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# Constrained local controllability of dynamic systems on time scales

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**Abstract**
The problem of controllability to a given convex target set of a linear time-varying control system on time scales is studied. Necessary and sufficient conditions of controllability with constrained controllers for such a system are given. To this aim the separation theorem is used.
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**Keywords:** time-varying control system; controllability with constrained control; separation theorem; time scale

## 1 Introduction

Controllability is one of the fundamental concepts in control theory. Kalman's classic result on the controllability assumes that controls are functions on time with values on some nonempty subset $\Omega$ of $\mathbb{R}^m$. In many practical situations this set should be bounded; see, for example, [1]. A restriction on controls poses some difficulties for controllability conditions. In [2, 3] necessary and sufficient conditions for controllability of linear time-varying continuous-time systems with control defined on some subset of $\mathbb{R}^m$ not containing zero are proved. For the discrete-time case, a similar problem has been studied, for example, in [4].

An analysis on time scale is nowadays recognized as a good tool for unifying and extending the existing results for continuous- and discrete-time dynamical models to nonhomogeneous time domains; see, for example, [5–8]. Time scale is a general model of time. Time, for example, may be continuous or discrete, or partially continuous and partially discrete, or nonhomogeneous. In particular, the time scale theory has been used for a description of hybrid systems or for the cases in which the nonuniform sampling is required; see, for example, [9–12].

One of the main concepts of time scale theory is the delta derivative, which is a generalization of the classical time derivative in the continuous time and the finite forward difference in the discrete time. This allows one to consider delta differential equations on an arbitrary time model. As a consequence, differential equations as well as difference equations are naturally accommodated in this theory [13, 14]. However, the discrete-time systems on time scales are based on the difference operator and not on the classical forward shift operator.

The goal of this paper is to study conditions under which a linear time-varying system with control constrains defined on a time scale is controllable. The considered problem

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concerns the question of the existence of an admissible control to a given convex target set $X$ from a specified initial state. We assume that a set of values of control $\Omega$ is convex and bounded. Using the separation hyperplane theorem, we give the necessary and sufficient conditions for constrained controllability. The idea of using this result comes from Schmitendorf and his coauthors; see, for example, [2–4] and [15]. Let us recall this theorem. Let $X$ be a real normed space and $A, B \subset X$.

**Theorem 1** [16] *Suppose that A and B are convex and disjoint sets. Let the interior of A be nonempty. Then sets A and B are separable.*

Theorem 1 implies the existence of a linear continuous functional $f$ such that

$$\sup_{y\in B} f(y) \le \inf_{x\in A} f(x). \tag{1}$$

The paper is organized as follows. Section 2 presents basic definitions and ideas of time scale calculus. Section 3 extends the classical notation of constrained controllability to the case of time-varying systems defined on any, also nonuniform, model of time. Sufficient and necessary conditions for constrained controllability on time scales are given. The condition for $\Omega$-controllability given in Section 4 shows that in a stationary case the obtained results can be approximated by the classical exponential function.

## 2 Time scales calculus

In this section the basic ideas from the time scale calculus are presented. More can be found in [13, 14].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. If $a < b$, then $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \le t \le b\}$.

For all $t \in \mathbb{T}$, the *forward jump operator* is defined as $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. If $t < \max \mathbb{T}$ and $\sigma(t) = t$, we say that $t$ is *right-dense*. If $t > \min \mathbb{T}$ and $\rho(t) = t$, we say that $t$ is *left-dense*. Finally, the graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, hence $\mu(t) = 0$. If $\mathbb{T} = h\mathbb{Z}$, where $h > 0$, then $\sigma(t) = t + h$, hence $\mu(t) = h$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The function $f$ is *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The function $f$ is *piecewise rd-continuous* if it is regulated and if it is rd-continuous at all, except possibly at finitely many, right-dense points $t \in \mathbb{T}$.

Let $\mathbb{T}^\kappa := \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ if $\sup \mathbb{T} < \infty$ and $\mathbb{T}^\kappa := \mathbb{T}$ if $\sup \mathbb{T} = \infty$. If $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, then *the delta derivative* of $x$ at $t$, denoted by $x^\Delta(t)$, is the real number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $V$ of $t$ such that $|[x(\sigma(t)) - x(s)] - x^\Delta(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|$ for all $s \in V$.

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) $D$, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$. For any regulated function $f$, there exists a function $F$ that is pre-differentiable with region of differentiation $D$ such that $F^\Delta(t) = f(t)$ for all $t \in D$. Such $F$ is called pre-antiderivative of $f$. *The Cauchy integral* for all $r, s \in \mathbb{T}^k$ is defined by $\int_r^s f(t)\Delta t :=$

$F(s) - F(r)$. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b x(t)\Delta t = \int_a^b x(t)\,dt$. If $[a,b]_{\mathbb{T}}$ consists of only isolated points, then $\int_a^b x(t)\Delta t = \sum_{t\in[a,b)_{\mathbb{T}}} \mu(t)x(t)$, for $a < b$.

