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Existence of positive solutions for a system of Caputo fractional difference equations depending on parameters

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Abstract

We consider the existence of at least two positive solutions for a system of Caputo fractional difference equations $\Delta_C^{\nu_j} y_j(t) = -\lambda_j f_j(y_1(t + \nu_1 - 1), \ldots, y_n(t + \nu_n - 1))$, subject to boundary conditions $y_j(\nu_j - 3) = \Delta y_j(\nu_j + b) = \Delta^2 y_j(\nu_j - 3) = 0$, where $2 < \nu_j \leq 3$, $j = 1, \ldots, n$. We use the Krasnosel'skiĭ fixed point theorem to obtain the sufficient conditions of the existence of two positive solutions for this boundary value problem of Caputo fractional difference equations depending on parameters.

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Keywords: Caputo fractional difference; boundary value problem; fixed point theory; positive solution

1 Introduction

In this paper we consider a system of Caputo fractional difference boundary value problem (FBVP) of the form:

$$\Delta_C^{\nu_j} y_j(t) = -\lambda_j f_j\big(y_1(t + \nu_1 - 1), \ldots, y_n(t + \nu_n - 1)\big), \tag{1.1}$$

$$y_j(\nu_j - 3) = \Delta y_j(\nu_j + b) = \Delta^2 y_j(\nu_j - 3) = 0, \tag{1.2}$$

where $t \in [0, b+1]_{\mathbb{N}_0} := \{0, 1, \ldots, b+1\}$, $b > 3$, $\lambda_j > 0$, $2 < \nu_j \leq 3$, $f_j : [0, +\infty) \times \cdots \times [0, +\infty) \to [0, +\infty)$ are continuous functions for each j ($j = 1, 2, \ldots, n$). $\Delta_C^{\nu} y(t)$ is the standard Caputo difference.

Fractional difference equations have been of great interest recently. It is caused by intensive development of the theory of discrete fractional calculus itself, see [1–19] and the references therein. Abdeljawad [1] defined left and right Caputo fractional sums and differences, studied some of their properties. Holm [2] introduced the fractional sum and difference operators. He developed and presented a complete and precise theory for composing fractional sums and differences. Atici and Sengül [3] provided some analysis of discrete fractional variational problems, their paper also provided some initial attempts at using the discrete fractional calculus to model biological processes. Abdeljawad and Baleanu [4] defined the right fractional sum and difference operators and obtained many of their properties. Then by using those properties they obtained a by-part formula analogous to that in the usual fractional calculus. In [5] the authors studied the stability of

discrete nonautonomous systems within the frame of the Caputo fractional difference by using the Lyapunov direct method. They discussed the conditions for uniform stability, uniform asymptotic stability, and uniform global stability. Mohammadi and Rezapour [6] discussed the existence and uniqueness of solutions for some nonlinear fractional differential equations via some boundary value problems by using fixed point results on ordered complete gauge spaces. Recently, Wu and Baleanu introduced some applications of the Caputo fractional difference to discrete chaotic maps in [7, 8].

In particular, the authors [9–19] developed some of the basic theory of fractional difference both IVPs and BVPs with delta derivative on the time scale $\mathbb{Z}$. In [10], we obtained some results on the existence of one or more positive solutions for the Caputo fractional boundary value problems by means of cone theoretic fixed point theorems. Thus, the fractional difference equation has recently attracted increasing attention from a growing number of researchers. However, systems of discrete fractional boundary value problems are limited (see [14–19]). Among them, Atici and Eloe [14] studied a linear system of fractional nabla difference equation with constant coefficients of the form

$$\nabla_0^\nu y(t) = Ay(t) + f(t), \quad t = 1, 2, \dots,$$

where $0 < \nu < 1$, A is an $n \times n$ matrix with constant entries, and f are n-vector valued functions. The operator ∇_0^ν is a Riemann-Liouville fractional difference. They constructed the fundamental matrix for the homogeneous system and the causal Green's function for the nonhomogeneous system.

In [15], the authors investigated the existence of solutions for a k-dimensional system of fractional finite difference equations:

$$\Delta^{\nu_1} y_1(t) + f_1\big(y_1(t + \nu_1 - 1), y_2(t + \nu_2 - 1), \dots, y_k(t + \nu_k - 1)\big) = 0,$$

$$\Delta^{\nu_2} y_2(t) + f_2\big(y_1(t + \nu_1 - 1), y_2(t + \nu_2 - 1), \dots, y_k(t + \nu_k - 1)\big) = 0,$$

$$\dots,$$

$$\Delta^{\nu_k} y_k(t) + f_k\big(y_1(t + \nu_1 - 1), y_2(t + \nu_2 - 1), \dots, y_k(t + \nu_k - 1)\big) = 0,$$

$$y_1(\nu_1 - 2) = \Delta y_1(\nu_1 + b) = 0,$$

$$y_2(\nu_2 - 2) = \Delta y_2(\nu_2 + b) = 0,$$

$$\dots,$$

$$y_k(\nu_k - 2) = \Delta y_k(\nu_k + b) = 0,$$

where $b \in \mathbb{N}_0$, $1 < \nu_i \leq 2$, $f_i : \mathbb{R}^k \to \mathbb{R}$ are continuous functions for $k = 1, 2, \dots$. They investigated the existence of solutions for this k-dimensional system of fractional finite difference equations by using the Krasnosel'skiĭ fixed point theorem.

In [16], Goodrich studied the following pair of discrete fractional boundary value problems:

$$-\Delta^{\nu_1} y_1(t) = \lambda_1 a_1(t + \nu_1 - 1) f_1\big(y_1(t + \nu_1 - 1), y_2(t + \nu_n - 1)\big),$$

$$-\Delta^{\nu_2} y_2(t) = \lambda_2 a_2(t + \nu_1 - 1) f_2\big(y_1(t + \nu_1 - 1), y_2(t + \nu_n - 1)\big),$$

$$y_1(\nu_1 - 2) = \psi_1(y_1), \qquad y_2(\nu_2 - 2) = \psi_2(y_2),$$

$$y_1(\nu_1 + b) = \phi_1(y_1), \qquad y_2(\nu_2 + b) = \phi_2(y_2),$$

where $t \in [0, b]_{\mathbb{N}_0} := \{0, 1, \ldots, b\}$, $\lambda_1, \lambda_2 > 0$, $\nu_1, \nu_2 \in (1, 2]$. Goodrich obtained the existence of at least one positive solution to this problem by means of the Krasnosel'skiĭ theorem for cons.

