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Recursive stochastic linear-quadratic optimal control and nonzero-sum differential game problems with random jumps

Na Li[1,2] and Zhiyong Yu[2*]

*Correspondence:
yuzhiyong@sdu.edu.cn
[2]School of Mathematics, Shandong University, Jinan, 250100, China
Full list of author information is available at the end of the article

Abstract

This paper presents an existence and uniqueness result for a kind of forward-backward stochastic differential equations (FBSDEs for short) driven by Brownian motion and Poisson process under some monotonicity conditions. By virtue of the conclusion of FBSDEs, we solve a linear-quadratic stochastic optimal control problem for forward-backward stochastic systems with random jumps. Moreover, we also solve a linear-quadratic nonzero-sum stochastic differential game problem. We obtain explicit forms of the unique optimal control and the unique Nash equilibrium point, respectively.

MSC: 60H10; 93E20; 91A15

Keywords: forward-backward stochastic differential equation; Poisson process; stochastic optimal control; linear-quadratic problem; nonzero-sum stochastic differential game; Nash equilibrium

1 Introduction

This paper is concerned with a kind of linear-quadratic stochastic optimal control (LQ SOC) problems and linear-quadratic nonzero-sum stochastic differential game (LQ NZSSDG) problems for forward-backward systems with random jumps. In detail, for the LQ NZSSDG problems, the controlled system is given by the following controlled linear forward-backward stochastic differential equation with Poisson process (FBSDE):

$$\begin{cases} dx_t = (A_t x_{t-} + B_t^1 v_t^1 + B_t^2 v_t^2 + \mu_t)\,dt + (C_t x_{t-} + D_t^1 v_t^1 + D_t^2 v_t^2 + \nu_t)\,dW_t \\ \qquad + \int_{\mathcal{E}} (E_t(e) x_{t-} + F_t^1(e) v_t^1 + F_t^2(e) v_t^2 + \psi_t(e)) \widetilde{N}(dt,de), \\ -dy_t = (G_t x_{t-} + H_t y_{t-} + I_t z_t + \int_{\mathcal{E}} J_t(e) k_t(e) \pi(de) + K_t^1 v_t^1 + K_t^2 v_t^2 + \phi_t)\,dt \\ \qquad - z_t\,dW_t - \int_{\mathcal{E}} k_t(e) \widetilde{N}(dt,de), \\ x_0 = a, \qquad y_T = \Upsilon x_T + \eta. \end{cases} \tag{1}$$

For simplicity of notation, here we only consider the case of two players. The corresponding conclusions for the case of n players can be obtained in the same way. In addition, we are given two cost (or utility, performance criterion) functionals for each player:

$$\mathcal{J}^i\big(v^1(\cdot),v^2(\cdot)\big)=\frac{1}{2}\mathbb{E}\int_0^T\bigg[\big\langle L_t^i x_t,x_t\big\rangle+\big\langle M_t^i y_t,y_t\big\rangle+\big\langle N_t^i z_t,z_t\big\rangle+\int_{\mathcal{E}}\big\langle O_t^i(e)k_t(e),k_t(e)\big\rangle\pi(de)$$
$$+\big\langle\Gamma_t^i v_t^i,v_t^i\big\rangle\bigg]\,dt+\frac{1}{2}\mathbb{E}\big\langle R^i x_T,x_T\big\rangle+\frac{1}{2}\big\langle S^i y_0,y_0\big\rangle\quad(i=1,2). \tag{2}$$

Under suitable conditions, the controlled system (1) and the cost functionals (2) are well defined. Our final objective is to find the unique Nash equilibrium point explicitly. It is well known that a stochastic optimal control problem can be regarded as a game problem with only one player. From this point of view, the LQ SOC problem is a special case of the LQ NZSSDG problem. We also study its unique optimal control.

We notice that in the controlled system (1), besides a forward stochastic differential equation (SDE), there is another backward stochastic differential equation (BSDE). The linear BSDE was introduced by Bismut [1] to solve some stochastic optimal control problems, and the general nonlinear BSDEs were introduced by Pardoux and Peng [2] and Duffie and Epstein [3] independently. It worth noting that in [3] and the subsequent paper [4] by El Karoui *et al.*, BSDEs were used to characterize a kind of stochastic differential recursive utility, which is an extension of the standard additive utility with the instantaneous utility depending not only on the instantaneous consumption rate but also on the future utility. From this viewpoint, the LQ SOC problem and LQ NZSSDG problem for the forward-backward system can be regarded as an extension of the recursive stochastic optimal control problem and the recursive differential game problem. On the other side, in the classical stochastic optimal control and differential game theory, the performance criterion is described by the linear expectation $\mathbb{E}$. However, sometimes the classical linear expectation $\mathbb{E}$ does not quite represent people's preferences (see Allais [5], Ellsberg [6]). A remedial way is to use the so-called *generalized expectation* introduced by Peng [7, 8] instead of the linear expectation. From the theory of the generalized expectation, it is also defined by some BSDEs. For research of stochastic optimization problems with generalized expectation, we refer to Yong [9] and Hui and Xiao [10]. Therefore, the LQ SOC problem and LQ NZSSDG problem studied in this paper can also be regarded as an extension of stochastic optimization problems with generalized expectations.

Recently, researchers paid much attention to the so-called *principal-agent problem* due to its wide applications in finance. Williams [11] constructed a model of dynamic principal-agent problems in continuous time. The actions of the principal and the agent affect the same controlled system, which is described by a forward SDE. By virtue of the classical stochastic maximum principle in the optimal control theory, the optimal control of the agent is given by a coupled FBSDE (called the stochastic Hamiltonian system). Then the principal's problem is formulated as a stochastic optimization problem for a coupled forward-backward system. The LQ SOC problem and the LQ NZSSDG problem (with forward-backward systems in the decoupled form) studied in this paper can be regarded as a simple case of those complicated problems.

We also notice that the controlled system (1) is in a linear form, and the cost functionals (2) are in quadratic forms. Such a kind of optimization problem is called a linear-quadratic (LQ) problem. The LQ problem is an important class of stochastic optimization problems, which provides a basic knowledge for more general problems since lots of nonlinear problems can be approximated by LQ problem reasonably. The LQ problem for forward system was firstly studied by Wonham [12]. Then Bismut [13] proved an existence result for optimal controls. By virtue of a square complete technique, Chen *et al.* [14] studied stochastic LQ problems with indefinite control weight costs. They gave the optimal feedback control by the solution of a stochastic Riccati equation. For the forward controlled system, Hamadène [15] linked an LQ nonzero-sum stochastic differential game problem to a linear coupled FBSDE by virtue of changing variables, and he also gave an existence result for Nash equilibrium points of the game problem. Wu [16] and Wu and Yu [17] improved and extended the result of [15]. Recently, Yu [18] considered the LQ problem for the forward-backward system, which is linked closely to a new type of FBSDEs: the linear coupled FBSDEs with double dimensions. The solution of this kind of FBSDEs played an important role in the construction of optimal controls and Nash equilibrium points.

In this paper, we adopt the model driven by both Brownian motion and Poisson process (see (1) and (2)), since jump diffusion processes characterize stochastic phenomena better than just Brownian motion diffusion processes, providing us more realistic models in practice. For example, in finance stock prices often exhibit some jump behavior. Moreover, financial markets with jump stock prices provide a rich family of incomplete financial models. For more information as regards jump diffusion models, interested readers may refer to Cont and Tankov [19] and Øksendal and Sulem [20]. We also refer to Tang and Li [21], Situ [22], and Wu and Wang [23] for some research on BSDEs driven by both Brownian motions and Poisson processes.

Inspired by the idea of stochastic maximum principle in optimal control theory and Hamadène's transform (see [15]), both the LQ SOC problem and the LQ NZSSDG problem are closed linked to a kind of coupled FBSDEs involving Poisson jumps, which is out of scope of the existing results in the FBSDEs theory. Historically, fully coupled FBSDEs were firstly studied by Antonelli [24]. He obtained a local solvability result by the fixed point theorem. For the global existence and uniqueness results, there exist two main methods. One concerns a kind of *four step scheme* approach introduced by Ma *et al.* [25], which can be regarded as a method combining partial differential equations and probability theory. In this method, the forward diffusion is required to be non-degenerate and the coefficients are required to be deterministic. The other one, named the *method of continuation*, is probabilistic; it was introduced originally by Hu and Peng [26], and then developed by Peng and Wu [27] and Yong [28, 29]. In this method, the conditions in [25] are relaxed, but some monotonicity assumption is proposed. However, the monotonicity assumption is naturally satisfied by many coupled FBSDEs arising from stochastic optimal control problems. Recently, Ma *et al.* [30] proposed a *unified approach*, which can be regarded as a combination of existing methods.

In the present paper, we use the method of continuation to obtain an existence and uniqueness result for a kind of FBSDEs with Poisson jumps under some monotonicity conditions. The solvability of FBSDEs is important for the following optimization problems and is also interesting in its own right. Then we use the solution of FBSDE to construct the unique optimal control for the LQ SOC problem. We develop Hamadène's transform to study the LQ NZSSDG problem. Under suitable conditions, we prove an existence and uniqueness result for the Nash equilibrium point. Moreover, we give an explicit form for it. Our method used in this paper is effective in studying the uniqueness of Nash equilibrium points, which is not considered in [15–17]. It is also worthy to point out that for the existence of Nash equilibrium point our conclusion is more general than the corresponding results in [15–18].

The rest of this paper is organized as follows. In Section 2, we give some notations. Section 3 is devoted to the proof of an existence and uniqueness theorem for adapted solutions of a new kind of FBSDEs driven by Brownian motion and a Poisson process. In Section 4, we give an explicit form of the unique optimal control for LQ SOC problem. In the last section we study the LQ NZSSDG problem and construct the unique Nash equilibrium point explicitly.

2 Notations

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathbb{R}^{m\times n}$ be the collection of $m\times n$ matrices with the inner product:

$$\langle A,B\rangle = \mathrm{tr}\{AB^\top\}, \quad \text{for any } A,B\in\mathbb{R}^{m\times n},$$

where $\top$ denotes the transpose of matrices.

Let $T>0$ is a constant and $[0,T]$ denote the finite time span. Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space. The filtration $\mathbb{F}=\{\mathcal{F}_t; 0\le t\le T\}$ is generated by two mutually independent stochastic processes. One is a 1-dimensional Brownian motion W, and the other one is a Poisson random measure N defined on $\mathbb{R}_+\times\mathcal{E}$, where $\mathcal{E}=\mathbb{R}^d-\{0\}$ is a nonempty Borel subset of some Euclidean space. The compensator of N is $\overline{N}(dt,de)=\pi(de)\,dt$, which makes $\{\widetilde{N}((0,t]\times A)=(N-\overline{N})((0,t]\times A)\}_{t\ge 0}$ a martingale for any A belonging to the Borel field $\mathcal{B}(\mathcal{E})$ with $\pi(A)<\infty$. Here π is a given σ-finite measure on the measurable space $(\mathcal{E},\mathcal{B}(\mathcal{E}))$ satisfying $\int_{\mathcal{E}}(1\wedge|e|^2)\pi(de)<\infty$. Then $\mathcal{F}_t$ is defined by

$$\sigma\{W_s: 0\le s\le t\}\vee\sigma\left\{\int_{\mathcal{E}}\widetilde{N}(s,de): 0\le s\le t\right\}\vee\mathcal{N}, \quad 0\le t\le T,$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets, and $\mathcal{F}=\mathcal{F}_T$. We note that in this paper we assume the dimension of Brownian motion $d_1=1$ and the number of Poisson random measure $d_2=1$ just for the simplicity of notations. Actually, all of the following conclusions still hold true for the case of $d_1>1$ and/or $d_2>1$.

