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Incomplete inverse spectral and nodal problems for Dirac operator

Zhaoying Wei^{1*}, Yongxia Guo² and Guangsheng Wei²

*Correspondence:
imwzhy@163.com

¹College of Science, Xi'an Shiyou University, Xi'an, 710065, P.R. China
Full list of author information is available at the end of the article

Abstract

The incomplete inverse spectral and inverse nodal problems for Dirac operator defined on a finite interval with separated boundary conditions are considered. We prove uniqueness theorems for the so-called incomplete inverse spectral problem. Using the obtained result we show that for a unique determination of the operator it is sufficient to specify the nodal points only on a part of the interval slightly exceeding its half.

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1 Introduction

Consider the eigenvalue problem corresponding to the Dirac operator, denoted by $L := L(Q(x); \alpha, \beta)$, of the form

$$ly := By' - Q(x)y = \lambda y, \quad 0 < x < 1, \quad (1.1)$$

with

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

subject to the boundary conditions

$$\begin{cases} U(y) := y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \\ V(y) := y_1(1) \cos \beta + y_2(1) \sin \beta = 0, \end{cases} \quad 0 \leq \alpha, \beta < \pi. \quad (1.2)$$

Here λ is a spectral parameter, $p(x), q(x) \in C^1[0, 1]$ and they are real-valued functions. The Dirac operator is the relativistic Schrödinger operator in quantum physics. Here we mainly investigate inverse spectral and inverse nodal problems for the Dirac operator (1.1)-(1.2) and establish a connection between them.

The basic and comprehensive results about Dirac operators were given in [1]. Furthermore, spectral problems for Sturm-Liouville or Dirac operators were extensively studied in various publications; see *e.g.* [2–5]. Moreover, sampling theory is one of the most impor-

tant mathematical tools used in communication engineering, the Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem is viewed as the fundamental result in information theory [6–8]. In the past years, this sampling theorem has been investigated and improved by several authors. In particular, Tharwat and Bhrawy, *etc.* [9–14]) used the derivative sampling theorem (Hermite interpolations and sinc-method) to compute the eigenvalues of the discontinuous Dirac systems.

Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems play an important role in mathematics and have many applications in natural sciences and engineering (see [2, 5] and the references therein). Some aspects of inverse spectral problems for the Dirac systems were studied in [1, 15–18] and other papers. In particular, in [17] it is proved that under some conditions finitely many partially known spectra and partial information on the potential entirely determine the potential. Inverse nodal problems, in turn, consist in constructing operators from given nodes (zeros) of their eigenfunctions (refer to [19–24]). From the physical point of view this corresponds to finding, *e.g.*, the density of a string or a beam from the zero-amplitude positions of the eigenvibrations. In [24] inverse nodal problems of reconstructing the Dirac operator on a finite interval were studied, where it was proved that the operator L is determined uniquely by specifying a dense set of nodal points.

In the present paper, we will consider inverse problems of recovering $Q(x)$, α , β from the given spectral and nodal characteristics. In what follows without loss of generality we always assume the mean value of $p(x) + q(x)$ is known *a priori*. Under this assumption we obtain uniqueness theorems and provide a constructive procedure for the solution. The novelty of this paper lies in the established connections between inverse nodal and spectral problems and the use of a set of nodal points of the components $y_1(x, \lambda_n)$ of the eigenfunctions $y(x, \lambda_n) = (y_1(x, \lambda_n), y_2(x, \lambda_n))^T$ as the given nodal data. As far as we know, incomplete inverse nodal problem for the Dirac system had not been considered before and the obtained results are natural generalizations of the well-known results on the classical inverse problems.

The paper is organized as follows. In Section 2, we prove the uniqueness theorems for so-called incomplete inverse spectral problem for the Dirac operator L . Using the obtained result we show that for unique determination of the operator it is sufficient to specify the nodal points only on $(b, 1)$ with $b < 1/2$ in Section 3.

2 Incomplete inverse spectral problem

In the first part of the paper we study the so-called incomplete inverse problem of recovering the coefficients of (1.1)-(1.2) from a part of the spectrum of L provided that they are known *a priori* on a part of the interval. We note that for recovering $Q(x)$ on the whole interval $[0, 1]$ it is necessary to specify two spectra of boundary value problems with different boundary conditions (see [15]). We also note that the analogous problem for the Sturm-Liouville operators and differential pencils were investigated in [17, 25–27] and other work.