## 3 Controllability with control constrains

Let $t_0 \in \mathbb{T}$ and $A(\cdot) \in \mathbb{R}^{n\times n}$, $X(\cdot) \in \mathbb{R}^{n\times n}$ be real-valued matrices and $A$ has rd-continuous elements. Let us consider the $\Delta$-differential equation

$$X^{\Delta}(t) = A(t)X(t). \tag{2}$$

A solution to equation (2) is a function $x_{ij} : [t_0, \sup\mathbb{T})_{\mathbb{T}} \to \mathbb{R}$ satisfying (2). For every initial condition $X(t_0) = X_0$, there exists a unique forward solution defined on $[t_0, t_1]_{\mathbb{T}}$. This forward matrix-valued solution of (2), for the initial condition $X(t_0) = I$, is called the *exponential function of* $A(t)$ (*a centered at* $t_0$). It is defined for all $t \in [t_0, \sup\mathbb{T})_{\mathbb{T}}$. Its value at $t$ is denoted by $e_{A(t)}(t, t_0)$. In particular, the exponential function on time scale is a real-valued function that is never equal to zero, but can be negative, see [13].

Let us consider a linear control system defined on a time scale $\mathbb{T}$:

$$x^{\Delta}(t) = A(t)x(t) + B(t)u(t), \tag{3}$$

where $x(\cdot) \in \mathbb{R}^n$ is a state vector of (3), $u(\cdot) \in \mathbb{R}^m$ is a piecewise rd-continuous control vector function, $A(\cdot) \in \mathbb{R}^{n\times n}$ and $B(\cdot) \in \mathbb{R}^{n\times m}$ are rd-continuous matrices. Solution of equation (3) for the initial state $x(t_0) = x_0$, $t_0 \in \mathbb{T}$, and control $u(\cdot)$, for any $t \in [t_0, \sup\mathbb{T})_{\mathbb{T}}$ is given by (see [17])

$$\psi(t, t_0, x_0, u) = e_{A(t)}(t, t_0)x_0 + \int_{t_0}^{t} e_{A(t)}\bigl(t, \sigma(s)\bigr)B(s)u(s)\Delta s.$$

Let us suppose that $t_0$ is an arbitrary point from $\mathbb{T}$. We assume that $\mathbb{T}$ consists of at least $n+1$ elements and $[t_0 - \rho(t_0), t_1 + \sigma(t_1)]_{\mathbb{T}}$ is compact. Let $\Omega$ be a set of all at least piecewise rd-continuous functions $u : \mathbb{T} \to \mathbb{R}^m$. The set of all states that can be reached by system (3) from $x_0 = x(t_0)$ in time $t_1 \in \mathbb{T}$ is called a *reachable set* from $x_0$ in $t_1$ and denoted by $\mathcal{R}_{x_0}(t_1)$, see [17], *i.e.*,

$$\mathcal{R}_{x_0}(t_1) := \bigl\{x \in \mathbb{R}^n : x(t_1) = \psi(t_1, t_0, x_0, u) \text{ for some } u \in \Omega\bigr\}.$$

**Proposition 2** *If* $\Omega \subset \mathbb{R}^m$ *is compact and convex, then* $\mathcal{R}_{x_0}(t_1)$ *is a compact and convex subset of* $\mathbb{R}^n$ *for any* $x_0 \in \mathbb{R}$.

*Proof* The result follows from the linearity of the map $\psi(t, t_0, x_0, u)$ and convexity of $\Omega$. □

Let $X \subset \mathbb{R}^n$ be a target set and $t_0 \in \mathbb{T}$ be arbitrary but fixed.

**Definition 3** System (3) is $\Omega$-*controllable to* $X$ from $x_0 = x(t_0) \in \mathbb{R}^n$ if there exist $t_1 \in \mathbb{T}$ and a piecewise rd-continuous control $u(t)$, $t \in [t_0, t_1]_{\mathbb{T}}$, such that for $x_0 = x(t_0)$ it holds

$$\psi(t_1, t_0, x_0, u) = x(t_1) \in X.$$

For a vector $\lambda \in \Lambda \subseteq \mathbb{R}^n$, let us define the following function, the so-called *cost function*:

$$I(x_0, t_0, t_1, \lambda) := \lambda^T e_{A(t)}(t_1, t_0)x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega}\bigl[\lambda^T e_{A(t)}\bigl(t_1, \sigma(s)\bigr)B(s)u(s)\bigr]\Delta s - \inf_{x\in X} \lambda x^T$$

for any $x \in X$ and $t \in \mathbb{T}$. By $\lambda^T$ is denoted the transposition of $\lambda \in \mathbb{R}^n$.

Let $A$ be a constant matrix.

- If $\mathbb{T} = \mathbb{R}$, then

$$I(x_0, t_0, t_1, \lambda) = \lambda^T e^{A(t_1-t_0)}x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega}\bigl[\lambda^T e^{A(t_1-s)}B(s)u(s)\bigr]\,ds - \inf_{x\in X} \lambda x^T.$$

So, it differs from the cost function used in the global $\Omega$-controllability case for the classical continuous linear time-varying system in [3].