For any Euclidean space $\mathbb{R}^m$, we introduce the following notations:

- $L^2(\mathcal{F}_T;\mathbb{R}^m)=\{\varsigma|\varsigma$ is an $\mathbb{R}^m$-valued $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[|\varsigma|^2]<\infty\}$,
- $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)=\{g:[0,T]\times\Omega\to\mathbb{R}^m|g(\cdot)$ is an $\mathbb{R}^m$-valued $\mathbb{F}$-adapted stochastic process such that $\|g\|^2=\mathbb{E}\int_0^T|g(t)|^2\,dt<\infty\}$,
- $\mathcal{C}^2_{\mathbb{F}}(0,T;\mathbb{R}^m)=\{g:[0,T]\times\Omega\to\mathbb{R}^m|g(\cdot)$ is a càdlàg process in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ such that $\|g\|^2=\mathbb{E}[\sup_{0\le t\le T}|g(t)|^2]<\infty\}$,
- $M^2_{\mathbb{F}}(0,T;\mathbb{R}^m)=\{r:[0,T]\times\mathcal{E}\times\Omega\to\mathbb{R}^m|\ r(\cdot,\cdot)$ is an $\mathbb{R}^m$-valued $\mathbb{F}$-adapted process such that $\|r\|^2=\mathbb{E}\int_0^T\int_{\mathcal{E}}|r(t,e)|^2\pi(de)\,dt<\infty\}$,

where $\|\cdot\|$ is the norm in each subspace.

Moreover, we define the space

$$\mathcal{R}=\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,$$

and the inner product in $\mathcal{R}$ is

$$\langle(\lambda,\xi),(\bar\lambda,\bar\xi)\rangle=\langle x,\bar x\rangle+\langle y,\bar y\rangle+\langle z,\bar z\rangle+\langle k,\bar k\rangle+\langle p,\bar p\rangle+\langle q,\bar q\rangle+\langle r,\bar r\rangle+\langle\theta,\bar\theta\rangle,$$

for any $\lambda = (x, y, z, k)$, $\xi = (p, q, r, \theta)$, $\bar{\lambda} = (\bar{x}, \bar{y}, \bar{z}, \bar{k})$ and $\bar{\xi} = (\bar{p}, \bar{q}, \bar{r}, \bar{\theta})$. Then the norm in $\mathcal{R}$ is deduced by $\|(\lambda, \xi)\| = \sqrt{\langle(\lambda, \xi), (\lambda, \xi)\rangle}$. We also define

$$\begin{aligned}\mathcal{L}[0,T] = {}& \mathcal{C}^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n\big) \times \mathcal{C}^2_{\mathbb{F}}\big(0,T;\mathbb{R}^m\big) \times L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^m\big) \times M^2_{\mathbb{F}}\big(0,T;\mathbb{R}^m\big) \\ & \times \mathcal{C}^2_{\mathbb{F}}\big(0,T;\mathbb{R}^m\big) \times \mathcal{C}^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n\big) \times L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n\big) \times M^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n\big)\end{aligned}$$

with the norm

$$\begin{aligned}\big\|\big(\lambda(\cdot),\xi(\cdot)\big)\big\| = {}& \bigg\{\mathbb{E}\Big(\sup_{t\in[0,T]}|x_t|^2 + \sup_{t\in[0,T]}|y_t|^2 + \sup_{t\in[0,T]}|p_t|^2 + \sup_{t\in[0,T]}|q_t|^2\Big) \\ & + \mathbb{E}\int_0^T \big(|z_t|^2 + |r_t|^2\big)\,dt + \mathbb{E}\int_0^T\int_{\mathcal{E}}\big(\big|k_t(e)\big|^2 + \big|\theta_t(e)\big|^2\big)\pi(de)\,dt\bigg\}^{\frac{1}{2}}\end{aligned}$$

for any $(\lambda(\cdot), \xi(\cdot)) = (x(\cdot), y(\cdot), z(\cdot), k(\cdot,\cdot), p(\cdot), q(\cdot), r(\cdot), \theta(\cdot,\cdot)) \in \mathcal{L}[0,T]$ and

$$\begin{aligned}\mathcal{G}[0,T] = {}& L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n\big) \times L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n\big) \times M^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n\big) \times L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^m\big) \\ & \times L^2\big(\mathcal{F}_T;\mathbb{R}^m\big) \times L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^m\big) \times L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^m\big) \times M^2_{\mathbb{F}}\big(0,T;\mathbb{R}^m\big) \\ & \times L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n\big) \times L^2\big(\mathcal{F}_T;\mathbb{R}^n\big) \times \mathbb{R}^m.\end{aligned}$$

The element of $\mathcal{G}[0,T]$ is often denoted by

$$\kappa = \big(\mu_1(\cdot), \nu_1(\cdot), \psi_1(\cdot,\cdot), \varphi_1(\cdot), \eta_1, \mu_2(\cdot), \nu_2(\cdot), \psi_2(\cdot,\cdot), \varphi_2(\cdot), \eta_2, b\big).$$

3 Solvability of forward-backward stochastic differential equation with Poisson jump

In this section, we consider the solvability of a new kind of forward-backward stochastic differential equation (FBSDE) with a Poisson jump, which is obtained by adjoint equations deduced from the stochastic maximum principle. The solution of FBSDE plays a key role to solve the LQ stochastic optimal control problems in the next section and the LQ nonzero-sum stochastic differential game problems in Section 5.

We consider the following FBSDE:

$$\begin{cases} dx_t = \alpha_1(t, x_{t-}, B_t p_{t-}, C_t q_{t-}, D_t r_t, \int_{\mathcal{E}} E_t(e)\theta_t(e)\pi(de))\,dt \\ \qquad + \beta_1(t, x_{t-}, B_t p_{t-}, C_t q_{t-}, D_t r_t, \int_{\mathcal{E}} E_t(e)\theta_t(e)\pi(de))\,dW_t \\ \qquad + \int_{\mathcal{E}} \gamma_1(t, e, x_{t-}, B_t p_{t-}, C_t q_{t-}, D_t r_t, \int_{\mathcal{E}} E_t(e)\theta_t(e)\pi(de))\widetilde{N}(dt, de), \\ -dy_t = f_1(t, x_{t-}, y_{t-}, z_t, k_t, B_t p_{t-}, C_t q_{t-}, D_t r_t, \int_{\mathcal{E}} E_t(e)\theta_t(e)\pi(de))\,dt - z_t\,dW_t \\ \qquad - \int_{\mathcal{E}} k_t(e)\widetilde{N}(dt, de), \\ dp_t = \alpha_2(t, x_{t-}, y_{t-}, z_t, k_t, p_{t-}, q_{t-}, r_t, \theta_t)\,dt + \beta_2(t, x_{t-}, y_{t-}, z_t, k_t, p_{t-}, q_{t-}, r_t, \theta_t)\,dW_t \\ \qquad + \int_{\mathcal{E}} \gamma_2(t, e, x_{t-}, y_{t-}, z_t, k_t, p_{t-}, q_{t-}, r_t, \theta_t)\widetilde{N}(dt, de), \\ -dq_t = f_2(t, x_{t-}, y_{t-}, z_t, k_t, p_{t-}, q_{t-}, r_t, \theta_t)\,dt - r_t\,dW_t - \int_{\mathcal{E}} \theta_t(e)\widetilde{N}(dt, de), \\ x_0 = a, \qquad q_T = \Phi_2(x_T, p_T), \\ p_0 = \Psi(y_0), \qquad y_T = \Phi_1(x_T), \end{cases} \tag{3}$$

where $(x, y, z, k, p, q, r, \theta)$ takes values in $\mathcal{R}$, B, C, D, E are $\mathbb{F}$-adapted matrix-valued bounded processes with appropriate dimensions, for each fixed $(x, y, z, k, p, q, r, \theta)$, α_1, β_1,

$\gamma_1, f_1, \alpha_2, \beta_2, \gamma_2, f_2, \Phi_1, \Phi_2, \Psi$ are $\mathbb{F}$-adapted mappings with appropriate dimensions, and $a \in \mathbb{R}^n$. For convenience, we use the following notations:

$$\begin{aligned} &\lambda = (x, y, z, k), \qquad \xi = (p, q, r, \theta), \\ &\xi_t^M = \left(B_t p, C_t q, D_t r, \int_{\mathcal{E}} E_t(e)\theta(e)\pi(de)\right), \\ &\xi_t^\Sigma = B_t p + C_t q + D_t r + \int_{\mathcal{E}} E_t(e)\theta(e)\pi(de). \end{aligned} \tag{4}$$

Definition 3.1 A process $(x(\cdot), y(\cdot), z(\cdot), k(\cdot,\cdot), p(\cdot), q(\cdot), r(\cdot), \theta(\cdot,\cdot)) \in \mathcal{L}[0,T]$ is called an adapted solution of FBSDE (3) if it satisfies (3). If there exists a unique adapted solution, then (3) is said to be uniquely solvable.

Similar to Hu and Peng [26] and Peng and Wu [27], for each $t \in [0,T]$, $(\lambda, \xi) \in \mathcal{R}$, we introduce the following notation:

$$\begin{aligned} A(t,\lambda,\xi) = \bigl(&-f_2(t,\lambda,\xi), \alpha_2(t,\lambda,\xi), \beta_2(t,\lambda,\xi), \gamma_2(t,\lambda,\xi), \\ &-f_1\bigl(t,\lambda,\xi_t^M\bigr), \alpha_1\bigl(t,x,\xi_t^M\bigr), \beta_1\bigl(t,x,\xi_t^M\bigr), \gamma_1\bigl(t,x,\xi_t^M\bigr)\bigr). \end{aligned}$$

Now we give assumptions on the coefficients of (3).

Assumption 1 (Lipschitz condition)

(i) For any $(\lambda,\xi) \in \mathcal{R}$, $A(\cdot,\lambda,\xi) \in L^2_{\mathbb{F}}(0,T;\mathcal{R})$. For any $x \in \mathbb{R}^n$, $\Phi_1(x) \in L^2(\mathcal{F}_T;\mathbb{R}^m)$. For any $(x,p) \in \mathbb{R}^n \times \mathbb{R}^m$, $\Phi_2(x,p) \in L^2(\mathcal{F}_T;\mathbb{R}^n)$. For any $y \in \mathbb{R}^m$, $\Psi(y) \in \mathbb{R}^m$.

(ii) The mappings A, Φ_1, Φ_2, and Ψ are uniformly Lipschitz continuous with respect to (λ,ξ), x, (x,p), and y, respectively.