Let $S(x, \lambda)$, $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the solutions of Eq. (1.1) under the initial conditions

$$S(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \varphi(0, \lambda) = \begin{pmatrix} \sin \alpha \\ -\cos \alpha \end{pmatrix}, \quad \psi(1, \lambda) = \begin{pmatrix} \sin \beta \\ -\cos \beta \end{pmatrix}.$$

It is clear that, for each fixed $x \in [0, 1]$, these solutions are entire in λ . Let $\tau = \text{Im } \lambda$. Then the following representations hold uniformly in x as $|\lambda| \rightarrow \infty$ (see [1], p.208, (5.11) and (5.12)):

$$\begin{cases} \varphi_1(x, \lambda) = \sin(\lambda x + \alpha + \eta(x)) + \frac{V_1(x, \lambda)}{\lambda} + O\left(\frac{e^{|\tau|x}}{\lambda^2}\right), \\ \varphi_2(x, \lambda) = -\cos(\lambda x + \alpha + \eta(x)) + \frac{V_2(x, \lambda)}{\lambda} + O\left(\frac{e^{|\tau|x}}{\lambda^2}\right). \end{cases} \tag{2.1}$$

Here

$$\begin{aligned} V_1(x, \lambda) &= \frac{(q-p)(x)}{4} \sin(\lambda x + \eta(x) + \alpha) + \frac{(q-p)(0)}{4} \sin(\lambda x + \eta(x) - \alpha) \\ &\quad - \frac{\int_0^x (p(t) - q(t))^2 dt}{8} \cos(\lambda x + \eta(x) + \alpha), \\ V_2(x, \lambda) &= \frac{(q-p)(x)}{4} \cos(\lambda x + \eta(x) + \alpha) + \frac{(p-q)(0)}{4} \cos(\lambda x + \eta(x) - \alpha) \\ &\quad - \frac{\int_0^x (p(t) - q(t))^2 dt}{8} \sin(\lambda x + \eta(x) + \alpha), \end{aligned}$$

and

$$\eta(x) = \frac{1}{2} \int_0^x (p(t) + q(t)) dt. \tag{2.2}$$

Denote $\Delta(\lambda) := \langle \psi(x, \lambda), \varphi(x, \lambda) \rangle$, where $\langle y(x), z(x) \rangle := y_1(x)z_2(x) - y_2(x)z_1(x)$. The function $\Delta(\lambda)$ is called the characteristic function of L and it does not depend on x . Substituting $x = 0$ and $x = 1$ into $\Delta(\lambda)$ we obtain $\Delta(\lambda) = V(\varphi) = -U(\psi)$. The function $\Delta(\lambda)$ is entire in λ and its zeros $\{\lambda_n\}_{n \in \mathbb{Z}}$ coincide with the eigenvalues of L . It follows from (2.1) that for $|\lambda| \rightarrow \infty$

$$\begin{aligned} \Delta(\lambda) &= \cos \beta \varphi_1(1, \lambda) + \sin \beta \varphi_2(1, \lambda) \\ &= \sin(\lambda + \alpha - \beta + \eta(1)) - \frac{\int_0^1 (p(t) - q(t))^2 dt}{8\lambda} \cos(\lambda + \alpha - \beta + \eta(1)) \\ &\quad + \frac{(q-p)(1)}{4\lambda} \sin(\lambda + \alpha + \beta + \eta(1)) + \frac{(q-p)(0)}{4\lambda} \sin(\lambda - \alpha - \beta + \eta(1)) \\ &\quad + O\left(\frac{e^{|\tau|}}{\lambda^2}\right), \end{aligned} \tag{2.3}$$

where $\eta(x)$ is defined in (2.2). It is well known that the spectrum of the problem (1.1)-(1.2) consists of the eigenvalues $\lambda_n, n \in \mathbb{Z}$, which are all real and simple, and the sequence $\{\lambda_n, n \in \mathbb{Z}\}$ satisfies the classical asymptotic form [1]

$$\lambda_n = n\pi + c_0 + \frac{c_1}{n\pi} + O\left(\frac{1}{n^2}\right), \tag{2.4}$$

where $c_0 = \beta - \alpha - \eta(1)$ and

$$c_1 = \frac{1}{4} \left[(p-q)(1) \sin 2\beta - (p-q)(0) \sin 2\alpha + \frac{1}{2} \int_0^1 (p-q)^2 dt \right].$$

For our purpose of this paper, together with the problem L defined by (1.1)-(1.2), we consider another problem \tilde{L} of the same form but with different coefficients $\tilde{p}(x), \tilde{q}(x), \tilde{\alpha}, \tilde{\beta}$. We agree that, everywhere below if a certain symbol δ denotes an object related to L , then $\tilde{\delta}$ will denote an analogous object related to \tilde{L} . Thus from the assumption we have $\eta(1) = \tilde{\eta}(1)$.

For the rest of the paper, we shall only consider the boundary condition $\alpha, \beta > 0$. The other cases can be treated similarly.

Denote $e_0(x) = (\exp(2ix), \exp(-2ix))^T$. The following theorem has been proven by Horváth [28] for the Sturm-Liouville operator. We show it also holds for the Dirac operator (1.1)-(1.2).