In [18], the authors considered the existence of at least one positive solution to the discrete fractional system:

$$\begin{cases} -\Delta^{\nu_1} y_1(t) = \lambda_1 f_1(t + \nu_1 - 1, y_1(t + \nu_1 - 1), y_2(t + \nu_2 - 1)), & t \in [1, b+1], \\ -\Delta^{\nu_2} y_2(t) = \lambda_2 f_2(t + \nu_2 - 1, y_1(t + \nu_1 - 1), y_2(t + \nu_2 - 1)), & t \in [1, b+1], \\ y_1(\nu_1 - 2) = y_1(\nu_1 + b + 1) = 0, \\ y_2(\nu_2 - 2) = y_2(\nu_2 + b + 1) = 0, \end{cases}$$

where $\nu_1, \nu_2 \in (1, 2]$.

Following this trend, in [19], we discussed the boundary value problems of fractional difference system of the form

$$-\Delta^{\nu_j} y_j(t) = \lambda_j f_j\big(y_1(t + \nu_1 - 1), \ldots, y_n(t + \nu_n - 1)\big),$$

$$y_j(\nu_j - 2) = \psi_j(y_j), \qquad y_j(\nu_j + b) = \phi_j(y_j),$$

where $t \in [0, b]_{\mathbb{N}_0} := \{0, 1, \ldots, b\}$, $\lambda_j > 0$, $1 < \nu_j \leq 2$, $f_j : [0, +\infty) \times \cdots \times [0, +\infty) \to [0, +\infty)$ are continuous functions. For each j we have that $\psi_j, \phi_j : \mathbb{R}^{b+3} \to \mathbb{R}$ ($j = 1, 2, \ldots, n$) are given functions. We obtained the sufficient conditions for the existence of two positive solutions to the boundary value problem of a fractional difference system. In this paper we open our studies in this field. We establish some conditions on parameters λ_i which are able to guarantee that FBVP (1.1)-(1.2) has at least two positive solutions and one positive solution, respectively, based on the Krasnosel'skiĭ theorem.

This paper is organized as follows. In Section 2, we provide basic definitions and demonstrate some lemmas in order to prove our main results. In Section 3, we establish some results for the existence of at least two positive solutions to FBVP (1.1)-(1.2), and we conclude with an example explicating our main result.

2 Preliminaries

In this section, we present some basic definitions in the discrete fractional calculus and establish some lemmas.

Definition 2.1 [1] We define

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$$

for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t + 1 - \nu$ is a pole of the gamma function and $t + 1$ is not a pole, then $t^{(\nu)} = 0$.

Definition 2.2 [1] The νth fractional sum of a function f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - s - 1)^{\underline{\nu-1}} f(s)$$

for $\nu > 0$ and $t \in \{a + \nu, a + \nu + 1, \ldots\} = \mathbb{N}_{a+\nu}$. We also define the νth Caputo fractional difference for $\nu > 0$ by

$$\Delta_C^\nu f(t) = \Delta^{-(n-\nu)} \Delta^n f(t) = \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-s-1)^{\underline{n-\nu-1}} \Delta^n f(s),$$

where $n - 1 < \nu \leq n$.

Lemma 2.3 [1, 2] *Assume that $\nu > 0$ and f is defined on domains $\mathbb{N}_a$, then*

$$\Delta_{a+(n-\nu)}^{-\nu} \Delta_C^\nu f(t) = f(t) - \sum_{k=0}^{n-1} c_k (t-a)^{\underline{k}},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$; $n - 1 < \nu \leq n$.

Lemma 2.4 [2] *Let $f : \mathbb{N}_{a+\nu} \times \mathbb{N}_a \to \mathbb{R}$ be given. Then*

$$\Delta \left(\sum_{s=a}^{t-\nu} f(t,s) \right) = \sum_{s=a}^{t-\nu} \Delta_t f(t,s) + f(t+1, t+1-\nu) \quad \text{for } t \in \mathbb{N}_{a+\nu}.$$

In order to get our main results, we now state an important lemma. This lemma gives a representation for the solution of (1.1)-(1.2), provided that the solution exists.

Lemma 2.5 [10] *Let $2 < \nu \leq 3$ and $g : [\nu - 2, \nu - 1, \ldots, \nu + b]_{\mathbb{N}_{\nu-2}} \to \mathbb{R}$ be given. Then the solution of the FBVP*

$$\Delta_C^\nu y(t) = -g(t + \nu - 1), \tag{2.1}$$

$$y(\nu - 3) = \Delta y(\nu + b) = \Delta^2 y(\nu - 3) = 0 \tag{2.2}$$

is given by

$$y(t) = \sum_{s=0}^{b+1} G(t,s) g(s + \nu - 1),$$

where the Green's function $G : [\nu - 2, \nu - 1, \ldots, \nu + b]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} (\nu-1)(t-\nu+3)(\nu+b-s-1)^{\underline{\nu-2}} \\ \quad - (t-s-1)^{\underline{\nu-1}}, & 0 \leq s < t - \nu + 1 \leq b + 1, \\ (\nu-1)(t-\nu+3)(\nu+b-s-1)^{\underline{\nu-2}}, & 0 \leq t - \nu + 1 \leq s \leq b + 1. \end{cases}$$

Remark Notice that $G(\nu - 3, s) = 0$, $G(t, b+2) = 0$. G could be extended to $[\nu - 3, \nu + b]_{\mathbb{N}_{\nu-3}} \times [0, b+2]_{\mathbb{N}_0}$, so we only discuss on $(t,s) \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0}$.