Assumption 2 (Domination condition) There exists a constant $L > 0$ such that, for any $\lambda = (x,y,z,k)$, any $\xi = (p,q,r,\theta)$ and any $\bar{\xi} = (\bar{p},\bar{q},\bar{r},\bar{\theta})$,

$$\bigl|g_1\bigl(t,x,\xi_t^M\bigr) - g_1\bigl(t,x,\bar{\xi}_t^M\bigr)\bigr| \le L\left|B_t\hat{p} + C_t\hat{q} + D_t\hat{r} + \int_{\mathcal{E}} E_t(e)\hat{\theta}(e)\pi(de)\right|, \quad \text{a.s. a.e.} \tag{5}$$

and

$$\bigl|f_1\bigl(t,\lambda,\xi_t^M\bigr) - f_1\bigl(t,\lambda,\bar{\xi}_t^M\bigr)\bigr| \le L\left|B_t\hat{p} + C_t\hat{q} + D_t\hat{r} + \int_{\mathcal{E}} E_t(e)\hat{\theta}(e)\pi(de)\right|, \quad \text{a.s. a.e.,} \tag{6}$$

where ξ_t^M and $\bar{\xi}_t^M$ are defined by (4), $\hat{\xi} = (\hat{p},\hat{q},\hat{r},\hat{\theta}) = (p-\bar{p}, q-\bar{q}, r-\bar{r}, \theta-\bar{\theta})$, and $g_1 = \alpha_1, \beta_1, \gamma_1$.

Assumption 3 (Monotonicity condition)

(i) There exists a constant $l > 0$ such that, for any $\lambda = (x,y,z,k)$, any $\bar{\lambda} = (\bar{x},\bar{y},\bar{z},\bar{k})$, any $\xi = (p,q,r,\theta)$, and any $\bar{\xi} = (\bar{p},\bar{q},\bar{r},\bar{\theta})$,

$$\begin{aligned} &\bigl\langle A(t,\lambda,\xi) - A(t,\bar{\lambda},\bar{\xi}), (\hat{\lambda},\hat{\xi})\bigr\rangle \\ &\quad \le -l\left|B_t\hat{p} + C_t\hat{q} + D_t\hat{r} + \int_{\mathcal{E}} E_t(e)\hat{\theta}(e)\pi(de)\right|^2, \quad \text{a.s. a.e.,} \end{aligned}$$

where $(\hat{\lambda},\hat{\xi}) = (\hat{x},\hat{y},\hat{z},\hat{k},\hat{p},\hat{q},\hat{r},\hat{\theta}) = (x-\bar{x}, y-\bar{y}, z-\bar{z}, k-\bar{k}, p-\bar{p}, q-\bar{q}, r-\bar{r}, \theta-\bar{\theta})$.

(ii) For any $x, \bar{x} \in \mathbb{R}^n$ and any $p, \bar{p} \in \mathbb{R}^m$,

$$\langle \Phi_1(x) - \Phi_1(\bar{x}), \hat{p} \rangle + \langle \Phi_2(x,p) - \Phi_2(\bar{x},\bar{p}), \hat{x} \rangle \geq 0, \quad \text{a.s.}$$

(iii) For any $y, \bar{y} \in \mathbb{R}^m$,

$$\langle \Psi(y) - \Psi(\bar{y}), \hat{y} \rangle \leq 0.$$

Now we are in the position to give the main result of this section.

Theorem 3.2 *Under Assumptions* 1, 2, *and* 3, *FBSDE* (3) *admits a unique solution*,

$$\big(\lambda(\cdot), \xi(\cdot)\big) = \big(x(\cdot), y(\cdot), z(\cdot), k(\cdot,\cdot), p(\cdot), q(\cdot), r(\cdot), \theta(\cdot,\cdot)\big) \in \mathcal{L}[0,T].$$

We shall employ the 'method of continuation' introduced by [26, 27] to prove Theorem 3.2. To this aim, we consider a family of FBSDEs parameterized by $\rho \in [0,1]$ as follows:

$$\begin{cases} dx_t^\rho = [-(1-\rho)lC_t^\top \xi_{t-}^{\rho\Sigma} + \rho\alpha_1(t, x_{t-}^\rho, \xi_{t-}^{\rho M}) + \mu_1(t)]\,dt \\ \qquad + [-(1-\rho)lD_t^\top \xi_{t-}^{\rho\Sigma} + \rho\beta_1(t, x_{t-}^\rho, \xi_{t-}^{\rho M}) + \nu_1(t)]\,dW_t \\ \qquad + \int_{\mathcal{E}} [-(1-\rho)lE_t^\top(e) \xi_{t-}^{\rho\Sigma} + \rho\gamma_1(t,e,x_{t-}^\rho, \xi_{t-}^{\rho M}) + \psi_1(t,e)]\widetilde{N}(dt,de), \\ -dy_t^\rho = [-(1-\rho)lB_t^\top \xi_{t-}^{\rho\Sigma} + \rho f_1(t, \lambda_{t-}^\rho, \xi_{t-}^{\rho M}) + \varphi_1(t)]\,dt - z_t^\rho\,dW_t \\ \qquad - \int_{\mathcal{E}} k_t^\rho(e)\widetilde{N}(dt,de), \\ dp_t^\rho = [\rho\alpha_2(t, \lambda_{t-}^\rho, \xi_{t-}^\rho) + \mu_2(t)]\,dt + [\rho\beta_2(t, \lambda_{t-}^\rho, \xi_{t-}^\rho) + \nu_2(t)]\,dW_t \\ \qquad + [\int_{\mathcal{E}} \rho\gamma_2(t,e,\lambda_{t-}^\rho, \xi_{t-}^\rho) + \psi_2(t,e)]\widetilde{N}(dt,de), \\ -dq_t^\rho = [\rho f_2(t, \lambda_{t-}^\rho, \xi_{t-}^\rho) + \varphi_2(t)]\,dt - r_t^\rho\,dW_t - \int_{\mathcal{E}} \theta_t^\rho(e)\widetilde{N}(dt,de), \\ x_0^\rho = a, \qquad q_T^\rho = \rho\Phi_2(x_T^\rho, q_T^\rho) + \eta_2, \\ p_0^\rho = \rho\Psi(y_0^\rho) + b, \qquad y_T^\rho = \rho\Phi_1(x_T^\rho) + \eta_1. \end{cases} \tag{7}$$

Clearly, when $\rho = 0$ the FBSDE (7) is in a decoupled form. From the classical theory of stochastic differential equations (SDEs) and backward stochastic differential equations (BSDEs), it is uniquely solvable. When $\rho = 1$ and $\kappa = (\mu_1(\cdot), \nu_1(\cdot), \psi_1(\cdot), \varphi_1(\cdot), \eta_1, \mu_2(\cdot), \nu_2(\cdot), \psi_2(\cdot), \varphi_2(\cdot), \eta_2, b)$ vanishes, FBSDE (7) coincides with FBSDE (3). The following lemma, which provides a path from the solvable case $\rho = 0$ to the desired unsolvable case $\rho = 1$, plays a key role.

Lemma 3.3 *Let Assumptions* 1, 2, *and* 3 *hold true. There exists a positive constant* δ_0 *such that, if for some* $\rho_0 \in [0,1)$, *there exists a unique solution* $(\lambda_t^{\rho_0}, \xi_t^{\rho_0})$ *of FBSDE* (7) *with any* $\kappa \in \mathcal{G}[0,T]$, *then there exists a unique solution* $(\lambda_t^{\rho_0+\delta}, \xi_t^{\rho_0+\delta})$ *of FBSDE* (7) *for* $\rho = \rho_0 + \delta$ *with* $\delta \in [0,\delta_0]$ *and* $\rho_0 + \delta \leq 1$.

Proof Let $\delta_0 > 0$ be undetermined, and $\delta \in [0,\delta_0]$. We consider the following FBSDE:

$$\begin{cases} dX_t = [-(1-\rho_0)lC_t^\top \Xi_{t-}^\Sigma + \rho_0\alpha_1(t,X_{t-},\Xi_{t-}^M) + \delta(lC_t^\top \xi_{t-}^\Sigma + \alpha_1(t,x_{t-},\xi_{t-}^M)) + \mu_1(t)]\,dt \\ \qquad + [-(1-\rho_0)lD_t^\top \Xi_{t-}^\Sigma + \rho_0\beta_1(t,X_{t-},\Xi_{t-}^M) + \delta(lD_t^\top \xi_{t-}^\Sigma + \beta_1(t,x_{t-},\xi_{t-}^M)) + \nu_1(t)]\,dW_t \\ \qquad + \int_{\mathcal{E}}[-(1-\rho_0)lE_t^\top(e)\Xi_{t-}^\Sigma + \rho_0\gamma_1(t,e,X_{t-},\Xi_{t-}^M) + \delta(lE_t^\top(e)\xi_{t-}^\Sigma + \gamma_1(t,e,x_{t-},\xi_{t-}^M)) \\ \qquad + \psi_1(t,e)]\widetilde{N}(dt,de), \\ -dY_t = [-(1-\rho_0)lB_t^\top \Xi_{t-}^\Sigma + \rho_0 f_1(t,\Lambda_t,\Xi_{t-}^M) + \delta(lB_t^\top \xi_{t-}^\Sigma + f_1(t,\lambda_t,\xi_{t-}^M)) + \varphi_1(t)]\,dt \\ \qquad - Z_t\,dW_t - \int_{\mathcal{E}} K_t(e)\widetilde{N}(dt,de), \\ dP_t = [\rho_0\alpha_2(t,\Lambda_{t-},\Xi_{t-}) + \delta\alpha_2(t,\lambda_{t-},\xi_{t-}) + \mu_2(t)]\,dt \\ \qquad + [\rho_0\beta_2(t,\Lambda_{t-},\Xi_{t-}) + \delta\beta_2(t,\lambda_{t-},\xi_{t-}) + \nu_2(t)]\,dW_t \\ \qquad + [\int_{\mathcal{E}} \rho_0\gamma_2(t,e,\Lambda_{t-},\Xi_{t-}) + \delta\gamma_2(t,e,\lambda_{t-},\xi_{t-}) + \psi_2(t,e)]\widetilde{N}(dt,de), \\ -dQ_t = [\rho_0 f_2(t,\Lambda_{t-},\Xi_{t-}) + \delta f_2(t,\lambda_{t-},\xi_{t-}) + \varphi_2(t)]\,dt - R_t\,dW_t - \int_{\mathcal{E}}\Theta_t(e)\widetilde{N}(dt,de), \\ X_0 = a, \qquad Q_T = \rho_0\Phi_2(X_T,P_T) + \delta\Phi_2(x_T,p_T) + \eta_2, \\ P_0 = \rho_0\Psi(Y_0) + \delta\Psi(y_0) + b, \qquad Y_T = \rho_0\Phi_1(X_T^\rho) + \delta\Phi_1(x_T^\rho) + \eta_1, \end{cases} \tag{8}$$

where, similar to (4), we denote

$$\begin{aligned} &\Lambda = (X,Y,Z,K), \qquad \Xi = (P,Q,R,\Theta), \\ &\Xi_t^M = \left(B_tP, C_tQ, D_tR, \int_{\mathcal{E}} E_t(e)\Theta(e)\pi(de)\right), \\ &\Xi_t^\Sigma = B_tP + C_tQ + D_tR + \int_{\mathcal{E}} E_t(e)\theta(e)\pi(de). \end{aligned}$$

Our assumption says, when $\rho = \rho_0$, for any $\kappa \in \mathcal{G}[0,T]$, FBSDE (7) admits a unique solution. Applying it to FBSDE (8), we have the result: for each $(\lambda(\cdot),\xi(\cdot)) = (x(\cdot),y(\cdot),z(\cdot),k(\cdot,\cdot), p(\cdot),q(\cdot),r(\cdot),\theta(\cdot,\cdot)) \in \mathcal{L}[0,T]$, FBSDE (8) admits a unique solution $(\Lambda(\cdot),\Xi(\cdot)) = (X(\cdot),Y(\cdot), Z(\cdot),K(\cdot,\cdot),P(\cdot),Q(\cdot),R(\cdot),\Theta(\cdot,\cdot)) \in \mathcal{L}[0,T]$. Consequently, this result implies a mapping:

$$(\Lambda,\Xi) = I_{\rho_0+\delta}(\lambda,\xi) : \mathcal{L}[0,T] \to \mathcal{L}[0,T]. \tag{9}$$

Next, we shall prove the above mapping is a contraction.