Theorem 2.1 Fix $b \in (0, 1/2]$. Let $\Lambda \subset \mathbb{Z}$ be a subset of integer numbers, and let $\Omega := \{\lambda_n\}_{n \in \Lambda}$ be a part of the spectrum of L such that the system of functions $\{e_0(\lambda_n x)\}_{n \in \Lambda}$ is complete in $\{L_2(0, b)\}^2$. If $p(x) = \tilde{p}(x), q(x) = \tilde{q}(x)$ on $[b, 1]$, and $\beta = \tilde{\beta}, \Omega = \tilde{\Omega}, \eta(1) = \tilde{\eta}(1)$, then $p(x) = \tilde{p}(x), q(x) = \tilde{q}(x)$ on $[0, 1]$ and $\alpha = \tilde{\alpha}$.

Remark 2.2 It is easy to see that if $b = 1/2$ then Theorem 2.1 reduces to the Hochstadt and Lieberman’s theorem [29] for the Dirac operator, which is studied by Malamud in [30]. At this point, our results are generalizations and improvements of the well-known results.

Proof Since $\Omega = \tilde{\Omega}$, it follows from (2.4) that $c_0 = \tilde{c}_0$, which together with the conditions $\beta = \tilde{\beta}$ and $\eta(1) = \tilde{\eta}(1)$ yields $\alpha = \tilde{\alpha}$. Let us denote by $\varphi(x, \lambda) = (\varphi_1(x, \lambda), \varphi_2(x, \lambda))^T$ the solution to the equation

$$By' - Q(x)y = \lambda y \tag{2.5}$$

with the initial condition

$$\begin{pmatrix} y_1(0, \lambda) \\ y_2(0, \lambda) \end{pmatrix} = \begin{pmatrix} \sin \alpha \\ -\cos \alpha \end{pmatrix}, \tag{2.6}$$

and by $\tilde{\varphi}(x, \lambda) = (\tilde{\varphi}_1(x, \lambda), \tilde{\varphi}_2(x, \lambda))^T$ the solution to the equation

$$B\tilde{y}' - \tilde{Q}(x)\tilde{y} = \lambda \tilde{y} \tag{2.7}$$

with the same initial condition as (2.6). Suppose $0 \leq a < b \leq \pi$, for all $(a_1, a_2)^T$ and $(b_1, b_2)^T$ in $(L_2[a, b])^2$, define the inner product

$$\langle (a_1, a_2)^T, (b_1, b_2)^T \rangle = a_1 b_1 + a_2 b_2.$$

Multiplying (2.5) by $\tilde{\varphi}(x, \lambda)$ and (2.7) by $\varphi(x, \lambda)$ (in the sense of scalar product in \mathbb{R}^2), respectively, and subtracting, we get

$$\frac{d}{dx} [\tilde{\varphi}_1(x, \lambda)\varphi_2(x, \lambda) - \tilde{\varphi}_2(x, \lambda)\varphi_1(x, \lambda)] + \langle (\tilde{Q}(x) - Q(x))\varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle = 0.$$

Integrating the above equality from 0 to 1 with respect to x , we infer from the conditions $p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$ on $[b, 1]$ and $\alpha = \tilde{\alpha}$ that

$$\begin{aligned} \int_0^b \langle (Q(t) - \tilde{Q}(t))\varphi(t, \lambda), \tilde{\varphi}(t, \lambda) \rangle dt &= \tilde{\varphi}_1(1, \lambda)\varphi_2(1, \lambda) - \tilde{\varphi}_2(1, \lambda)\varphi_1(1, \lambda) \\ &= [\tilde{\varphi}_1(1, \lambda)\Delta(\lambda) - \tilde{\Delta}(\lambda)\varphi_1(1, \lambda)]/\sin \beta. \end{aligned} \tag{2.8}$$

Define

$$p_1(x) = p(x) - \tilde{p}(x), \quad q_1(x) = q(x) - \tilde{q}(x)$$

and

$$H(\lambda) = \int_0^b \langle (Q(t) - \tilde{Q}(t))\varphi(t, \lambda), \tilde{\varphi}(t, \lambda) \rangle dt.$$

Since $\Delta(\lambda_n) = \tilde{\Delta}(\lambda_n) = 0$ for $n \in \Lambda$, it follows from (2.8) that

$$H(\lambda_n) = 0, \quad n \in \Lambda. \tag{2.9}$$

It is known [1] that there exist kernels $K(x, t) = (K_{ij}(x, t))_{i,j=1}^2$ and $\tilde{K}(x, t) = (\tilde{K}_{ij}(x, t))_{i,j=1}^2$ with entries continuously differentiable on $0 \leq t \leq x \leq 1$ such that

$$\begin{cases} \varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x K(x, t)\varphi_0(t, \lambda) dt, \\ \tilde{\varphi}(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x \tilde{K}(x, t)\varphi_0(t, \lambda) dt, \end{cases} \tag{2.10}$$

where $\varphi_0(x, \lambda) = (\sin(\lambda x + \alpha), -\cos(\lambda x + \alpha))^T$. Thus we can show from (2.10) that