Lemma 2.6 [10] *The Green's function G satisfies the following conditions*:
  (i) $G(t,s) > 0$, $(t,s) \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0}$.
  (ii) $\max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = G(\nu + b, s)$, $s \in [0, b+1]_{\mathbb{N}_0}$.
  (iii) $\min_{\frac{\nu+b}{4} \leq t \leq \frac{3(\nu+b)}{4}} G(t,s) \geq \frac{1}{4} \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = \frac{1}{4} G(\nu + b, s)$, $s \in [0, b+1]_{\mathbb{N}_0}$.

The proofs of Lemma 2.5 and Lemma 2.6 can be found in [10], so we omit their proofs. Let

$$\mathcal{B}_j := \left\{ y_j : [\nu_j - 3, \nu_j + b]_{\mathbb{N}_{\nu_j - 3}} \to \mathbb{R}, y_j(\nu_j - 3) = \Delta y_j(\nu_j + b) = \Delta^2 y_j(\nu_j - 3) = 0 \right\}$$

be equipped with the usual maximum norm $\| \cdot \|$, it is easy to verify that $\mathcal{B}_j$ is the Banach space. Then we put $\mathcal{K} := \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \times \mathcal{B}_n$. By equipping $\mathcal{K}$ with the norm

$$\left\| (y_1, \ldots, y_n) \right\| = \|y_1\| + \cdots + \|y_n\|,$$

it follows that $(\mathcal{K}, \| \cdot \|)$ is a Banach space.

Now consider the operator $T : \mathcal{K} \to \mathcal{K}$ defined by

$$T(y_1, \ldots, y_n)(t_1, \ldots, t_n) = \left(T_1(y_1, \ldots, y_n)(t_1), \ldots, T_n(y_1, \ldots, y_n)(t_n) \right), \tag{2.3}$$

where we define $T_j : \mathcal{K} \to \mathcal{B}_j$ by

$$T_j(y_1, \ldots, y_n)(t_j) = \lambda_j \sum_{s=0}^{b+1} G_j(t_j, s) f_j \big(y_1(s + \nu_1 - 1), \ldots, y_n(s + \nu_n - 1) \big). \tag{2.4}$$

Let $I := [\frac{\nu_1 + b}{4}, \frac{3(\nu_1 + b)}{4}] \times \cdots \times [\frac{\nu_n + b}{4}, \frac{3(\nu_n + b)}{4}]$. In the sequel, we shall also make use of the cone

$$\Lambda := \left\{ (y_1, \ldots, y_n) \in \mathcal{K} : y_1, \ldots, y_n \geq 0, \right.$$

$$\left. \min_{(t_1, \ldots, t_n) \in I} \left[y_1(t_1) + \cdots + y_n(t_n) \right] \geq \frac{1}{4} \left\| (y_1, \ldots, y_n) \right\| \right\}.$$

Lemma 2.7 *Let T be the operator defined as in (2.3). Then $T : \Lambda \to \Lambda$.*

Proof We show first that for $(y_1, \ldots, y_n) \in \mathcal{K}$, by the definitions T_j ($j = 1, 2, \ldots, n$), it is clear that

$$T_j(y_1, \ldots, y_n)(t_j) \geq 0, \quad j = 1, 2, \ldots, n.$$

On the other hand, we show that

$$\min_{(t_1, \ldots, t_n) \in I} \left[T_1(y_1, \ldots, y_n)(t_1) + \cdots + T_n(y_1, \ldots, y_n)(t_n) \right] \geq \frac{1}{4} \left\| T(y_1, \ldots, y_n) \right\|$$

for $(y_1, \ldots, y_n) \in \mathcal{K}$. In fact, by Lemma 2.6(iii), we have

$$\min_{t_j \in [\frac{\nu_j + b}{4}, \frac{3(\nu_j + b)}{4}]} T_j(y_1, \ldots, y_n)(t_j)$$

$$\geq \min_{t_j \in [\frac{b + \nu_j}{4}, \frac{3(\nu_j + b)}{4}]} \lambda_j \sum_{s=0}^{b+1} G_j(t_j, s) f_j \big(y_1(s + \nu_1 - 1), \ldots, y_n(s + \nu_n - 1) \big)$$

$$\geq \lambda_j \sum_{s=0}^{b+1} \frac{1}{4} G_j(\nu_j + b, s) f_j\big(y_1(s + \nu_1 - 1), \ldots, y_n(s + \nu_n - 1)\big)$$

$$\geq \frac{1}{4} \max_{t_j \in [\nu_j - 2, \nu_j + b]_{\mathbb{N}_{\nu_j - 2}}} \lambda_j \sum_{s=0}^{b+1} G_j(t_j, s) f_j\big(y_1(s + \nu_1 - 1), \ldots, y(s + \nu_n - 1)\big)$$

$$= \frac{1}{4} \big\| T_j(y_1, \ldots, y_n) \big\|$$

for $(y_1, \ldots, y_n) \in \mathcal{K}$ and $j = 1, 2, \ldots, n$. Then we obtain

$$\min_{(t_1, \ldots, t_n) \in I} \big[T_1(y_1, \ldots, y_n)(t_1) + \cdots + T_n(y_1, \ldots, y_n)(t_n) \big]$$

$$\geq \min_{(t_1, \ldots, t_n) \in I} T_1(y_1, \ldots, y_n)(t_1) + \cdots + \min_{(t_1, \ldots, t_n) \in I} T_n(y_1, \ldots, y_n)(t_n)$$

$$\geq \frac{1}{4} \big\| T_1(y_1, \ldots, y_n) \big\| + \cdots + \frac{1}{4} \big\| T_n(y_1, \ldots, y_n) \big\|$$

$$\geq \frac{1}{4} \big\{ \big\| T_1(y_1, \ldots, y_n) \big\| + \cdots + \big\| T_n(y_1, \ldots, y_n) \big\| \big\}$$

$$= \frac{1}{4} \big\| T(y_1, \ldots, y_n) \big\|$$

for $(y_1, \ldots, y_n) \in \mathcal{K}$. So, we conclude that $T : \Lambda \to \Lambda$. This completes the proof. $\square$

Theorem 2.8 *Let* $f_j : [0, +\infty) \times \cdots \times [0, +\infty) \to [0, +\infty)$ *be given for* $j = 1, \ldots, n$. *If* $(y_1, \ldots, y_n) \in \mathcal{K}$ *is a fixed point of* T, *then* $(y_1, \ldots, y_n) \in \mathcal{K}$ *is a solution of FBVP* (1.1)-(1.2).