Let $(\bar{\lambda},\bar{\xi}) = (\bar{x},\bar{y},\bar{z},\bar{k},\bar{p},\bar{k},\bar{q},\bar{\theta}) \in \mathcal{L}[0,T]$ and $(\bar{\Lambda},\overline{\Xi}) = (\bar{X},\bar{Y},\bar{Z},\bar{K},\bar{P},\bar{Q},\bar{R},\bar{\Theta}) = I_{\rho_0+\delta}(\bar{\lambda},\bar{\xi})$. We set

$$\begin{aligned} (\hat{\lambda},\hat{\xi}) &= (\lambda-\bar{\lambda},\xi-\bar{\xi}) = (\hat{x},\hat{y},\hat{z},\hat{k},\hat{p},\hat{q},\hat{r},\hat{\theta}) \\ &= (x-\bar{x},y-\bar{y},z-\bar{z},k-\bar{k},p-\bar{p},q-\bar{q},r-\bar{r},\theta-\bar{\theta}) \end{aligned}$$

and

$$\begin{aligned} (\hat{\Lambda},\hat{\Xi}) &= (\Lambda-\bar{\Lambda},\Xi-\overline{\Xi}) = (\hat{X},\hat{Y},\hat{Z},\hat{K},\hat{P},\hat{Q},\hat{R},\hat{\Theta}) \\ &= (X-\bar{X},Y-\bar{Y},Z-\bar{Z},K-\bar{K},P-\bar{P},Q-\bar{Q},R-\bar{R},\Theta-\bar{\Theta}). \end{aligned}$$

Applying Itô's formula to $\langle\hat{X}_t,\hat{Q}_t\rangle + \langle\hat{Y}_t,\hat{P}_t\rangle$ yields

$$\begin{aligned} &\rho_0\mathbb{E}\bigl\{\bigl\langle\Phi_2(X_T,P_T) - \Phi_2(\bar{X}_T,\bar{P}_T),\hat{X}_T\bigr\rangle + \bigl\langle\Phi_1(X_T) - \Phi_1(\bar{X}_T),\hat{P}_T\bigr\rangle\bigr\} \\ &\quad - \rho_0\bigl\langle\Psi(Y_0) - \Psi(\bar{Y}_0),\hat{Y}_0\bigr\rangle + \delta\bigl[\mathbb{E}\bigl\{\bigl\langle\Phi_2(x_T,p_T) - \Phi_2(\bar{x}_T,\bar{p}_T),\hat{X}_T\bigr\rangle \\ &\quad + \bigl\langle\Phi_1(x_T) - \Phi_1(\bar{x}_T),\hat{P}_T\bigr\rangle\bigr\}\bigr] - \delta\bigl\langle\Psi(y_0) - \Psi(\bar{y}_0),\hat{Y}_0\bigr\rangle \end{aligned}$$

$$
\begin{aligned}
&= -(1-\rho_0)l\mathbb{E}\int_0^T \big|\hat{\Xi}_t^{\Sigma}\big|^2\,dt + \rho_0\mathbb{E}\int_0^T \big\langle A(t,\Lambda_t,\Xi_t) - A(t,\bar{\Lambda}_t,\overline{\Xi}_t),(\hat{\Lambda}_t,\hat{\Xi}_t)\big\rangle\,dt \\
&\quad + \delta l\mathbb{E}\int_0^T \bigg[\big\langle \hat{P}_t, B_t^\top \hat{\xi}_{t-}^{\Sigma}\big\rangle + \big\langle \hat{Q}_t, C_t^\top \hat{\xi}_{t-}^{\Sigma}\big\rangle + \big\langle \hat{R}_t, D_t^\top \hat{\xi}_{t-}^{\Sigma}\big\rangle + \int_{\mathcal{E}}\big\langle \hat{\Theta}_t(e), E_t^\top(e)\hat{\xi}_{t-}^{\Sigma}\big\rangle \pi(de)\bigg]\,dt \\
&\quad + \delta\mathbb{E}\int_0^T \bigg[\big\langle \hat{X}_t, -\hat{f}_2(t)\big\rangle + \big\langle \hat{Y}_t, \hat{\alpha}_2(t)\big\rangle + \big\langle \hat{Z}_t, \hat{\beta}_2(t)\big\rangle + \int_{\mathcal{E}}\big\langle \hat{K}_t(e), \hat{\gamma}_2(t,e)\big\rangle \pi(de) \\
&\quad + \big\langle \hat{P}_t, -\hat{f}_1(t)\big\rangle + \big\langle \hat{Q}_t, \hat{\alpha}_1(t)\big\rangle + \big\langle \hat{R}_t, \hat{\beta}_1(t)\big\rangle + \int_{\mathcal{E}}\big\langle \hat{\Theta}_t(e), \hat{\gamma}_1(t,e)\big\rangle \pi(de)\bigg]\,dt,
\end{aligned}
$$

where

$$
\begin{aligned}
&\hat{g}_1(t) = g_1\big(t,x,\xi_t^M\big) - g_1\big(t,\bar{x},\bar{\xi}_t^M\big), \qquad \hat{f}_1(t) = f_1(t,\lambda,\xi) - f_1(t,\bar{\lambda},\bar{\xi}), \\
&\hat{g}_2(t) = g_2(t,\lambda,\xi) - g_2(t,\bar{\lambda},\bar{\xi}),
\end{aligned}
$$

with $g_1 = \alpha_1, \beta_1, \gamma_1$ and $g_2 = \alpha_2, \beta_2, \gamma_2, f_2$. By Assumptions 1, 2, and 3, for any $\varepsilon > 0$ we have

$$
\begin{aligned}
&\mathbb{E}\int_0^T \bigg|B_t\hat{P}_{t-} + C_t\hat{Q}_{t-} + D_t\hat{R}_t + \int_{\mathcal{E}} E_t(e)\hat{\Theta}_t\pi(de)\bigg|^2\,dt \\
&\quad \le \varepsilon C\big\|(\hat{\Lambda}_t,\hat{\Xi}_t)\big\|^2 + \frac{\delta C}{\varepsilon}\big\|(\hat{\lambda}_t,\hat{\xi}_t)\big\|^2,
\end{aligned}
\tag{10}
$$

where $C > 0$ is a constant. Here and hereafter, C represents some generic constant which can be changed from line to line. We also apply Itô's formula to $|\hat{X}_t|^2$. By Assumptions 1 and 2, combining with Gronwall's inequality and Burkholder-Davis-Gundy's inequality, we have

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}|\hat{X}_t|^2 &\le C\mathbb{E}\int_0^T \bigg|B_t\hat{P}_{t-} + C_t\hat{Q}_{t-} + D_t\hat{R}_t + \int_{\mathcal{E}} E_t(e)\hat{\Theta}_t\pi(de)\bigg|^2\,dt \\
&\quad + \delta C\Big[\mathbb{E}\sup_{t\in[0,T]}|\hat{x}_t|^2 + \|\hat{\xi}_t\|^2\Big].
\end{aligned}
\tag{11}
$$

Similarly, by applying Itô's formula to $|\hat{Y}_t|^2$, $|\hat{P}_t|^2$ and $|\hat{Q}_t|^2$, respectively, we obtain three other estimates:

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|\hat{Y}_t|^2 + \mathbb{E}\int_0^T|\hat{Z}_t|^2\,dt + \mathbb{E}\int_0^T\int_{\mathcal{E}}\big|\hat{K}_t(e)\big|^2\pi(de)\,dt \\
&\quad \le C\mathbb{E}\int_0^T \bigg|B_t\hat{P}_{t-} + C_t\hat{Q}_{t-} + D_t\hat{R}_t + \int_{\mathcal{E}} E_t(e)\hat{\Theta}_t\pi(de)\bigg|^2\,dt \\
&\qquad + C\mathbb{E}\sup_{t\in[0,T]}|\hat{X}_t|^2 + \delta C\big\|(\hat{\lambda}_t,\hat{\xi}_t)\big\|^2,
\end{aligned}
\tag{12}
$$

$$
\mathbb{E}\sup_{t\in[0,T]}|\hat{P}_t|^2 \le C\|\hat{\Lambda}_t\|^2 + \delta C\big\|(\hat{\lambda}_t,\hat{\xi}_t)\big\|^2,
\tag{13}
$$

and

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|\hat{Q}_t|^2 + \mathbb{E}\int_0^T|\hat{R}_t|^2\,dt + \mathbb{E}\int_0^T\int_{\mathcal{E}}\big|\hat{\Theta}_t(e)\big|^2\pi(de)\,dt \\
&\quad \le C\|\hat{\Lambda}_t\|^2 + C\mathbb{E}\sup_{t\in[0,T]}|\hat{P}_t|^2 + \delta C\big\|(\hat{\lambda}_t,\hat{\xi}_t)\big\|^2.
\end{aligned}
\tag{14}
$$

By the combination of (10), (11), (12), (13), and (14) and taking $\varepsilon = 1/(2C)$, we have

$$\|(\hat{\Lambda}_t, \hat{\Xi}_t)\|^2 \leq \delta C \|(\hat{\lambda}_t, \hat{\xi}_t)\|^2. \tag{15}$$

We emphasize that the constant $C > 0$ appearing in the above inequality is independent of ρ_0 and δ. Now, we choose $\delta_0 > 0$ such that $C\delta_0 \leq 1/2$; then, for any $\delta \in [0, \delta_0]$,

$$\|(\hat{\Lambda}_t, \hat{\Xi}_t)\|^2 \leq \frac{1}{2} \|(\hat{\lambda}_t, \hat{\xi}_t)\|^2. \tag{16}$$

This implies that the mapping $I_{\rho_0+\delta}$ is a contraction. It follows immediately that the mapping admits a unique fixed point which is exactly the unique solution of (7) for $\rho = \rho_0 + \delta$. We thus complete the proof. □

Proof of Theorem 3.2 By Lemma 3.3, there exists a fixed step $\delta_0 > 0$ such that, if for a given $\rho_0 \in [0, 1)$, FBSDE (7) is uniquely solvable for any $\kappa \in \mathcal{G}[0, T]$ and any $a \in \mathbb{R}^n$, then (7) is also uniquely solvable for $\rho = \rho_0 + \delta$ with $\delta \in [0, \delta_0]$. Obviously, for $\rho = 0$, FBSDE (7) is in a decoupled form and then can be uniquely solved. So we can increase the parameter ρ step by step from $\rho = 0$ to $\rho = 1$. Especially, taking $\kappa = 0$ and $\rho = 1$, we get the desired conclusion: FBSDE (3) is uniquely solvable. □