$$\begin{aligned} H(\lambda) &= \int_0^b \langle (Q(t) - \tilde{Q}(t))\varphi(t, \lambda), \tilde{\varphi}(t, \lambda) \rangle dt \\ &= \int_0^b p_1(t) \left[-\cos 2(\lambda t + \alpha) + \int_0^t R_1(t, s)e^{2i\lambda s} ds + \int_0^t R_2(t, s)e^{-2i\lambda s} ds \right] dt \\ &\quad + \int_0^b q_1(t) \left[-\sin 2(\lambda t + \alpha) + \int_0^t R_3(t, s)e^{2i\lambda s} ds + \int_0^t R_4(t, s)e^{-2i\lambda s} ds \right] dt \\ &= \int_0^b f_1(t) \left[e^{2i\lambda t} + \int_0^t S_{11}(t, s)e^{2i\lambda s} ds + \int_0^t S_{12}(t, s)e^{-2i\lambda s} ds \right] dt \\ &\quad + \int_0^b f_2(t) \left[e^{-2i\lambda t} + \int_0^t S_{21}(t, s)e^{2i\lambda s} ds + \int_0^t S_{22}(t, s)e^{-2i\lambda s} ds \right] dt \\ &= \int_0^b e^{2i\lambda s} \left[f_1(s) + \int_s^b f_1(t)S_{11}(t, s) + f_2(t)S_{21}(t, s) dt \right] ds \\ &\quad + \int_0^b e^{-2i\lambda s} \left[f_2(s) + \int_s^b f_1(t)S_{12}(t, s) + f_2(t)S_{22}(t, s) dt \right] ds \\ &= \int_0^b \left\langle e_0(\lambda s), f(s) + \int_s^b S(t, s)f(t) dt \right\rangle ds, \end{aligned}$$

where

$$f(x) = (f_1(x), f_2(x))^T = \left(-\frac{e^{2i\alpha}}{2i} (q_1(x) + ip_1(x)), \frac{e^{-2i\alpha}}{2i} (q_1(x) - ip_1(x)) \right)^T,$$

and $R_l(x, t), l = 1, \dots, 4, S(x, t) = (S_{ij}(x, t)), i, j = 1, 2$ are matrix functions with entries which are piecewise-continuously differentiable on $0 \leq t \leq x \leq 1$. Taking (2.9) into account, it yields

$$0 = H(\lambda_n) = \int_0^b \left\langle e_0(\lambda_n s), f(s) + \int_s^b S(t, s) f(t) dt \right\rangle ds, \quad n \in \Lambda.$$

Thus from the completeness of the functions $\{e_0(\lambda_n x)\}_{n \in \Lambda}$ in $\{L_2(0, b)\}^2$, it follows that

$$f(s) + \int_s^b S(t, s) f(t) dt = 0 \quad \text{for } s \in (0, b).$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus we have obtained $f(x) = (f_1(x), f_2(x))^T = 0$ on $[0, b]$ and consequently $p_1(x) = q_1(x) = 0$, i.e. $p(x) = \tilde{p}(x), q(x) = \tilde{q}(x)$ on $[0, b]$. This completes the proof. \square

In Section 3 the following theorem will be used.

Theorem 2.3 Fix $b < 1/2$ and $N \in \mathbb{N}$. Let $\beta = \tilde{\beta}, p(x) = \tilde{p}(x), q(x) = \tilde{q}(x)$ on $[b, 1]$. If $\{\lambda_n\}_{|n| \geq N} = \{\tilde{\lambda}_n\}_{|n| \geq N}$ and $\eta(1) = \tilde{\eta}(1)$, then $L = \tilde{L}$, i.e., $p(x) = \tilde{p}(x), q(x) = \tilde{q}(x)$ on $[0, 1]$ and $\alpha = \tilde{\alpha}$.

Proof It follows from the conditions of the theorem that

$$\alpha = \tilde{\alpha}, \quad \eta(a) = \tilde{\eta}(a). \tag{2.11}$$

Let $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ be the solutions of Eq. (1.1) under the boundary conditions

$$U(\Phi) = 1, \quad V(\Phi) = 0; \quad \Psi_1(0, \lambda) = 1, \quad V(\Psi) = 0.$$

Then a direct calculation yields

$$\begin{cases} \Phi(x, \lambda) = -\frac{\psi(x, \lambda)}{\Delta(\lambda)} = \frac{1}{\sin \alpha} [S(x, \lambda) + M(\lambda)\varphi(x, \lambda)], \\ \Psi(x, \lambda) = \frac{\psi(x, \lambda)}{\psi_1(0, \lambda)} = \frac{1}{\sin \alpha} [\varphi(x, \lambda) + M_0(\lambda)S(x, \lambda)], \end{cases} \tag{2.12}$$

where

$$M(\lambda) = -\frac{\psi_1(0, \lambda)}{\Delta(\lambda)} \quad \text{and} \quad M_0(\lambda) = -\frac{\Delta(\lambda)}{\psi_1(0, \lambda)} = \frac{1}{M(\lambda)}. \tag{2.13}$$

Denote

$$D(\lambda) = \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle|_{x=b}. \tag{2.14}$$