Proof Suppose that the operator T has a fixed point, say $(y_1, \ldots, y_n) \in \mathcal{K}$. Let $(t_1, \ldots, t_n) \in \mathbb{N}_{\nu_1 - 2} \times \cdots \times \mathbb{N}_{\nu_n - 2}$, then we have

$$y_j(t_j) = T_j(y_1, \ldots, y_n)(t_j), \quad j = 1, 2, \ldots, n,$$

where T_j is defined as in (2.4). It is easy to check that

$$T_j(y_1, \ldots, y_n)(\nu_j - 3) = 0$$

and

$$\Delta T_j(y_1, \ldots, y_n)(\nu_j + b)$$

$$= T_j(y_1, \ldots, y_n)(\nu_j + b + 1) - T_j(y_1, \ldots, y_n)(\nu_j + b)$$

$$= \lambda_j \sum_{s=0}^{b+1} G_j(\nu_j + b + 1, s) f_j\big(y_1(s + \nu_1 - 1), \ldots, y_n(s + \nu_n - 1)\big)$$

$$\quad - \lambda_j \sum_{s=0}^{b+1} G_j(\nu_j + b, s) f_j\big(y_1(s + \nu_1 - 1), \ldots, y_n(s + \nu_n - 1)\big)$$

$$= \lambda_j \sum_{s=0}^{b+1} \big[G_j(\nu_j + b + 1, s) - G_j(\nu_j + b, s) \big] f_j\big(y_1(s + \nu_1 - 1), \ldots, y_n(s + \nu_n - 1)\big)$$

$$= \frac{\lambda_j}{\Gamma(\nu)} \sum_{s=0}^{b+1} \big[(\nu_j - 1)(\nu_j + b + 1 - \nu_j + 3)(\nu_j + b - s - 1)^{\underline{\nu_j - 2}} - (\nu_j + b + 1 - s - 1)^{\underline{\nu_j - 1}}$$

$$- (v_j - 1)(v_j + b - v_j + 3)(v_j + b - s - 1)^{\underline{v_j - 2}} + (v_j + b - s - 1)^{\underline{v_j - 1}}\big]$$

$$\times f_j\big(y_1(s + v_1 - 1), \ldots, y_n(s + v_n - 1)\big)$$

$$= \frac{\lambda_j}{\Gamma(v)} \sum_{s=0}^{b+1} \bigg\{ (v_j - 1)(v_j + b - s - 1)^{\underline{v_j - 2}}$$

$$- \bigg[\frac{(v_j + b - s)\Gamma(v_j + b - s)}{(b - s + 1)\Gamma(b - s + 1)} - \frac{\Gamma(v_j + b - s)}{\Gamma(b - s + 1)} \bigg] \bigg\}$$

$$\times f_j\big(y_1(s + v_1 - 1), \ldots, y_n(s + v_n - 1)\big)$$

$$= \frac{\lambda_j}{\Gamma(v)} \sum_{s=0}^{b+1} \bigg[(v_j - 1)(v_j + b - s - 1)^{\underline{v_j - 2}} - \frac{(v_j - 1)\Gamma(v_j + b - s)}{(b - s + 1)\Gamma(b - s + 1)} \bigg]$$

$$\times f_j\big(y_1(s + v_1 - 1), \ldots, y_n(s + v_n - 1)\big)$$

$$= 0.$$

Finally, when $0 \le t_j - v_j + 1 \le s \le b + 1$,

$$G_j(t_j, s) = \frac{1}{\Gamma(v_j)} (v_j - 1)(t_j - v_j + 3)(v_j + b - s - 1)^{\underline{v_j - 2}},$$

then

$$\Delta_{t_j}^2 G_j(t_j, s) = 0.$$

Therefore, we can get

$$\Delta^2 T_j(y_1, \ldots, y_n)(v_j - 3) = 0.$$

So the boundary conditions are satisfied, which completes the proof. □

Finally, to accomplish proof of our main results, we state cones theory. In particular, we require the following well-known fixed point theorem for cons in [20].

Theorem 2.9 [20] *Let $\mathcal{B}$ a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume further that $T : \mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{K}$ is a completely continuous operator. If either*

(i) $\|Ty\| \le \|y\|$ *for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \ge \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$, or*

(ii) $\|Ty\| \ge \|y\|$ *for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \le \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$,*

then the operator T has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main results

In this section, we state and prove the existence of at least two positive solutions regarding FBVP (1.1)-(1.2). Then we conclude this section with examples to illustrate our main results. First, denote

$$\alpha_j = \sum_{s=0}^{b+1} G_j(v_j + b, s), \qquad \beta_j = \sum_{s=[\frac{v_j+b}{4} - v_j + 1]}^{[\frac{3(v_j+b)}{4} - v_j + 1]} G_j\bigg(\bigg[\frac{b - v_j}{2} \bigg] + v_j, s \bigg),$$

$$\Omega_\xi = \left\{ (y_1,\ldots,y_n) \in \Lambda : \left\| (y_1,\ldots,y_n) \right\| < \xi \right\},$$

$$\partial\Omega_\xi = \left\{ (y_1,\ldots,y_n) \in \Lambda : \left\| (y_1,\ldots,y_n) \right\| = \xi \right\}.$$

For convenience, we now present the conditions that we presume in the sequel.