4 LQ SOC problem

In this section, we apply the solvability result of FBSDEs studied in the above section to deal with a linear-quadratic stochastic optimal control (LQ SOC) problem for a forward-backward system with Poisson random jumps, in which the controlled system is given by

$$\begin{cases} dx_t = (A_t x_{t-} + B_t v_t + \mu_t)\,dt + (C_t x_{t-} + D_t v_t + \nu_t)\,dW_t \\ \qquad + \int_{\mathcal{E}} (E_t(e) x_{t-} + F_t(e) v_t + \psi_t(e)) \widetilde{N}(dt, de), \\ -dy_t = (G_t x_{t-} + H_t y_{t-} + I_t z_t + \int_{\mathcal{E}} J_t(e) k_t(e) \pi(de) + K_t v_t + \phi_t)\,dt, \\ \qquad - z_t\,dW_t - \int_{\mathcal{E}} k_t(e) \widetilde{N}(dt, de), \\ x_0 = a, \qquad y_T = \Upsilon x_T + \eta, \end{cases} \tag{17}$$

where A, B, C, D, E, F, G, H, I, J, K are $\mathbb{F}$-adapted matrix-valued bounded processes with appropriate dimensions, Υ is an $\mathcal{F}_T$-measurable $m \times n$ matrix-valued bounded random variable, $\mu, \nu \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, $\psi \in M^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, $\phi \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, $a \in \mathbb{R}^n$, and $\eta \in L^2(\mathbb{F}_T; \mathbb{R}^m)$. The admissible control set is denoted by $\mathcal{V} = L^2_{\mathbb{F}}(0, T; \mathbb{R}^k)$, in which each element $v(\cdot)$ is called an admissible control. Clearly, for any admissible control $v(\cdot)$, FBSDE (17) admits a unique adapted solution

$$\begin{aligned} \big(x^v(\cdot), y^v(\cdot), z^v(\cdot), k^v(\cdot,\cdot)\big) \in\ & \mathcal{C}^2_{\mathbb{F}}\big(0, T; \mathbb{R}^n\big) \times \mathcal{C}^2_{\mathbb{F}}\big(0, T; \mathbb{R}^m\big) \\ & \times L^2_{\mathbb{F}}\big(0, T; \mathbb{R}^m\big) \times M^2_{\mathbb{F}}\big(0, T; \mathbb{R}^m\big), \end{aligned}$$

and we called it the state trajectory corresponding to $v(\cdot)$. Additionally, we give a cost functional associated with $v(\cdot)$ in quadratic form as follows:

$$\begin{aligned} \mathcal{J}\big(v(\cdot)\big) = \frac{1}{2}\mathbb{E}\int_0^T \bigg[& \langle L_t x_t, x_t \rangle + \langle M_t y_t, y_t \rangle + \langle N_t z_t, z_t \rangle + \int_{\mathcal{E}} \big\langle O_t(e) k_t(e), k_t(e) \big\rangle \pi(de) \\ & + \langle \Gamma_t v_t, v_t \rangle \bigg] dt + \frac{1}{2}\mathbb{E}\langle R x_T, x_T \rangle + \frac{1}{2}\langle S y_0, y_0 \rangle. \end{aligned} \tag{18}$$

Here L, M, N, O are $\mathbb{F}$-adapted symmetric and nonnegative definite matrix-valued bounded processes with appropriate dimensions, Γ is an $\mathbb{F}$-adapted $k \times k$ symmetric and positive definite matrix-valued bounded process, and there exists a constant $\delta > 0$ such that

$$\langle \Gamma_t \nu, \nu \rangle \geq \delta |\nu|^2, \quad \text{a.s.}$$

for any $\nu \in \mathbb{R}^k$ and for almost every $t \in [0, T]$, R is an $\mathcal{F}_T$-measurable $n \times n$ symmetric and nonnegative definite matrix-valued bounded random variable, and S is an $m \times m$ symmetric and nonnegative definite matrix.

Problem (LQ SOC) The problem is to look for an admissible control $u(\cdot) \in \mathcal{V}$ which satisfies

$$\mathcal{J}\big(u(\cdot)\big) = \inf_{\nu(\cdot) \in \mathcal{V}} \mathcal{J}\big(\nu(\cdot)\big). \tag{19}$$

Such an admissible control $u(\cdot)$ is called an optimal control, and $(x(\cdot), y(\cdot), z(\cdot), k(\cdot, \cdot)) = (x^u(\cdot), y^u(\cdot), z^u(\cdot), k^u(\cdot, \cdot))$ is called the corresponding optimal trajectory.

The following theorem gives an explicit characterization of the unique optimal control.

Theorem 4.1 *Problem* (*LQ SOC*) *admits a unique optimal control which is in the following form*:

$$u_t = -\Gamma_t^{-1}\left(B_t^\top q_t + D_t^\top r_t + \int_{\mathcal{E}} F_t^\top(e)\theta_t(e)\pi(de) - K_t^\top p_t\right), \quad t \in [0, T], \tag{20}$$

where $(x(\cdot), y(\cdot), z(\cdot), k(\cdot, \cdot), p(\cdot), q(\cdot), r(\cdot), \theta(\cdot, \cdot)) \in \mathcal{L}[0, T]$ *is the unique solution of the following FBSDE* (*for the sake of simplicity, we denote it by FBSDE-LQ-SOC*):

$$\begin{cases} dx_t = (A_t x_{t-} - B_t \Gamma_t^{-1}(B_t^\top q_{t-} + D_t^\top r_t + \int_{\mathcal{E}} F_t^\top(e)\theta_t(e)\pi(de) - K_t^\top p_t) + \mu_t)\, dt \\ \qquad + (C_t x_{t-} - D_t \Gamma_t^{-1}(B_t^\top q_{t-} + D_t^\top r_t + \int_{\mathcal{E}} F_t^\top(e)\theta_t(e)\pi(de) - K_t^\top p_t) + \nu_t)\, dW_t \\ \qquad + \int_{\mathcal{E}}(E_t(e) x_{t-} - F_t(e)\Gamma_t^{-1}(B_t^\top q_{t-} + D_t^\top r_t + \int_{\mathcal{E}} F_t^\top(e)\theta_t(e)\pi(de) - K_t^\top p_t) \\ \qquad + \psi_t(e))\widetilde{N}(dt, de), \\ -dy_t = (G_t x_{t-} + H_t y_{t-} + I_t z_t + \int_{\mathcal{E}} J_t(e) k_t(e)\pi(de) - K_t \Gamma_t^{-1}(B_t^\top q_{t-} + D_t^\top r_t \\ \qquad + \int_{\mathcal{E}} F_t^\top(e)\theta_t(e)\pi(de) - K_t^\top p_{t-}) + \phi_t)\, dt - z_t\, dW_t - \int_{\mathcal{E}} k_t(e)\widetilde{N}(dt, de), \\ dp_t = (H_t^\top p_{t-} - M_t y_t)\, dt + (I_t^\top p_{t-} - N_t z_t)\, dW_t \\ \qquad + \int_{\mathcal{E}}(J_t^\top(e) p_{t-} - O_t(e) k_t(e))\widetilde{N}(dt, de), \\ -dq_t = (A_t^\top q_{t-} + C_t^\top r_t + \int_{\mathcal{E}} E_t^\top(e)\theta_t(e)\pi(de) - G_t^\top p_{t-} + L_t x_t)\, dt - r_t\, dW_t \\ \qquad - \int_{\mathcal{E}} \theta_t(e)\widetilde{N}(dt, de), \\ x_0 = a, \qquad p_0 = -S y_0, \\ y_T = \Upsilon x_T + \eta, \qquad q_T = R x_T - \Upsilon^\top p_T. \end{cases} \tag{21}$$

Proof By Theorem 3.2, FBSDE-LQ-SOC (21) admits a unique solution $(x(\cdot), y(\cdot), z(\cdot), k(\cdot, \cdot), p(\cdot), q(\cdot), r(\cdot), \theta(\cdot, \cdot)) \in \mathcal{L}[0, T]$.

Next, we prove that $u(\cdot)$ defined by (20) is an optimal control. For any given $v(\cdot) \in \mathcal{V}$, we denote the corresponding state trajectory by $(x^v(\cdot), y^v(\cdot), z^v(\cdot), k^v(\cdot,\cdot))$. By applying Itô's formula to $\langle x_t^v - x_t, q_t\rangle + \langle p_t, y_t^v - y_t\rangle$, we have

$$\begin{aligned}
&\mathbb{E}\langle Rx_T, x_T^v - x_T\rangle + \langle Sy_0, y_0^v - y_0\rangle \\
&\quad = -\mathbb{E}\int_0^T \biggl[\langle L_t x_t, x_t^v - x_t\rangle + \langle M_t y_t, y_t^v - y_t\rangle + \langle N_t z_t, z_t^v - z_t\rangle \\
&\qquad + \int_{\mathcal{E}} \langle O_t(e)k_t(e), k_t^v(e) - k_t(e)\rangle \pi(de) + \langle \Gamma_t u_t, v_t - u_t\rangle\biggr]\,dt. \qquad (22)
\end{aligned}$$

Then we analyze the difference between $\mathcal{J}(v(\cdot))$ and $\mathcal{J}(u(\cdot))$:

$$\begin{aligned}
&\mathcal{J}\bigl(v(\cdot)\bigr) - \mathcal{J}\bigl(u(\cdot)\bigr) \\
&\quad = \frac{1}{2}\mathbb{E}\int_0^T \biggl[\langle L_t x_t^v, x_t^v\rangle - \langle L_t x_t, x_t\rangle + \langle M_t y_t^v, y_t^v\rangle - \langle M_t y_t, y_t\rangle \\
&\qquad + \langle N_t z_t^v, z_t^v\rangle - \langle N_t z_t, z_t\rangle + \int_{\mathcal{E}} \langle O_t(e)k_t^v(e), k_t^v(e)\rangle \pi(de) - \int_{\mathcal{E}} \langle O_t(e)k_t(e), k_t(e)\rangle \pi(de) \\
&\qquad + \langle \Gamma_t v_t, v_t\rangle - \langle \Gamma_t u_t, u_t\rangle\biggr]\,dt + \frac{1}{2}\mathbb{E}\bigl[\langle Rx_T^v, x_T^v\rangle - \langle Rx_T, x_T\rangle\bigr] \\
&\qquad + \frac{1}{2}\bigl[\langle Sy_0^v, y_0^v\rangle - \langle Sy_0, y_0\rangle\bigr] \\
&\quad = \frac{1}{2}\mathbb{E}\int_0^T \biggl[\langle L_t(x_t^v - x_t), x_t^v - x_t\rangle + \langle M_t(y_t^v - y_t), y_t^v - y_t\rangle + \langle N_t(z_t^v - z_t), z_t^v - z_t\rangle \\
&\qquad + \int_{\mathcal{E}} \langle O_t(e)(k_t^v(e) - k_t(e)), k_t^v(e) - k_t(e)\rangle \pi(de) + \langle \Gamma_t(v_t - u_t), v_t - u_t\rangle\biggr]\,dt \\
&\qquad + \frac{1}{2}\mathbb{E}\langle R(x_T^v - x_T), x_T^v - x_T\rangle + \frac{1}{2}\langle S(y_0^v - y_0), y_0^v - y_0\rangle + \triangle, \qquad (23)
\end{aligned}$$

where

$$\begin{aligned}
\triangle &= \mathbb{E}\int_0^T \biggl[\langle L_t x_t, x_t^v - x_t\rangle + \langle M_t y_t, y_t^v - y_t\rangle + \langle N_t z_t, z_t^v - z_t\rangle \\
&\quad + \int_{\mathcal{E}} \langle O_t(e)k_t(e), k_t^v(e) - k_t(e)\rangle \pi(de) + \langle \Gamma_t u_t, v_t - u_t\rangle\biggr]\,dt \\
&\quad + \mathbb{E}\langle Rx_T, x_T^v - x_T\rangle + \langle Sy_0, y_0^v - y_0\rangle.
\end{aligned}$$

From (22), we have $\triangle = 0$. Moreover, since L, M, N, O, R, S are nonnegative definite and Γ_t is positive definite, from (23), we get

$$\mathcal{J}\bigl(v(\cdot)\bigr) - \mathcal{J}\bigl(u(\cdot)\bigr) \geq 0.$$

This implies $u(\cdot)$ defined by (20) is an optimal control.