With the help of (2.12)-(2.13), we calculate for all $|n| \geq N$

$$\frac{D(\lambda)}{\sin \alpha} \Big|_{\lambda=\lambda_n} = \langle \Psi(b, \lambda), \tilde{\Psi}(b, \lambda) \rangle \Big|_{\lambda=\lambda_n} = \frac{\langle \psi(b, \lambda), \tilde{\psi}(b, \lambda) \rangle}{\psi_1(0, \lambda) \tilde{\psi}_1(0, \lambda)} \Big|_{\lambda=\lambda_n}.$$

Under the conditions of the theorem we have $\psi(x, \lambda) = \tilde{\psi}(x, \lambda)$ for $x \in [b, 1]$, hence

$$D(\lambda_n) = 0, \quad |n| \geq N. \tag{2.15}$$

Moreover, we can infer from (2.1), (2.11) and (2.14) that

$$D(\lambda) = O\left(\frac{e^{2|\tau|a}}{\lambda}\right). \tag{2.16}$$

Consider the function

$$F(\lambda) = \frac{D(\lambda)}{\Delta(\lambda)} (\lambda - \lambda_0) \prod_{n=1}^{N-1} (\lambda - \lambda_{-n})(\lambda - \lambda_n),$$

which by virtue of (2.15) is entire in λ . On the other hand, according to (2.3) and with the help of [31], pp.118-119, we have for sufficiently large $|\lambda|$

$$|\Delta(\lambda)| \geq C_\delta e^{|\tau|}, \quad \lambda \in G_\delta, \tag{2.17}$$

where $G_\delta = \{\lambda : |\lambda - n\pi - c_0| \geq \delta, n \in \mathbb{Z}\}$, which together with (2.16) implies that

$$F(\lambda) = O(\lambda^{2N-2} \exp((2a-1)|\tau|)), \quad |\lambda| \rightarrow \infty, \lambda \in G_\delta.$$

Using Phragmen-Lindelöf and Liouville’s theorems we arrive at $F(\lambda) \equiv 0$. Consequently, $D(\lambda) \equiv 0$, which means

$$m_-(b, \lambda) = -\frac{\varphi_2(b, \lambda)}{\varphi_1(b, \lambda)} = -\frac{\tilde{\varphi}_2(b, \lambda)}{\tilde{\varphi}_1(b, \lambda)} = \tilde{m}_-(b, \lambda). \tag{2.18}$$

The result is obtained immediately from the uniqueness theorem in [17] and this completes the proof. □

3 Incomplete inverse nodal problem

In the first part of this section we obtain uniqueness theorem of recovering the potential $Q(x)$ on the whole interval $[0, 1]$ and the parameters α, β in the boundary conditions from a dense subset of nodal points. Further, using the obtained result in Section 2 and developing the idea of the works [25, 27] we prove that for a unique determination of L it is sufficient to specify the set of nodal points only on $[b, 1]$ with $b < 1/2$.

First we study the oscillation property of the first component $y_1(x, \lambda_n)$ of the eigenfunction $y(x, \lambda_n)$ of the Dirac system for sufficiently large $|n|$.

Lemma 3.1 For sufficiently large $|n|$, the first component $y_1(x, \lambda_n)$ of the eigenfunction $y(x, \lambda_n)$ of the Dirac system has exactly $|n|$ nodes in the interval $(0, 1)$:

$$\begin{aligned} 0 < x_n^1 < \dots < x_n^n < 1 \quad \text{for } n > 0, \\ 0 < x_n^0 < \dots < x_n^{n+1} < 1 \quad \text{for } n < 0. \end{aligned} \tag{3.1}$$

Moreover,

$$\begin{aligned} x_n^j = & \frac{j\pi - \alpha - \eta(x_n^j)}{n\pi} - \frac{c_0(j\pi - \alpha - \eta(x_n^j)) - c_2}{(n\pi)^2} \\ & + \frac{(c_0^2 - c_1)(j\pi - \alpha - \eta(x_n^j))}{(n\pi)^3} + O\left(\frac{1}{n^3}\right) \end{aligned} \tag{3.2}$$

uniformly with respect to $j \in \mathbb{Z}$, where

$$c_2 = \frac{\sin 2\alpha}{4}(q - p)(0) + \frac{1}{8} \int_0^{x_n^j} (p(t) - q(t))^2 dt. \tag{3.3}$$

Proof We note that the eigenfunctions $(y_1(x, \lambda_n), y_2(x, \lambda_n))^T$ of the Dirac operator L are real-valued. From (2.1) we see that the function $y_1(x, \lambda_n)$ has the following asymptotic formula for sufficiently large $|n|$, uniformly in x :

$$\begin{aligned} y_1(x, \lambda_n) = & \sin(\lambda_n x + \alpha + \eta(x)) - \frac{\int_0^x (p(t) - q(t))^2 dt}{8\lambda_n} \cos(\lambda_n x + \alpha + \eta(x)) \\ & + \frac{(q - p)(x)}{4\lambda_n} \sin(\lambda_n x + \alpha + \eta(x)) + \frac{(q - p)(0)}{4\lambda_n} \sin(\lambda_n x - \alpha + \eta(x)) + O\left(\frac{1}{\lambda_n^2}\right). \end{aligned}$$