(H$_1$) $\lim_{(y_1+\cdots+y_n)\to 0^+} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = \infty$, $t_j \in [v_j - 2, v_j + b]_{\mathbb{N}_{v_j-2}}, j = 1,2,\ldots,n$,

(H$_2$) $\lim_{(y_1+\cdots+y_n)\to \infty} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = \infty$, $t_j \in [v_j - 2, v_j + b]_{\mathbb{N}_{v_j-2}}, j = 1,2,\ldots,n$,

(H$_3$) $\lim_{(y_1+\cdots+y_n)\to 0^+} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = 0$, $t_j \in [v_j - 2, v_j + b]_{\mathbb{N}_{v_j-2}}, j = 1,2,\ldots,n$,

(H$_4$) $\lim_{(y_1+\cdots+y_n)\to \infty} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = 0$, $t_j \in [v_j - 2, v_j + b]_{\mathbb{N}_{v_j-2}}, j = 1,2,\ldots,n$,

(H$_5$) $\lim_{(y_1+\cdots+y_n)\to 0^+} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = l_j$, $t_j \in [v_j - 2, v_j + b]_{\mathbb{N}_{v_j-2}}, j = 1,2,\ldots,n$,

(H$_6$) $\lim_{(y_1+\cdots+y_n)\to \infty} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = L_j$, $t_j \in [v_j - 2, v_j + b]_{\mathbb{N}_{v_j-2}}, j = 1,2,\ldots,n$,

where $0 < l_j, L_j < +\infty$.

Theorem 3.1 *Suppose that there exist two different positive numbers r_1 and r_2 ($r_1 < r_2$) such that*

$$\max_{0 \le y_1+\cdots+y_n \le r_1} f_j(y_1,\ldots,y_n) \le \frac{r_1}{n\lambda_j\alpha_j},$$

$$\min_{\frac{1}{4}r_2 \le y_1+\cdots+y_n \le r_2} f_j(y_1,\ldots,y_n) \ge \frac{r_2}{n\lambda_j\beta_j}.$$

Then the operator T has a fixed point $(\bar{y}_1,\ldots,\bar{y}_n) \in \Lambda$ such that

$$r_1 \le \left\| (\bar{y}_1,\ldots,\bar{y}_n) \right\| \le r_2.$$

Proof For any $(y_1,\ldots,y_n) \in \Omega_{r_1}$ and $\|(y_1,\ldots,y_n)\| = r_1$, we have

$$\left\| T_j(y_1,\ldots,y_n) \right\|$$

$$= \max_{t_j \in [v_j-2, v_j+b]_{\mathbb{N}_{v_j-2}}} \left| \lambda_j \sum_{s=0}^{b+1} G_j(t_j,s) f_j\big(y_1(s + v_1 - 1), \ldots, y_n(s + v_n - 1) \big) \right|$$

$$\le \lambda_j \sum_{s=0}^{b+1} \max_{t_j \in [v_j-2, v_j+b]_{\mathbb{N}_{v_j-2}}} G_j(t_j,s) f_j\big(y_1(s + v_1 - 1), \ldots, y_n(s + v_n - 1) \big)$$

$$\le \lambda_j \sum_{s=0}^{b+1} G_j(v_j + b, s) \frac{r_1}{n\lambda_j\alpha_j}$$

$$= \frac{r_1}{n}$$

$$= \frac{1}{n} \left\| (y_1,\ldots,y_n) \right\|,$$

$j = 1,\ldots,n$. That is,

$$\left\| T(y_1,\ldots,y_n)(t_1,\ldots,t_n) \right\|$$

$$= \left\| \big(T_1(y_1,\ldots,y_n)(t_1), \ldots, T_n(y_1,\ldots,y_n)(t_n) \big) \right\|$$

$$= \left\| T_1(y_1, \ldots, y_n) \right\| + \cdots + \left\| T_n(y_1, \ldots, y_n) \right\|$$

$$\leq \frac{1}{n} \left\| (y_1, \ldots, y_n) \right\| + \cdots + \frac{1}{n} \left\| (y_1, \ldots, y_n) \right\|$$

$$= \left\| (y_1, \ldots, y_n) \right\|$$

for $(y_1, \ldots, y_n) \in \partial \Omega_{r_1}$.

On the other hand, for any $(y_1, \ldots, y_n) \in \Omega_{r_2}$ and $\frac{v_j + b}{4} \leq t_j \leq \frac{3(v_j + b)}{4}$, note that $[\frac{b - v_j}{2}] + v_j \in [\frac{v_j + b}{4}, \frac{3(v_j + b)}{4}]$, we have

$$\left(T_j(y_1, \ldots, y_n) \right) \left(\left[\frac{b - v_j}{2} \right] + v_j \right)$$

$$= \lambda_j \sum_{s=0}^{b+1} G_j \left(\left[\frac{b - v_j}{2} \right] + v_j, s \right) f_j \left(y_1(s + v_1 - 1), \ldots, y_n(s + v_n - 1) \right)$$

$$\geq \lambda_j \sum_{s=[\frac{v_j + b}{4} - v_j + 1]}^{[\frac{3(v_j + b)}{4} - v_j + 1]} G_j \left(\left[\frac{b - v_j}{2} \right] + v_j, s \right) f_j \left(y_1(s + v_1 - 1), \ldots, y_n(s + v_n - 1) \right)$$

$$\geq \lambda_j \sum_{s=[\frac{v_j + b}{4} - v_j + 1]}^{[\frac{3(v_j + b)}{4} - v_j + 1]} G_j \left(\left[\frac{b - v_j}{2} \right] + v_j, s \right) \frac{r_2}{n \lambda_j \beta_j}$$

$$= \frac{r_2}{n}.$$

Then

$$\left\| T_j(y_1, \ldots, y_n) \right\| \geq \frac{r_2}{n} = \frac{1}{n} \left\| (y_1, \ldots, y_n) \right\|, \quad j = 1, \ldots, n.$$

That is,

$$\left\| T(y_1, \ldots, y_n)(t_1, \ldots, t_n) \right\|$$

$$= \left\| \left(T_1(y_1, \ldots, y_n)(t_1), \ldots, T_n(y_1, \ldots, y_n)(t_n) \right) \right\|$$

$$= \left\| T_1(y_1, \ldots, y_n) \right\| + \cdots + \left\| T_n(y_1, \ldots, y_n) \right\|$$

$$\geq \frac{1}{n} \left\| (y_1, \ldots, y_n) \right\| + \cdots + \frac{1}{n} \left\| (y_1, \ldots, y_n) \right\|$$

$$= \left\| (y_1, \ldots, y_n) \right\|$$

for $(y_1, \ldots, y_n) \in \partial \Omega_{r_2}$.