Finally, we show the uniqueness of optimal control. As we proved above, u defined by (20) is an optimal control of Problem (LQ SOC). Let $\bar{u}$ be another optimal control, and denote the corresponding state trajectory by $(x^{\bar{u}}(\cdot), y^{\bar{u}}(\cdot), z^{\bar{u}}(\cdot), k^{\bar{u}}(\cdot,\cdot))$. Obviously, $\mathcal{J}(\bar{u}(\cdot)) =$

$\mathcal{J}(u(\cdot))$. Coming back to (23), we have

$$\begin{aligned}
0 &= \mathcal{J}\big(\bar{u}(\cdot)\big) - \mathcal{J}\big(u(\cdot)\big) \\
&= \frac{1}{2}\mathbb{E}\int_0^T \Big[\big\langle L_t\big(x_t^{\bar{u}} - x_t\big), x_t^{\bar{u}} - x_t\big\rangle + \big\langle M_t\big(y_t^{\bar{u}} - y_t\big), y_t^{\bar{u}} - y_t\big\rangle + \big\langle N_t\big(z_t^{\bar{u}} - z_t\big), z_t^{\bar{u}} - z_t\big\rangle \\
&\quad + \int_{\mathcal{E}} \big\langle O_t(e)\big(k_t^{\bar{u}}(e) - k_t(e)\big), k_t^{\bar{u}}(e) - k_t(e)\big\rangle \pi(de) + \big\langle \Gamma_t(\bar{u}_t - u_t), \bar{u}_t - u_t\big\rangle \Big] dt \\
&\quad + \frac{1}{2}\mathbb{E}\big\langle R\big(x_T^{\bar{u}} - x_T\big), x_T^{\bar{u}} - x_T\big\rangle + \frac{1}{2}\big\langle S\big(y_0^{v} - y_0\big), y_0^{v} - y_0\big\rangle \\
&\geq \mathbb{E}\int_0^T \big\langle \Gamma_t(\bar{u}_t - u_t), \bar{u}_t - u_t\big\rangle dt.
\end{aligned}$$

Because Γ_t is positive, we get $\bar{u}(\cdot) = u(\cdot)$. Due to the arbitrariness of $\bar{u}$, we obtain the uniqueness of the optimal control. □

5 LQ NZSSDG problem

Now we extend the LQ SOC problem to a linear-quadratic nonzero-sum stochastic differential game (LQ NZSSDG) problem for a forward-backward system with random jumps. Without loss of generality, we only consider the case of two players in this paper. The case of n (> 2) players can be treated in the same way. In detail, the game system is described by

$$\begin{cases} dx_t = (A_t x_{t-} + B_t^1 v_t^1 + B_t^2 v_t^2 + \mu_t)\,dt + (C_t x_{t-} + D_t^1 v_t^1 + D_t^2 v_t^2 + \nu_t)\,dW_t \\ \qquad + \int_{\mathcal{E}} (E_t(e) x_{t-} + F_t^1(e) v_t^1 + F_t^2(e) v_t^2 + \psi_t(e)) \widetilde{N}(dt, de), \\ -dy_t = (G_t x_{t-} + H_t y_{t-} + I_t z_t + \int_{\mathcal{E}} J_t(e) k_t(e) \pi(de) + K_t^1 v_t^1 + K_t^2 v_t^2 + \phi_t)\,dt, \\ \qquad - z_t\,dW_t - \int_{\mathcal{E}} k_t(e) \widetilde{N}(dt, de), \\ x_0 = a, \qquad y_T = \Upsilon x_T + \eta, \end{cases} \tag{24}$$

where A, B^1, B^2, C, D^1, D^2, E, F^1, F^2, G, H, I, J, K^1, K^2 are $\mathbb{F}$-adapted matrix-valued bounded processes with appropriate dimensions, Υ is an $\mathcal{F}_T$-measurable $m \times n$ matrix-valued bounded random variable, $\mu, \nu \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, $\psi \in M^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, $\phi \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, $a \in \mathbb{R}^n$, and $\eta \in L^2(\mathcal{F}_T; \mathbb{R}^m)$. Different from Problem (LQ SOC), there exist two control processes $v^1(\cdot)$ and $v^2(\cdot)$ belonging to two players, respectively, to affect the state of the game system at the same time. Let $\mathcal{V}^i = L^2_{\mathbb{F}}(0, T; \mathbb{R}^{k_i})$ $(i = 1, 2)$ denote the set of admissible controls. Each element $v^i(\cdot) \in \mathcal{V}^i$ is called an admissible control for Player i $(i = 1, 2)$. Moreover, $\mathcal{V}^1 \times \mathcal{V}^2$ is called the set of admissible controls for the players. For any given pair of admissible controls $(v^1(\cdot), v^2(\cdot)) \in \mathcal{V}^1 \times \mathcal{V}^2$, FBSDE (24) admits a unique solution

$$\begin{aligned}
\big(x^{v_1,v_2}(\cdot), y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot), k^{v_1,v_2}(\cdot,\cdot)\big) &\in \mathcal{C}^2_{\mathbb{F}}\big(0, T; \mathbb{R}^n\big) \times \mathcal{C}^2_{\mathbb{F}}\big(0, T; \mathbb{R}^m\big) \times L^2_{\mathbb{F}}\big(0, T; \mathbb{R}^m\big) \\
&\quad \times M^2_{\mathbb{F}}\big(0, T; \mathbb{R}^m\big),
\end{aligned}$$

which is called the state trajectory corresponding to $(v^1(\cdot), v^2(\cdot))$. Additionally, each player has his own cost functional:

$$\begin{aligned}
&\mathcal{J}^i\big(v^1(\cdot),v^2(\cdot)\big)\\
&\quad=\frac{1}{2}\mathbb{E}\int_0^T\bigg[\langle L_t^i x_t,x_t\rangle+\langle M_t^i y_t,y_t\rangle+\langle N_t^i z_t,z_t\rangle+\int_{\mathcal{E}}\langle O_t^i(e)k_t(e),k_t(e)\rangle\pi(de)\\
&\qquad+\langle\Gamma_t^i v_t^i,v_t^i\rangle\bigg]\,dt+\frac{1}{2}\mathbb{E}\langle R^i x_T,x_T\rangle+\frac{1}{2}\langle S^i y_0,y_0\rangle\quad(i=1,2).
\end{aligned}\tag{25}$$

Here for $i=1,2$, L^i, M^i, N^i, O^i are $\mathbb{F}$-adapted symmetric and nonnegative definite matrix-valued bounded processes with appropriate dimensions, Γ^i is an $\mathbb{F}$-adapted $k\times k$ symmetric and positive definite matrix-valued bounded process, and there exists a constant $\delta>0$ such that

$$\langle\Gamma_t^i v,v\rangle\geq\delta|v|^2,\quad\text{a.s.}$$

for any $v\in\mathbb{R}^k$ and for almost every $t\in[0,T]$, R^i is an $\mathcal{F}_T$-measurable $n\times n$ symmetric and nonnegative definite matrix-valued bounded random variable, and S^i is an $m\times m$ symmetric and nonnegative definite matrix.

Suppose each player hopes to minimize his cost functional $\mathcal{J}^i(v^1(\cdot),v^2(\cdot))$ by selecting an appropriate admissible control $v^i(\cdot)$ $(i=1,2)$, then the game problem (24)-(25) is formulated as follows.

Problem (LQ NZSSDG) The problem is to find a pair of admissible controls $(u^1(\cdot),u^2(\cdot))\in\mathcal{V}^1\times\mathcal{V}^2$ such that

$$\begin{aligned}
&\mathcal{J}^1\big(u^1(\cdot),u^2(\cdot)\big)=\inf_{v^1(\cdot)\in\mathcal{V}^1}\mathcal{J}^1\big(v^1(\cdot),u^2(\cdot)\big),\\
&\mathcal{J}^2\big(u^1(\cdot),u^2(\cdot)\big)=\inf_{v^2(\cdot)\in\mathcal{V}^2}\mathcal{J}^2\big(u^1(\cdot),v^2(\cdot)\big).
\end{aligned}\tag{26}$$

Such a pair of admissible controls $(u^1(\cdot),u^2(\cdot))$ is called a Nash equilibrium point. For notational convenience, we denote by $(x(\cdot),y(\cdot),z(\cdot),k(\cdot,\cdot))=(x^{u^1,u^2}(\cdot),y^{u^1,u^2}(\cdot),z^{u^1,u^2}(\cdot),k^{u^1,u^2}(\cdot,\cdot))$ the state trajectory corresponding to a Nash equilibrium point $(u^1(\cdot),u^2(\cdot))$.

Similar to Problem (LQ SOC) studied in the above section, we will link Nash equilibrium points of Problem (LQ NZSSDG) to solutions of some FBSDE.