From $y_1(x_n^j, \lambda_n) = 0$, we get

$$\tan(\lambda_n x_n^j + \alpha + \eta(x_n^j)) = \frac{\int_0^{x_n^j} (p - q)^2 dt + 2 \sin 2\alpha (q - p)(0)}{8\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right).$$

Using Taylor’s expansion for the arctangent, we have

$$x_n^j = \frac{1}{\lambda_n} [j\pi - \alpha - \eta(x_n^j)] + \frac{c_2}{\lambda_n^2} + O\left(\frac{1}{\lambda_n^3}\right),$$

where c_2 is defined in (3.3). Furthermore, using the asymptotic formula

$$\lambda_n^{-1} = \frac{1}{n\pi} - \frac{c_0}{(n\pi)^2} + \frac{c_0^2 - c_1}{(n\pi)^3} + O\left(\frac{1}{n^4}\right)$$

we conclude that equality (3.2) holds.

Equality (3.2) gives the asymptotic expansion for the nodal lengths $l_n^j := x_n^{j+1} - x_n^j$; we have uniformly with respect to j

$$\begin{aligned} l_n^j = & \frac{\pi - \frac{1}{2} \int_{x_n^j}^{x_n^{j+1}} (p(t) + q(t)) dt}{n\pi} - \frac{c_0(\pi - \frac{1}{2} \int_{x_n^j}^{x_n^{j+1}} (p(t) + q(t)) dt)}{(n\pi)^2} + O\left(\frac{1}{n^3}\right) \\ = & \frac{1}{n} + o\left(\frac{1}{n}\right), \quad |n| \rightarrow \infty. \end{aligned} \tag{3.4}$$

Consequently, for large $|n|$ we have $x_n^j < x_n^{j+1}$ for positive n and $x_n^j > x_n^{j+1}$ for negative n . The asymptotic formula (3.2) gives for $j = 0, \pm 1, n, n + 1$ that

$$\begin{aligned} x_n^{-1} &= \frac{-\pi - \alpha}{n\pi} + O\left(\frac{1}{n^2}\right), & x_n^0 &= \frac{-\alpha}{n\pi} + O\left(\frac{1}{n^2}\right), \\ x_n^1 &= \frac{\pi - \alpha}{n\pi} + O\left(\frac{1}{n^2}\right), & \dots, & & x_n^n &= 1 - \frac{\alpha}{n\pi} + O\left(\frac{1}{n^2}\right), \\ x_n^{n+1} &= 1 + \frac{\pi - \alpha}{n\pi} + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Thus, according to the order of x_n^j , for large $|n|$, the first component $y_1(x, \lambda_n)$ of the eigenfunction $y(x, \lambda_n)$ of the Dirac system has exactly $|n|$ nodes in the interval $(0, 1)$, i.e., x_n^j , $j = \overline{1, n}$ for positive n and x_n^j , $j = \overline{n + 1, 0}$ for negative n . The proof is complete. \square

Corollary 3.2 *From Lemma 3.1 and (3.4) it follows that the sets $X_1 = \{x_n^j\}_{n>0}$, $X_2 = \{x_n^j\}_{n<0}$ and $X_0 = X_1 \cup X_2$ are all dense in $(0, 1)$.*

Definition 3.3 Let $X \subset X_j$, $j = 0, 1, 2$. The set X is called twin if together with each of its points x_n^j the set X contains at least one of adjacent nodal points x_n^{j-1} and/or x_n^{j+1} .

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

Theorem 3.4 *Fix $i = 1, 2$. Let $X \subset X_i$ be a subset nodal points which is dense and twin in $(0, 1)$. If $X = \tilde{X}$ and $\eta(1) = \tilde{\eta}(1)$, then $p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$ on $[0, 1]$ and $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$.*

Remark 3.5 Note that in this theorem the dense subset X only need to be known for $n > 0$ or $n < 0$ rather than both, i.e., the known subset X do not need to be contained in X_0 . Similar results can be obtained using a dense subset of nodal points of the second component $y_2(x, \lambda_n)$ as the given spectral data for recovering the Dirac system.