By the use of Theorem 2.9, there exists $(\bar{y}_1, \ldots, \bar{y}_n) \in \Lambda$ such that $T(\bar{y}_1, \ldots, \bar{y}_n) = (\bar{y}_1, \ldots, \bar{y}_n)$, the proof is complete. □

Theorem 3.2 *Suppose that conditions* (H$_1$) *and* (H$_2$) *hold. Then, for every* $\lambda_j \in (0, \lambda_j^*)$, *FBVP* (1.1)-(1.2) *has at least two positive solutions, where*

$$\lambda_j^* = \frac{1}{n \alpha_j} \sup_{r>0} \frac{r}{\max_{0 \leq y_1 + \cdots + y_n \leq r} f_j(y_1, \ldots, y_n)}.$$

Proof Define the function

$$\varphi_j(r) = \frac{r}{n\alpha_j \max_{0 \leq y_1 + \cdots + y_n \leq r} f_j(y_1, \ldots, y_n)}, \quad j = 1, \ldots, n.$$

It is easy to know that $\varphi_j : (0, +\infty) \to (0, +\infty)$ is a continuous function. From (H_1), we see that $\lim_{r \to 0} \frac{r}{f_j(r)} = 0$, that is, $\lim_{r \to 0} \frac{r}{n\alpha_j f_j(r)} = 0$, and

$$0 < \varphi_j(r) = \frac{r}{n\alpha_j \max_{0 \leq y_1 + \cdots + y_n \leq r} f_j(y_1, \ldots, y_n)} \leq \frac{r}{n\alpha_j f_j(r)},$$

so $\lim_{r \to 0} \varphi_j(r) = 0$.

From (H_2), we see further that $\lim_{r \to \infty} \varphi_j(r) = 0$. Then there exists $r_0 > 0$ such that $\varphi_j(r_0) = \max_{r > 0} \varphi_j(r) = \lambda_j^*$, $j = 1, \ldots, n$. For any $\lambda_j \in (0, \lambda_j^*)$, by the intermediate value theorem, there exist two points $d_1 \in (0, r_0)$, $d_2 \in (r_0, \infty)$ such that $\varphi_j(d_1) = \varphi_j(d_2) = \lambda_j$. Thus, we have

$$f_j(y_1, \ldots, y_n) \leq \frac{d_1}{n\lambda_j \alpha_j}, \quad y_1 + \cdots + y_n \in [0, d_1],$$

$$f_j(y_1, \ldots, y_n) \leq \frac{d_2}{n\lambda_j \alpha_j}, \quad y_1 + \cdots + y_n \in [0, d_2].$$

On the other hand, since (H_1) and (H_2) hold, there exist $e_1 \in (0, d_1)$, $e_2 \in (d_2, \infty)$ such that

$$\frac{f_j(y_1, \ldots, y_n)}{y_1 + \cdots + y_n} \geq \frac{4}{n\lambda_j \beta_j}, \quad y_1 + \cdots + y_n \in (0, e_1] \cup \left[\frac{1}{4}e_2, \infty\right).$$

Thus

$$f_j(y_1, \ldots, y_n) \geq \frac{e_1}{n\lambda_j \beta_j}, \quad y_1 + \cdots + y_n \in \left[\frac{1}{4}e_1, e_1\right],$$

$$f_j(y_1, \ldots, y_n) \geq \frac{e_2}{n\lambda_j \beta_j}, \quad y_1 + \cdots + y_n \in \left[\frac{1}{4}e_2, e_2\right].$$

Application of Theorem 3.1 and Theorem 2.8 leads to two distinct positive solutions of FBVP (1.1)-(1.2) which satisfy

$$e_1 \leq \left\|(\bar{y}_1, \ldots, \bar{y}_n)\right\| \leq d_1, \qquad d_2 \leq \left\|(\bar{y}'_1, \ldots, \bar{y}'_n)\right\| \leq e_2.$$

The proof is complete. □

By the proof of Theorem 3.2, we obtain the following.

Corollary 3.3 *If one of conditions* (H_1) *and* (H_2) *is satisfied, then for every* $0 < \lambda_j < \lambda_j^*$, *FBVP* (1.1)-(1.2) *has at least one positive solution.*

Theorem 3.4 *Suppose that* (H_3), (H_4) *hold. Then, for any* $\lambda_j \geq \lambda_j^{**}$, *FBVP* (1.1)-(1.2) *has at least two positive solutions, where*

$$\lambda_j^{**} = \frac{1}{n\beta_j} \inf_{r > 0} \frac{r}{\min_{\frac{1}{4}r \leq y_1 + \cdots + y_n \leq r} f_j(y_1, \ldots, y_n)}.$$

Proof Define the function

$$\psi_j(r) = \frac{r}{n\beta_j \min_{r/4 \le y_1 + \cdots + y_n \le r} f_j(y_1, \ldots, y_n)}, \quad j = 1, \ldots, n.$$

We know that $\psi_j : (0, +\infty) \to (0, +\infty)$ is a continuous function. For $\lambda_j > \lambda_j^{**}$, there exists $0 < e_3 < +\infty$ such that

$$f_j(y_1, \ldots, y_n) \ge \frac{e_3}{n\lambda_j\beta_j}, \quad y_1 + \cdots + y_n \in \left[\frac{1}{4}e_3, e_3\right].$$

