}(x_j,t) = \begin{cases} \frac{1}{2(1-\theta_j)}(q_j(t)x_j^{(2)})^2, & j = i, \\ 0, & j \neq i, \end{cases}$$

$$F_{ij}(x,y,t) = \frac{1}{2}\left(x_j^{(2)}\right)^2 - \frac{1}{2}\left(x_i^{(2)}\right)^2 - \frac{1}{2}\left(x_i^{(2)} - x_j^{(2)}\right)^2.$$

Compute

$$w_i(x,t) - \sum_{j=1}^{l}(1-\theta_j)\bar{w}_{i,j}(x_j,t) = \left[p_i(t) - \frac{1}{2(1-\theta_i)}q_i^2(t)\right]\left(x_i^{(2)}\right)^2.$$

Assume along each directed cycle $\mathcal{C}_\mathcal{Q}$ of the weighted digraph $(\mathcal{G},A)$,

$$\sum_{(u,v)\in\mathcal{E}(\mathcal{C}_\mathcal{Q})} F_{vu}(x,y,t) \leq -\frac{1}{2}\sum_{(u,v)\in\mathcal{E}(\mathcal{C}_\mathcal{Q})}\left(x_u^{(2)} - x_v^{(2)}\right)^2.$$

Suppose that there exists a $1 \leq k \leq l$ such that $p_k(t) - \frac{1}{2(1-\theta_k)}q_k^2(t) \in \Psi(\mathbb{R}_+;\mathbb{R}_+)$, by Theorem 4.1, the point $(x_1,x_2) \in S$ satisfies $x_k^{(2)} = 0$. By the strong connectivity of $(\mathcal{Q},A)$, each vertex i belongs to a connective circle $\mathcal{C}_\mathcal{Q}$ including vertex k. Using Theorem 4.1 again, we know $x_i^{(2)} = 0$, $i = 1,2,\ldots,l$. Denote by $(x_1(t),x_2(t))$ the global solution of SCS (4.38) with an initial value, then

$$\lim_{t\to\infty} x_i^{(2)}(t) = 0 \quad \text{a.s. } i = 1,2,\ldots,l.$$

Furthermore, by the second equation of (4.38), the stochastic oscillator has the property

$$\lim_{t\to\infty} x_i^{(1)}(t) = 0 \quad \text{a.s. } i = 1,2,\ldots,l.$$

For example, see Figure 1 for a computer simulation of the paths of three independent oscillators described by (4.37), where $f_1(x) = x|x|$, $\theta_1 = 0.5$, $p_1 = 1 + 0.3\cos t$, $q_1 = 0.5\sin t$; $f_2(x) = x$, $\theta_2 = 0.6$, $p_2 = 0.05 + 1.25e^{-t}$, $q_2 = 0.2 - e^{-\frac{1}{2}t}$; $f_3(x) = x$, $\theta_3 = 0.25$, $p_3 = 1 + e^{-t}$, $q_3 = 0.2 + e^{-t}$.

Couple these oscillators based on directed arcs in the digraph described by Figure 2, simulate the paths of the coupled oscillators and obtain Figure 3, which supports the main results clearly.

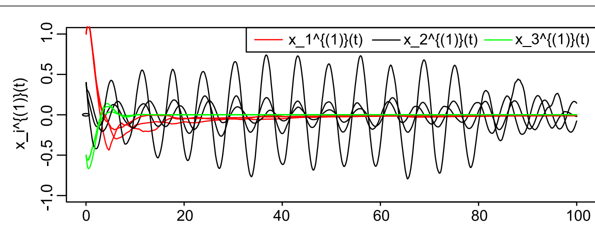
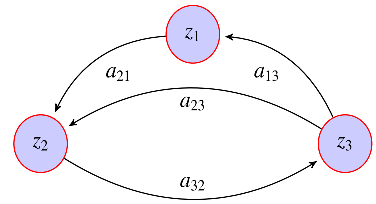


Figure 1 Computer simulation of the paths of $x_1^{(1)}(t)$, $x_2^{(1)}(t)$ and $x_3^{(1)}(t)$ for the three discrete-time independent oscillators described by (4.37) using the Euler-Maruyama method with step size 0.02 and initial values $x_1^{(1)}(0) = 1$, $x_1^{(2)}(0) = 0.6$, $x_2^{(1)}(0) = 0.4$, $x_2^{(2)}(0) = 0.7$, $x_3^{(1)} = -0.5$, $x_3^{(2)}(0) = -0.5$.

Figure 2 A strongly connected digraph with three vertices where $a_{13} = 3$, $a_{21} = 2$, $a_{23} = 4$, $a_{32} = 1$, $a_{11} = a_{12} = a_{22} = a_{31} = a_{33} = 0$.



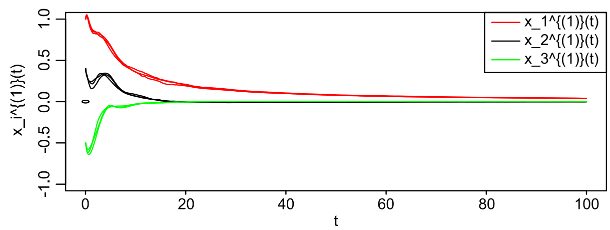


Figure 3 Computer simulation of the paths of $x_1^{(1)}(t)$, $x_2^{(1)}(t)$, and $x_3^{(1)}(t)$ for the discrete-time coupled oscillators described by (4.38) using the Euler-Maruyama method with the same step size, parameters and initial values as those of Figure 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors indicated in parentheses made substantial contributions to the following tasks of research: drafting the manuscript (RW); participating in the design of the study (WL, XL); writing and revising of the paper (XL). All authors read and approved the final manuscript.

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