Proof of Theorem 3.4 We first prove the given nodal set X uniquely determines the parameters α, β and the function $p(x) + q(x)$ on $[0, 1]$. For each fixed $x \in [0, 1]$ choose a sequence $\{x_n^j\} \subset X_i$ such that $x_n^j \rightarrow x$ as $|n| \rightarrow \infty$. Using the asymptotic expansion (3.2), we get

$$n\pi x_n^j - j\pi = -\alpha - \eta(x_n^j) - \frac{j}{n}c_0 + O\left(\frac{1}{n}\right). \tag{3.5}$$

Also the fact $\lim_{|n| \rightarrow \infty} x_n^j = x$ implies that $j/n \rightarrow x$ and $\eta(x_n^j) \rightarrow \eta(x)$. From this it follows that as $|n| \rightarrow \infty$ the limit of the left-hand side of (3.5) exists and

$$g_i(x) := \lim_{|n| \rightarrow \infty} [n\pi x_n^j - j\pi] = -\alpha - \eta(x) - c_0x. \tag{3.6}$$

Taking the values for $g_i(x)$ at $x = 0$ and $x = 1$, respectively, we obtain

$$\alpha = -g_i(0), \quad \beta = -g_i(1). \tag{3.7}$$

After α and β are reconstructed, taking derivatives of the function $g_i(x)$, we derive

$$p(x) + q(x) - \int_0^1 (p(t) + q(t)) dt = -2[g'_i(x) - \alpha + \beta]. \tag{3.8}$$

Note that if $X = \tilde{X}$, then (3.6) yields $g_i(x) \equiv \tilde{g}_i(x)$, $x \in [0, 1]$. By virtue of (3.7)-(3.8) and the assumption $\eta(1) = \tilde{\eta}(1)$, we get

$$\alpha = \tilde{\alpha}, \quad \beta = \tilde{\beta}, \quad p(x) + q(x) = \tilde{p}(x) + \tilde{q}(x) \quad \text{on } [0, 1]. \tag{3.9}$$

Next we are going to show $p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$ on $[0, 1]$. Multiply (2.5) with $\lambda = \lambda_n$ by $\tilde{\varphi}(x, \tilde{\lambda}_n)$ and (2.7) with $\lambda = \tilde{\lambda}_n$ by $\varphi(x, \lambda_n)$ (in the sense of scalar product in \mathbb{R}^2), respectively, subtract the two equations and integrate it from x_n^j to x_n^{j+1} . This result is

$$\begin{aligned} 0 &= \int_{x_n^j}^{x_n^{j+1}} \langle (\tilde{Q}(t) - Q(t) - (\lambda_n - \tilde{\lambda}_n)I)\varphi(t, \lambda_n), \tilde{\varphi}(t, \tilde{\lambda}_n) \rangle dt \\ &= \int_{x_n^j}^{x_n^{j+1}} (\tilde{p}(t) - p(t))\varphi_1(t, \lambda_n)\tilde{\varphi}_1(t, \tilde{\lambda}_n) dt + \int_{x_n^j}^{x_n^{j+1}} (\tilde{q}(t) - q(t))\varphi_2(t, \lambda_n)\tilde{\varphi}_2(t, \tilde{\lambda}_n) dt \\ &\quad + (\tilde{\lambda}_n - \lambda_n) \int_{x_n^j}^{x_n^{j+1}} [\tilde{\varphi}_1(t, \tilde{\lambda}_n)\varphi_1(t, \lambda_n) + \varphi_2(t, \lambda_n)\tilde{\varphi}_2(t, \tilde{\lambda}_n)] dt. \end{aligned} \tag{3.10}$$

Since $\eta(1) = \tilde{\eta}(1)$, it yields from (2.1), (2.4) and (3.9)

$$\lambda_n - \tilde{\lambda}_n = O\left(\frac{1}{n}\right) \tag{3.11}$$

and

$$\begin{cases} \tilde{y}_1(x, \tilde{\lambda}_n)y_1(x, \lambda_n) = \frac{1}{2}[1 - \cos((\lambda_n + \tilde{\lambda}_n)x + 2(\alpha + \eta(x)))] + O\left(\frac{1}{n}\right), \\ \tilde{y}_2(x, \tilde{\lambda}_n)y_2(x, \lambda_n) = \frac{1}{2}[1 + \cos((\lambda_n + \tilde{\lambda}_n)x + 2(\alpha + \eta(x)))] + O\left(\frac{1}{n}\right). \end{cases}$$

From the above equalities we get from (3.10) and (3.11)

$$\begin{aligned} 0 &= \int_{x_n^j}^{x_n^{j+1}} [(\tilde{q}(t) - q(t)) + (\tilde{p}(t) - p(t))] dt \\ &\quad + \int_{x_n^j}^{x_n^{j+1}} [(\tilde{q}(t) - q(t)) - (\tilde{p}(t) - p(t))] \cos(2(\alpha + \eta(t)) + (\lambda_n + \tilde{\lambda}_n)t) dt + O\left(\frac{1}{n^2}\right) \\ &= \frac{1}{j} \int_{x_n^j}^{x_n^{j+1}} [(\tilde{q}(t) - q(t)) - (\tilde{p}(t) - p(t))] \cos(2(\alpha + \eta(t)) + (\lambda_n + \tilde{\lambda}_n)t) dt + O\left(\frac{1}{n}\right). \end{aligned}$$