By condition (H_3), there exists $0 < d_3 < e_3$ such that

$$f_j(y_1, \ldots, y_n) \le \frac{d_3}{n\lambda_j\alpha_j}, \quad y_1 + \cdots + y_n \in [0, d_3].$$

From condition (H_4), there exists $e_3 < d_0 < +\infty$ such that

$$\frac{f_j(y_1, \ldots, y_n)}{y_1 + \cdots + y_n} \le \frac{1}{n\lambda_j\alpha_j}, \quad y_1 + \cdots + y_n \in [d_0, +\infty).$$

Let $M_j = \max_{0 \le y_1 + \cdots + y_n \le d_0} f_j(y_1, \ldots, y_n)$. Choose $d_4 > d_0$ such that $d_4 \ge \lambda_j M_j \alpha_j$. Then

$$f_j(y_1, \ldots, y_n) \le \frac{d_4}{n\lambda_j\alpha_j}, \quad y_1 + \cdots + y_n \in [0, d_4].$$

By Theorem 3.1 and Theorem 2.8, the proof is complete. $\qquad\square$

From the proof of Theorem 3.4, we get the following.

Corollary 3.5 *Suppose that one of conditions* (H_3) *and* (H_4) *holds. Then, for every* $\lambda_j > \lambda_j^{**}$, *FBVP* (1.1)-(1.2) *has at least one positive solution.*

Theorem 3.6 *Suppose that one of the following cases is satisfied*:
  (1) (H_1), (H_6) *hold, and* $0 < \lambda_j < \frac{1}{n\alpha_j L_j}$;
  (2) (H_2), (H_5) *hold, and* $0 < \lambda_j < \frac{1}{n\alpha_j l_j}$.
Then FBVP (1.1)-(1.2) *has at least one positive solution.*

Proof (1) From (H_6) (namely $\lim_{(y_1 + \cdots + y_n) \to \infty} \frac{f_j(y_1, \ldots, y_n)}{y_1 + \cdots + y_n} = L_j$, $t_j \in [v_j - 2, v_j + b]_{\mathbb{N}_{v_j - 2}}$, $j = 1, 2,$ $\ldots, n$), for any $\epsilon_j > 0$, there exists a number $R_0 > 0$, for $y_1 + \cdots + y_n \in (R_0, +\infty)$, we have

$$f_j(y_1, \ldots, y_n) < (L_j + \epsilon_j)(y_1 + \cdots + y_n).$$

Let $M_j = \max_{0 \le y_1 + \cdots + y_n \le R_0} f_j(y_1, \ldots, y_n)$. Choose $R > \max\{R_0, \frac{M_j}{L_j + \epsilon_j}\}$, then

$$f_j(y_1, \ldots, y_n) < (L_j + \epsilon_j)(y_1 + \cdots + y_n)$$

for $y_1 + \cdots + y_n \in [0, R]$. As arbitrarily of ϵ_j, so $f_j(y_1, \ldots, y_n) \le L_j R$. Note that $0 < \lambda_j < \frac{1}{n\alpha_j L_j}$, then

$$\lambda_j^* = \frac{1}{n\alpha_j} \sup_{R > 0} \frac{R}{\max_{0 \le y_1 + \cdots + y_n \le R} f(y)} > \frac{1}{n\alpha_j} \frac{R}{LR} = \frac{1}{n\alpha_j L_j} > \lambda_j > 0.$$

Namely $0 < \lambda_j < \lambda_j^*$. By means of Corollary 3.3, FBVP (1.1)-(1.2) has at least one positive solution.

The proof is similar to (2) and hence omitted. □

Similarly, we have the following.

Theorem 3.7 *Suppose that one of the following cases is satisfied*:
  (1) (H_3), (H_6) *hold, and* $\frac{4}{nL_j\beta_j} < \lambda_j < +\infty$;
  (2) (H_4), (H_5) *hold, and* $\frac{4}{nl_j\beta_j} < \lambda_j < +\infty$.
Then FBVP (1.1)-(1.2) has at least one positive solution.

Theorem 3.8 *Suppose that conditions (H_5) and (H_6) hold. If λ_j satisfies* $\frac{4}{n\beta_jL_j} < \lambda_j < \frac{1}{n\alpha_jl_j}$ *or* $\frac{4}{n\beta_jl_j} < \lambda_j < \frac{1}{n\alpha_jL_j}$, *then FBVP (1.1)-(1.2) has at least one positive solution.*

Proof Suppose that $\frac{4}{n\beta_jL_j} < \lambda_j < \frac{1}{n\alpha_jl_j}$ holds. Choose $\epsilon_j > 0$ such that $\frac{4}{n\beta_j(L_j-\epsilon_j)} \le \lambda_j \le \frac{1}{n\alpha_j(l_j+\epsilon_j)}$. With condition (H_5), there exists $\tau > 0$ such that $f_j(y_1,\ldots,y_n) \le (l_j + \epsilon_j)(y_1 + \cdots + y_n)$ for $y_1 + \cdots + y_n \in (0, \tau)$. Thus, for $y_1 + \cdots + y_n \in \partial\Omega_\tau$,

$$\|T_j(y_1,\ldots,y_n)\|$$

$$= \max_{t_j \in [v_j-2,v_j+b]_{\mathbb{N}_{v_j-2}}} \left| \lambda_j \sum_{s=0}^{b+1} G_j(t_j,s) f_j\big(y_1(s + v_1 - 1),\ldots,y_n(s + v_n - 1)\big) \right|$$

$$\le \lambda_j(l_j + \epsilon_j)(y_1 + \cdots + y_n) \sum_{s=0}^{b+1} \max_{t_j \in [v_j-2,v_j+b]_{\mathbb{N}_{v_j-2}}} G_j(t_j,s)$$

$$= \lambda_j(l_j + \epsilon_j)(y_1 + \cdots + y_n) \sum_{s=0}^{b+1} G_j(v_j + b, s)$$