Theorem 5.1 *$(u^1(\cdot),u^2(\cdot))$ is a Nash equilibrium point of Problem (LQ NZSSDG), if and only if $(u^1(\cdot),u^2(\cdot))$ has the form*

$$\begin{pmatrix}u_t^1\\u_t^2\end{pmatrix}=\begin{pmatrix}-(\Gamma_t^1)^{-1}[(B_t^1)^\top q_t^1+(D_t^1)^\top r_t^1+\int_{\mathcal{E}}(F_t^1)^\top(e)\theta_t^1(e)\pi(de)-(K_t^1)^\top p_t^1]\\-(\Gamma_t^2)^{-1}[(B_t^2)^\top q_t^2+(D_t^2)^\top r_t^2+\int_{\mathcal{E}}(F_t^2)^\top(e)\theta_t^2(e)\pi(de)-(K_t^2)^\top p_t^2]\end{pmatrix},\tag{27}$$

$t\in[0,T]$, where $(x(\cdot),y(\cdot),z(\cdot),k(\cdot,\cdot),p^1(\cdot),q^1(\cdot),r^1(\cdot),\theta^1(\cdot,\cdot),p^2(\cdot),q^2(\cdot),r^2(\cdot),\theta^2(\cdot,\cdot))$ satisfies the following FBSDE (for simplicity, we denote it by FBSDE-LQ-NZSSDG):

$$
\begin{cases}
dx_t = \{A_t x_{t-} - B_t^1(\Gamma_t^1)^{-1}[(B_t^1)^\top q_{t-}^1 + (D_t^1)^\top r_t^1 + \int_{\mathcal{E}} (F_t^1)^\top(e) k_t^1(e)\pi(de) - (K_t^1)^\top p_{t-}^1] \\
\qquad - B_t^2(\Gamma_t^2)^{-1}[(B_t^2)^\top q_{t-}^2 + (D_t^2)^\top r_t^2 + \int_{\mathcal{E}} (F_t^2)^\top(e) k_t^2(e)\pi(de) - (K_t^2)^\top p_{t-}^2] + \mu_t\}\,dt \\
\qquad + \{C_t x_{t-} - D_t^1(\Gamma_t^1)^{-1}[(B_t^1)^\top q_{t-}^1 + (D_t^1)^\top r_t^1 + \int_{\mathcal{E}} (F_t^1)^\top(e) k_t^1(e)\pi(de) - (K_t^1)^\top p_{t-}^1] \\
\qquad - D_t^2(\Gamma_t^2)^{-1}[(B_t^2)^\top q_{t-}^2 + (D_t^2)^\top r_t^2 + \int_{\mathcal{E}} (F_t^2)^\top(e) k_t^2(e)\pi(de) - (K_t^2)^\top p_{t-}^2] + \nu_t\}\,dW_t \\
\qquad + \int_{\mathcal{E}} \{E_t(e) x_{t-} - F_t^1(e)(\Gamma_t^1)^{-1}[(B_t^1)^\top q_{t-}^1 + (D_t^1)^\top r_t^1 + \int_{\mathcal{E}} (F_t^1)^\top(e) k_t^1(e)\pi(de) \\
\qquad - (K_t^1)^\top p_{t-}^1] - F_t^2(e)(\Gamma_t^2)^{-1}((B_t^2)^\top q_{t-}^2 + (D_t^2)^\top r_t^2 + \int_{\mathcal{E}} (F_t^2)^\top(e) k_t^2(e)\pi(de) \\
\qquad - (K_t^2)^\top p_{t-}^2) + \psi_t(e)\}\widetilde{N}(dt,de), \\
-dy_t = \{G_t x_{t-} + H_t y_{t-} + I_t z_t + \int_{\mathcal{E}} J_t(e) k_t(e)\pi(de) - K_t^1(\Gamma_t^1)^{-1}[(B_t^1)^\top q_{t-}^1 + (D_t^1)^\top r_t^1 \\
\qquad + \int_{\mathcal{E}} (F_t^1)^\top(e) k_t^1(e)\pi(de) - (K_t^1)^\top p_{t-}^1] - K_t^2(\Gamma_t^2)^{-1}[(B_t^2)^\top q_{t-}^2 + (D_t^2)^\top r_t^2 \\
\qquad + \int_{\mathcal{E}} (F_t^2)^\top(e) k_t^2(e)\pi(de) - (K_t^2)^\top p_{t-}^2] + \phi_t\}\,dt - z_t\,dW_t - \int_{\mathcal{E}} k_t(e)\widetilde{N}(dt,de), \\
dp_t^1 = (H_t^\top p_{t-}^1 - M_t^1 y_{t-})\,dt + (I_t^\top p_{t-}^1 - N_t^1 z_t)\,dW_t + \int_{\mathcal{E}} (J_t^\top(e) p_{t-}^1 - O_t^1(e) k_t(e))\widetilde{N}(dt,de), \\
dp_t^2 = (H_t^\top p_{t-}^2 - M_t^2 y_{t-})\,dt + (I_t^\top p_{t-}^2 - N_t^2 z_t)\,dW_t \\
\qquad + \int_{\mathcal{E}} (J_t^\top(e) p_{t-}^2 - O_t^2(e) k_t(e))\widetilde{N}(dt,de), \\
-dq_t^1 = (A_t^\top q_{t-}^1 + C_t^\top r_t^1 + \int_{\mathcal{E}} E_t^\top(e)\theta_t^1(e)\pi(de) - G_t^\top p_{t-}^1 + L_t^1 x_{t-})\,dt - r_t^1\,dW_t \\
\qquad - \int_{\mathcal{E}} \theta_t^1(e)\widetilde{N}(dt,de), \\
-dq_t^2 = (A_t^\top q_{t-}^2 + C_t^\top r_t^2 + \int_{\mathcal{E}} E_t^\top(e)\theta_t^2(e)\pi(de) - G_t^\top p_{t-}^2 + L_t^2 x_{t-})\,dt - r_t^2\,dW_t \\
\qquad - \int_{\mathcal{E}} \theta_t^2(e)\widetilde{N}(dt,de), \\
x_0 = a, \qquad y_T = \Upsilon x_T + \eta, \\
p_0^1 = -S^1 y_0, \qquad q_T^1 = R^1 x_T - \Upsilon^\top p_T^1, \\
p_0^2 = -S^2 y_0, \qquad q_T^2 = R^2 x_T - \Upsilon^\top p_T^2.
\end{cases}
\tag{28}
$$

Proof Noticing the definition of Nash equilibrium point, we link Problem (LQ NZSSDG) with two Problems (LQ SOC) studied in the above section. Precisely, for $i = 1,2$, we denote $j = 3 - i$. Fix $u^j(\cdot)$, to minimize the following cost functional:

$$
\mathcal{J}^i\big(v^i(\cdot), u^j(\cdot)\big) = \frac{1}{2}\mathbb{E}\int_0^T \bigg[\langle L_t^i x_t, x_t\rangle + \langle M_t^i y_t, y_t\rangle + \langle N_t^i z_t, z_t\rangle + \int_{\mathcal{E}} \langle O_t^i(e) k_t(e), k_t(e)\rangle \pi(de) \\
+ \langle \Gamma_t^i v_t^i, v_t^i\rangle\bigg]\,dt + \frac{1}{2}\mathbb{E}\langle R^i x_T, x_T\rangle + \frac{1}{2}\langle S^i y_0, y_0\rangle, \tag{29}
$$

subject to

$$
\begin{cases}
dx_t = [A_t x_{t-} + B_t^i v_t^i + (B_t^j v_t^j + \mu_t)]\,dt + [C_t x_{t-} + D_t^i v_t^i + (D_t^j v_t^j + \nu_t)]\,dW_t \\
\qquad + \int_{\mathcal{E}} [E_t(e) x_{t-} + F_t^i(e) v_t^i + (F_t^j(e) v_t^j + \psi_t(e))]\widetilde{N}(dt,de), \\
-dy_t = [G_t x_{t-} + H_t y_{t-} + I_t z_t + \int_{\mathcal{E}} J_t(e) k_t(e)\pi(de) + K_t^i v_t^i + (K_t^j v_t^j + \phi_t)]\,dt \\
\qquad - z_t\,dW_t - \int_{\mathcal{E}} k_t(e)\widetilde{N}(dt,de), \\
x_0 = a, \qquad y_T = \Upsilon x_T + \eta,
\end{cases}
\tag{30}
$$

over $\mathcal{V}^i$ is an LQ SOC problem. By Theorem 4.1, the unique optimal control of LQ SOC problem (29)-(30) has the following form:

$$
u_t^i = -\big(\Gamma_t^i\big)^{-1}\bigg[\big(B_t^i\big)^\top q_t^i + \big(D_t^i\big)^\top r_t^i + \int_{\mathcal{E}} \big(F_t^i\big)^\top(e)\theta_t^i(e)\pi(de) + \big(K_t^i\big)^\top p_t^i\bigg], \quad t \in [0,T],
$$

where $(x(\cdot), y(\cdot), z(\cdot), k(\cdot,\cdot), p^i(\cdot), q^i(\cdot), r^i(\cdot), \theta^i(\cdot,\cdot))$ is the unique solution of the following FBSDEF-LQ-SOC ($i = 1,2$):

$$
\begin{cases}
dx_t = \{A_t x_{t-} - B_t^i(\Gamma_t^i)^{-1}[(B_t^i)^\top q_{t-}^i + (D_t^i)^\top r_t^i + \int_{\mathcal{E}}(F_t^i)^\top(e)\theta_t^i(e)\pi(de) - (K_t^i)^\top p_{t-}^i] \\
\qquad + (B_t^j u_t^j + \mu_t)\}\,dt + \{C_t x_{t-} - D_t^i(\Gamma_t^i)^{-1}[(B_t^i)^\top q_{t-}^i + (D_t^i)^\top r_t^i \\
\qquad + \int_{\mathcal{E}}(F_t^i)^\top(e)\theta_t^i(e)\pi(de) \\
\qquad - (K_t^i)^\top p_{t-}^i] + (D_t^j u_t^j + \nu_t)\}\,dW_t + \int_{\mathcal{E}}\{E_t(e)x_{t-} - F_t^i(e)(\Gamma_t^i)^{-1}[(B_t^i)^\top q_{t-}^i + (D_t^i)^\top r_t^i \\
\qquad + \int_{\mathcal{E}}(F_t^i)^\top(e)k_t^i(e)\pi(de) - (K_t^i)^\top p_{t-}^i] + (F_t^j(e)u_t^j + \psi_t(e))\}\widetilde{N}(dt,de), \\
-dy_t = \{G_t x_{t-} + H_t y_{t-} + I_t z_t + \int_{\mathcal{E}} J_t(e)k_t(e)\pi(de) - K_t^i(\Gamma_t^i)^{-1}[(B_t^i)^\top q_{t-}^i + (D_t^i)^\top r_t^1 \\
\qquad + \int_{\mathcal{E}}(F_t^i)^\top(e)k_t^i(e)\pi(de) - (K_t^i)^\top p_{t-}^i] + (K_t^j u_t^j + \phi_t)\}\,dt - z_t\,dW_t \\
\qquad - \int_{\mathcal{E}} k_t(e)\widetilde{N}(dt,de), \\
dp_t^i = (H_t^\top p_{t-}^i - M_t^i y_{t-})\,dt + (I_t^\top p_{t-}^i - N_t^i z_t)\,dW_t + \int_{\mathcal{E}}(J_t^\top(e)p_{t-}^i - O_t^i(e)k_t(e))\widetilde{N}(dt,de), \\
-dq_t^i = (A_t^\top q_{t-}^i + C_t^\top r_t^i + \int_{\mathcal{E}} E_t^\top(e)\theta_t^i(e)\pi(de) - G_t^\top p_{t-}^i + L_t^i x_{t-})\,dt - r_t^i\,dW_t \\
\qquad - \int_{\mathcal{E}}\theta_t^i(e)\widetilde{N}(dt,de), \\
p_0^i = -S^i y_0, \qquad q_T^i = R^i x_T - \Upsilon^\top p_T^i.
\end{cases}
\tag{31}
$$

Combining the two FBSDE-LQ-SOCs (31) for the cases $i = 1$ and $i = 2$, we get FBSDE-LQ-NZSSDG (28), and we thus finish the proof. □

Next, we would like to employ and extend the linear transform introduce by Hamadène [15] to discuss the solvability of FBSDE-LQ-NZSSDG (28). To this aim, we introduce the following assumptions.