Thus applying the result in [32], we derive with $j = j_n(x)$ that for $n \rightarrow \infty$

$$[(\tilde{q}(x) - q(x)) - (\tilde{p}(x) - p(x))] \cos(2(\alpha + \eta(x)) + (\lambda_n + \tilde{\lambda}_n)x) = 0, \tag{3.12}$$

which means $q(x) - p(x) = \tilde{q}(x) - \tilde{p}(x)$, since the quantity $\cos(2(\alpha + \eta(x)) + (\lambda_n + \tilde{\lambda}_n)x)$ cannot be identically zero for all $n \in \mathbb{Z}$ and $x \in [0, 1]$. Hence we obtain from (3.9) that $p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$ on $[0, 1]$. The proof is complete. \square

According to Lemma 3.1 one can choose sufficiently large N_1 such that for all $n > N_1$ there are exactly the first component $y_1(x, \lambda_n)$ of the eigenfunction $y(x, \lambda_n)$ and the first component $y_1(x, \lambda_{-n})$ of the eigenfunction $y(x, \lambda_{-n})$, corresponding to the eigenvalues $\lambda_n > 0$ and $\lambda_{-n} < 0$, respectively, of the operator L that posses precisely n nodes in the interval $(0, 1)$. Analogous to the proof of [25], Theorem 6, the same assertion with the same N_1 is also valid for the operator L_a of the form

$$By' - Q(x)y = \lambda y, \quad 0 \leq a < x < 1, \quad y_1(a) = V(y) = 0, \tag{3.13}$$

which can be formulated thus: for all $a \in [0, 1)$ and $n > N_1$ there are exactly the first component $y_1(x, \lambda_{n,a})$ of the eigenfunctions $y(x, \lambda_{n,a})$ and the first component $y_1(x, \lambda_{-n,a})$ of the eigenfunctions $y(x, \lambda_{-n,a})$, corresponding to the eigenvalues $\lambda_{n,a} > 0$ and $\lambda_{-n,a} < 0$, respectively, for the operator L_a possess precisely n zeros in the interval $(a, 1)$.

Let us go on to the investigation of an incomplete inverse nodal problem when the nodal points are given only on a part of the interval.

Theorem 3.6 *Fix $b < 1/2$. If $X_0 \cap (b, 1) = \tilde{X}_0 \cap (b, 1)$ and $\eta(1) = \tilde{\eta}(1)$, then $L = \tilde{L}$. Thus, the specification of the nodes on any interval $(b, 1)$, $b < 1/2$, together with the mean values of $p + q$, uniquely determines the functions $p(x)$, $q(x)$, and the coefficients α , β of the boundary conditions.*

Proof By the same arguments in the proof of Theorem 3.4, it is easy to see that

$$\beta = \tilde{\beta}, \quad p(x) = \tilde{p}(x), \quad q(x) = \tilde{q}(x) \quad \text{on } [b, 1]. \tag{3.14}$$

Choose $N \geq N_1$ so that $x_n^{n-N_1-1} > b$ and $x_{-n}^{-n+N_1+2} > b$ for $n \geq N$. Fix $n \geq N$ and put $a := x_n^{n-N_1-1}$. Consider the operator L_a , then for $n \geq N$ the component-function $y_1(x, \lambda_{n,a})$ of the eigenfunction $y(x, \lambda_{n,a})$ and the component-function $y_1(x, \lambda_{-n,a})$ of the eigenfunction $y(x, \lambda_{-n,a})$ for the operator L_a both have precisely n zeros in the interval $(a, 1)$, corresponding to the eigenvalues $\lambda_{n,a} > 0$ and $\lambda_{-n,a} < 0$, respectively. Moreover, these component-functions also satisfy the boundary conditions in (3.13) for the corresponding eigenvalues. Since the function $y_1(x, \lambda_n)$ has precisely N_1 zeros in the interval $(a, 1)$ and $V(y) = y_1(a, \lambda_n) = 0$, thus we have $\lambda_n = \lambda_{N_1,a}$ or $\lambda_n = \lambda_{-N_1,a}$. Furthermore, (3.14) yields $\tilde{\lambda}_n = \lambda_{N_1,a}$ or $\tilde{\lambda}_n = \lambda_{-N_1,a}$. Since $\lambda_n \tilde{\lambda}_n > 0$ and $\lambda_{N_1,a} \lambda_{-N_1,a} < 0$, we get $\lambda_n = \tilde{\lambda}_n$. Analogously putting $a := x_{-n}^{-n+N_1+2}$ we arrive at $\lambda_{-n} = \tilde{\lambda}_{-n}$. Thus, we have $\lambda_n = \tilde{\lambda}_n$ for $|n| \geq N$ and it remains to apply Theorem 2.3 to get $p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$ on $[0, b]$ and $\alpha = \tilde{\alpha}$. The proof is complete. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors have made equal contributions to each part of this article. All the authors read and approved the final manuscript.

Author details

¹College of Science, Xi'an Shiyou University, Xi'an, 710065, P.R. China. ²College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710062, P.R. China.

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