$$\le \lambda_j \alpha_j \frac{\tau}{n\lambda_j\alpha_j}$$

$$= \frac{\tau}{n}$$

$$= \frac{1}{n}\|(y_1,\ldots,y_n)\|,$$

$j = 1,\ldots,n$. That is,

$$\|T(y_1,\ldots,y_n)(t_1,\ldots,t_n)\|$$

$$= \big\|\big(T_1(y_1,\ldots,y_n)(t_1),\ldots,T_n(y_1,\ldots,y_n)(t_n)\big)\big\|$$

$$= \|T_1(y_1,\ldots,y_n)\| + \cdots + \|T_n(y_1,\ldots,y_n)\|$$

$$\le \frac{1}{n}\|(y_1,\ldots,y_n)\| + \cdots + \frac{1}{n}\|(y_1,\ldots,y_n)\|$$

$$= \|(y_1,\ldots,y_n)\|$$

for $(y_1,\ldots,y_n) \in \partial\Omega_\tau$.

By condition (H_6), there exists $R_1 > 0$ such that $f_j(y_1,\ldots,y_n) \ge (L_j - \epsilon_j)(y_1 + \cdots + y_n)$ for $y_1 + \cdots + y_n \ge \frac{1}{4}R_1$.

Let $R_2 = \max\{2\tau, R_1\}$, for $(y_1, \ldots, y_n) \in \partial\Omega_{R_2}$, we get

$$
\left(T_j(y_1, \ldots, y_n)\right)\left(\left[\frac{b - \nu_j}{2}\right] + \nu_j\right)
$$

$$
= \lambda_j \sum_{s=0}^{b+1} G_j\left(\left[\frac{b - \nu_j}{2}\right] + \nu_j, s\right) f_j\left(y_1(s + \nu_1 - 1), \ldots, y_n(s + \nu_n - 1)\right)
$$

$$
\geq \lambda_j (L_j - \epsilon_j) \sum_{s=[\frac{\nu_j + b}{4} - \nu_j + 1]}^{[\frac{3(\nu_j + b)}{4} - \nu_j + 1]} G_j\left(\left[\frac{b - \nu_j}{2}\right] + \nu_j, s\right)(y_1 + \cdots + y_n)
$$

$$
\geq \lambda_j \frac{1}{4}(L_j - \epsilon_j) R_2 \sum_{s=[\frac{\nu_j + b}{4} - \nu_j + 1]}^{[\frac{3(\nu_j + b)}{4} - \nu_j + 1]} G_j\left(\left[\frac{b - \nu_j}{2}\right] + \nu_j, s\right)
$$

$$
\geq \frac{R_2}{n}.
$$

Then

$$
\left\| T_j(y_1, \ldots, y_n) \right\| \geq \frac{R_2}{n} = \frac{1}{n}\left\|(y_1, \ldots, y_n)\right\|, \quad j = 1, \ldots, n.
$$

That is,

$$
\left\| T(y_1, \ldots, y_n)(t_1, \ldots, t_n) \right\|
$$

$$
= \left\| \left(T_1(y_1, \ldots, y_n)(t_1), \ldots, T_n(y_1, \ldots, y_n)(t_n)\right) \right\|
$$

$$
= \left\| T_1(y_1, \ldots, y_n) \right\| + \cdots + \left\| T_n(y_1, \ldots, y_n) \right\|
$$

$$
\geq \frac{1}{n}\left\|(y_1, \ldots, y_n)\right\| + \cdots + \frac{1}{n}\left\|(y_1, \ldots, y_n)\right\|
$$

$$
= \left\|(y_1, \ldots, y_n)\right\|
$$

for $(y_1, \ldots, y_n) \in \partial\Omega_{R_2}$

By using Theorem 2.9, we obtain the conclusion.

A similar proof holds when $\frac{4}{n\beta_j l_j} < \lambda_j < \frac{1}{n\alpha_j L_j}$. The proof is complete. □

We now present an example illustrating the sorts of boundary conditions that can be treated by Theorem 3.2.

Example 3.1 Consider the following boundary value problems:

$$
\begin{cases}
\Delta^{\frac{17}{8}} y_1(t) = -\lambda_1 f_1(y_1(t + \frac{9}{8}), y_2(t + \frac{17}{16})), \\
\Delta^{\frac{33}{16}} y_2(t) = -\lambda_2 f_2(y_1(t + \frac{9}{8}), y_2(t + \frac{17}{16})), \\
y_1(-\frac{7}{8}) = \Delta y_1(\frac{169}{8}) = \Delta^2 y_1(-\frac{7}{8}) = 0, \\
y_2(-\frac{15}{16}) = \Delta y_2(\frac{337}{16}) = \Delta^2 y_2(-\frac{15}{16}) = 0,
\end{cases}
\tag{3.1}
$$

where $b = 19$, $\nu_1 = \frac{17}{8}$, $\nu_2 = \frac{33}{16}$, we take

$$
f_1(y_1, y_2) = \frac{1}{64\sqrt{2}}(y_1 + y_2)^{\frac{1}{2}} + (y_1 + y_2)^2, \qquad f_2(y_1, y_2) = \frac{1}{25}(y_1 + y_2)^{\frac{1}{2}} + \frac{1}{64}(y_1 + y_2)^{\frac{3}{2}},
$$

$f_1, f_2 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, and y_1 is defined on the time scale $\{-\frac{7}{8}, \frac{1}{8}, \ldots, \frac{169}{8}\}$, y_2 is defined on the time scale $\{-\frac{15}{16}, \frac{1}{16}, \ldots, \frac{337}{16}\}$. f_1 and f_2 satisfy conditions of Theorem 3.2. A computation shows that $\lambda_1^* \approx 0.01456$, $\lambda_2^* \approx 0.032845$, then, for every $\lambda_j \in (0, \lambda_j^*)$ ($j = 1, 2$), problem (3.1) has at least two positive solutions.

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