Assumption 4

(i) The dimension of x is equal to that of y, *i.e.* $n = m$.

(ii) There exist four constants $\zeta_B, \zeta_D, \zeta_F, \zeta_K \in \mathbb{R}$ and two $\mathbb{F}$-adapted matrix-valued bounded processes $\mathcal{Q}_t^1$ and $\mathcal{Q}_t^2$ with appropriate dimensions, such that

$$
\begin{aligned}
&B_t^i = \zeta_B \mathcal{Q}_t^i, \qquad D_t^i = \zeta_D \mathcal{Q}_t^i, \\
&F_t^i(e) = \zeta_F \mathcal{Q}_t^i, \qquad K_t^i = \zeta_K \mathcal{Q}_t^i, \quad t \in [0,T], i = 1,2.
\end{aligned}
\tag{32}
$$

(iii) The matrix-valued processes $\mathcal{Q}_t^i(\Gamma_t^i)^{-1}(\mathcal{Q}_t^i)^\top$ $(i = 1,2)$, are independent of t.

(iv) The following commutation relations among matrices hold true:

$$
\mathcal{Q}_t^i\big(\Gamma_t^i\big)^{-1}\big(\mathcal{Q}_t^i\big)^\top \mathcal{P} = \mathcal{P}\mathcal{Q}_t^i\big(\Gamma_t^i\big)^{-1}\big(\mathcal{Q}_t^i\big)^\top, \quad t \in [0,T], i = 1,2,
\tag{33}
$$

where $\mathcal{P} = A_t^\top, C_t^\top, E_t^\top, G_t^\top, H_t^\top, I_t^\top, J_t^\top, L_t^i, M_t^i, N_t^i, O_t^i, R^i, S^i, K_t^i, \Gamma_t^i$.

We note that Assumption 4 is easy to satisfy when the matrix-valued processes appearing in the game system (24) and the cost functionals (25) are valued in $\mathbb{R}$ and independent of t.

Theorem 5.2 *Under Assumption* 4, *FBSDE-LQ-NZSSDG* (28) *is uniquely solvable. Moreover,* $(u^1(\cdot), u^2(\cdot))$ *defined by* (27) *is the unique Nash equilibrium point of Problem* (*LQ NZSSDG*).

Proof Due to Theorem 5.1, we only need to prove the existence and uniqueness of FBSDE-LQ-NZSSDG (28). To this aim, we introduce another FBSDE:

$$
\begin{cases}
d\tilde{x}_t = [A_t\tilde{x}_{t-} - (\zeta_B)^2\tilde{q}_{t-} - \zeta_B\zeta_D\tilde{r}_t - \zeta_B\zeta_F\int_{\mathcal{E}}\tilde{\theta}_t(e)\pi(de) + \zeta_B\zeta_K\tilde{p}_{t-} + \mu_t]\,dt \\
\qquad + [C_t\tilde{x}_{t-} - \zeta_B\zeta_D\tilde{q}_{t-} - (\zeta_D)^2\tilde{r}_t - \zeta_D\zeta_F\int_{\mathcal{E}}\tilde{\theta}_t(e)\pi(de) + \zeta_D\zeta_K\tilde{p}_{t-} + \nu_t]\,dW_t \\
\qquad + \int_{\mathcal{E}}[E_t(e)\tilde{x}_{t-} - \zeta_B\zeta_F\tilde{q}_{t-} - \zeta_D\zeta_F\tilde{r}_t \\
\qquad - (\zeta_F)^2\int_{\mathcal{E}}\tilde{\theta}_t(e)\pi(de) + \zeta_F\zeta_K\tilde{p}_{t-} + \psi_t(e)]\widetilde{N}(dt,de), \\
-d\tilde{y}_t = [G_t\tilde{x}_{t-} + H_t\tilde{y}_{t-} + I_t\tilde{z}_t + \int_{\mathcal{E}}J_t(e)\tilde{k}_t(e)\pi(de) \\
\qquad - \zeta_B\zeta_K\tilde{q}_{t-} - \zeta_D\zeta_K\tilde{r}_t - \zeta_F\zeta_K\int_{\mathcal{E}}\tilde{\theta}_t(e)\pi(de) \\
\qquad + (\zeta_K)^2\tilde{p}_{t-} + \phi_t]\,dt - \tilde{z}_t\,dW_t - \int_{\mathcal{E}}\tilde{k}_t(e)\widetilde{N}(dt,de), \\
d\tilde{p}_t = \{H_t^\top\tilde{p}_{t-} - [\mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top M_t^1 + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top M_t^2]\tilde{y}_{t-}\}\,dt \\
\qquad + \{I_t^\top\tilde{p}_{t-} - [\mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top N_t^1 + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top N_t^2]\tilde{z}_t\}\,dW_t \\
\qquad + \int_{\mathcal{E}}\{J_t^\top(e)\tilde{p}_{t-} - [\mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top O_t^1(e) + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top O_t^2(e)]\tilde{k}_t(e)\}\widetilde{N}(dt,de), \\
-d\tilde{q}_t = \{A_t^\top\tilde{q}_{t-} + C_t^\top\tilde{r}_t + \int_{\mathcal{E}}E_t^\top(e)\tilde{\theta}_t(e)\pi(de) - G_t^\top\tilde{p}_{t-} + [\mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top L_t^1 \\
\qquad + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top L_t^2]\tilde{x}_{t-}\}\,dt - \tilde{r}_t\,dW_t - \int_{\mathcal{E}}\tilde{\theta}_t(e)\widetilde{N}(dt,de), \\
\tilde{x}_0 = a, \qquad \tilde{p}_0 = -[\mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top S^1 + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top S^2]\tilde{y}_0, \\
\tilde{y}_T = \Upsilon\tilde{x}_T + \eta, \qquad \tilde{q}_T = [\mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top R^1 + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top R^2]\tilde{x}_T - \Upsilon^\top\tilde{p}_T,
\end{cases}
\tag{34}
$$

and we shall show that the unique solvability of (28) is equivalent to that of (34). On the one hand, based on the commutation relations among matrices (see (33)), if $(x,y,z,k,p^1,q^1,r^1,\theta^1,p^2,q^2,r^2,\theta^2)$ is a solution of FBSDE-LQ-NZSSDG (28), then $(\tilde{x},\tilde{y},\tilde{z},\tilde{k},\tilde{p},\tilde{q},\tilde{r},\tilde{\theta})$ solves FBSDE (34), where

$$
\begin{cases}
\tilde{x}_t = x_t, \qquad \tilde{y}_t = y_t, \qquad \tilde{z}_t = z_t, \qquad \tilde{k}_t = k_t, \\
\tilde{p}_t = \mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top p_t^1 + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top p_t^2, \\
\tilde{q}_t = \mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top q_t^1 + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top q_t^2, \\
\tilde{r}_t = \mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top r_t^1 + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top r_t^2, \\
\tilde{\theta}_t = \mathcal{Q}_t^1(\Gamma_t^1)^{-1}(\mathcal{Q}_t^1)^\top \theta_t^1 + \mathcal{Q}_t^2(\Gamma_t^2)^{-1}(\mathcal{Q}_t^2)^\top \theta_t^2.
\end{cases}
$$

On the other hand, when $(\tilde{x},\tilde{y},\tilde{z},\tilde{k},\tilde{p},\tilde{q},\tilde{r},\tilde{\theta})$ is a solution of FBSDE (34), we let $x(\cdot) = \tilde{x}(\cdot)$, $y(\cdot) = \tilde{y}(\cdot)$, $z(\cdot) = \tilde{z}(\cdot)$, $k(\cdot,\cdot) = \tilde{k}(\cdot,\cdot)$. From the existence and uniqueness results of SDEs and BSDEs, the equation

$$
\begin{cases}
dp_t^1 = (H_t^\top p_{t-}^1 - M_t^1\tilde{y}_{t-})\,dt + (I_t^\top p_{t-}^1 - N_t^1\tilde{z}_t)\,dW_t \\
\qquad + \int_{\mathcal{E}}(J_t^\top(e)p_{t-}^1 - O_t^1(e)\tilde{k}_t(e))\widetilde{N}(dt,de), \\
dp_t^2 = (H_t^\top p_{t-}^2 - M_t^2\tilde{y}_{t-})\,dt + (I_t^\top p_{t-}^2 - N_t^2\tilde{z}_t)\,dW_t \\
\qquad + \int_{\mathcal{E}}(J_t^\top(e)p_{t-}^2 - O_t^2(e)\tilde{k}_t(e))\widetilde{N}(dt,de), \\
-dq_t^1 = (A_t^\top q_{t-}^1 + C_t^\top r_t^1 + \int_{\mathcal{E}}E_t^\top(e)\theta_t^1(e)\pi(de) - G_t^\top p_{t-}^1 + L_t^1\tilde{x}_{t-})\,dt - r_t^1\,dW_t \\
\qquad - \int_{\mathcal{E}}\theta_t^1(e)\widetilde{N}(dt,de), \\
-dq_t^2 = (A_t^\top q_{t-}^2 + C_t^\top r_t^2 + \int_{\mathcal{E}}E_t^\top(e)\theta_t^2(e)\pi(de) - G_t^\top p_{t-}^2 + L_t^2\tilde{x}_{t-})\,dt - r_t^2\,dW_t \\
\qquad - \int_{\mathcal{E}}\theta_t^2(e)\widetilde{N}(dt,de), \\
p_0^1 = -S^1\tilde{y}_0, \qquad q_T^1 = R^1\tilde{x}_T - \Upsilon^\top p_T^1, \\
p_0^2 = -S^2\tilde{y}_0, \qquad q_T^2 = R^2\tilde{x}_T - \Upsilon^\top p_T^2,
\end{cases}
$$

admits a unique solution denoted by $(p^1(\cdot), q^1(\cdot), r^1(\cdot), \theta^1(\cdot,\cdot), p^2(\cdot), q^2(\cdot), r^2(\cdot), \theta^2(\cdot,\cdot))$. It is easy to check $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}(\cdot), p^1(\cdot), q^1(\cdot), r^1(\cdot), \theta^1(\cdot,\cdot), p^2(\cdot), q^2(\cdot), r^2(\cdot), \theta^2(\cdot,\cdot))$ defined above is a solution of FBSDE-LQ-NZSSDG (28), *i.e.* the existence of (28) is equivalent to that of (34). Moreover, in a similar way, one can prove that the uniqueness of (28) is also equivalent to that of (34).

It is easy to check the coefficients of FBSDE (34) satisfy Assumption 1, Assumption 2, and Assumption 3. By Theorem 3.2, FBSDE (34) admits a unique solution, then the same is true for FBSDE-LQ-NZSSDG (34). The proof is completed. □

Remark 5.3 In [15–18], when they studied nonzero-sum game problems, some stronger assumptions, such as the diffusion of the forward equation in the game system (24) does not depend on the controls $u_1(\cdot)$ and $u_2(\cdot)$ (*i.e.*, $D_i = 0$, $i = 1, 2$) and so on, are imposed. Here we relax this kind of conditions to (32).

Competing interests
The authors declare that there are no competing interest regarding the publication of this paper.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, QiLu Normal University, Jinan, 250013, China. [2]School of Mathematics, Shandong University, Jinan, 250100, China.

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