- If $\mathbb{T} = \mathbb{Z}$, then

$$I(x_0, t_0, t_1, \lambda) = \lambda^T (I + A)^{(t_1-t_0)}x_0 + \sum_{s=t_0}^{t_1-1} \max_{u\in\Omega}\bigl[\lambda^T (I + A)^{(t_1-s-1)}B(s)u(s)\bigr] - \inf_{x\in X} \lambda x^T.$$

Function $I$ differs from the coast function used in $\Omega$-controllability conditions in the classical discrete linear time-varying case in [4].

**Theorem 4** *If system* (3) *is $\Omega$-controllable to an open and convex target set $X$ in time $t_1 \in \mathbb{T}$ from the initial state $x(t_0) = x_0$, then there exists a vector $\lambda \in \mathbb{R}^n$ such that $I(x_0, t_0, t_1, \lambda) > 0$.*

*Proof* Suppose that system (3) is $\Omega$-controllable to $X$ in time $t_1 \in \mathbb{T}$ from $x_0$. Then $\mathcal{R}_{x_0}(t_1) \cap X \neq \emptyset$. Moreover, from Theorem 1 and inequality (1) (by contradiction) it follows that

$$\sup_{\psi(t_1,t_0,x_0,u)\in\mathcal{R}_{x_0}(t_1)} \lambda\psi^T(t_1, t_0, x_0, u) > \inf_{x\in X} \lambda x^T,$$

where $\psi^T$ means the transposition of the vector function $\psi$ described in Definition 3. This means that

$$\begin{aligned}&\sup_{\psi(t_1,t_0,x_0,u)\in\mathcal{R}_{x_0}(t_1)} \lambda\psi^T(t_1, t_0, x_0, u) - \inf_{x\in X} \lambda x^T \\ &\quad= \sup_{\psi(t_1,t_0,x_0,u)\in\mathcal{R}_{x_0}(t_1)} \biggl[\lambda^T e_{A(t)}(t_1, t_0)x_0 + \int_{t_0}^{t_1} \lambda^T e_{A(t)}\bigl(t_1, \sigma(s)\bigr)B(s)u(s)\Delta s\biggr] - \inf_{x\in X} \lambda x^T > 0.\end{aligned} \tag{4}$$

Compactness of the set $\Omega$ and equality (4) imply that

$$\lambda^T e_{A(t)}(t_1, t_0)x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega} \lambda^T e_{A(t)}\bigl(t_1, \sigma(s)\bigr)B(s)u(s)\Delta s - \inf_{x\in X} \lambda x^T > 0.$$

Hence there exists $\lambda \in \mathbb{R}^n$ such that $I(x_0, t_0, t_1, \lambda) > 0$. □

**Theorem 5** *If there exists a vector $\lambda \in \mathbb{R}^n$ such that $I(x_0, t_0, t_1, \lambda) > 0$, then system* (3) *is $\Omega$-controllable to an open and convex target set $X$ in time $t_1 \in \mathbb{T}$ from the initial state $x(t_0) = x_0$.*

*Proof* Suppose that system (3) is not $\Omega$-controllable to an open and convex set $X$ in $t_1 \in \mathbb{T}$ from $x(t_0) = x_0$. This means that $\mathcal{R}_{x_0}(t_1)$ and $X$ are disjoint. From Theorem 1 it follows that

$$\sup_{\psi(t_1,t_0,x_0,u)\in\mathcal{R}_{x_0}(t_1)} \lambda\psi^T(t_1, t_0, x_0, u) \le \inf_{x\in X} \lambda x^T. \tag{5}$$

On the other hand, if there exists a vector $\lambda \in \mathbb{R}^n$ such that $I(x_0, t_0, t_1, \lambda) > 0$, then the compactness of the set $\Omega$ implies the following:

$$\begin{aligned} 0 &< \lambda^T e_{A(t)}(t_1, t_0)x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega}\bigl[\lambda^T e_{A(t)}\bigl(t_1, \sigma(s)\bigr)B(s)u(s)\bigr]\Delta s - \inf_{x\in X} \lambda x^T \\ &= \sup_{u\in\Omega}\left[\lambda^T e_{A(t)}(t_1, t_0)x_0 + \int_{t_0}^{t_1} \lambda^T e_{A(t)}\bigl(t_1, \sigma(s)\bigr)B(s)u(s)\Delta s\right] - \inf_{x\in X} \lambda x^T \\ &= \sup_{\psi(t_1,t_0,x_0,u)\in\mathcal{R}_{x_0}(t_1)} \lambda^T\psi(t_1, t_0, x_0, u) - \inf_{x\in X} \lambda x^T. \end{aligned}$$

Then

$$\sup_{\psi(t_1,t_0,x_0,u)\in\mathcal{R}_{x_0}(t_1)} \lambda\psi^T(t_1, t_0, x_0, u) > \inf_{x\in X} \lambda x^T$$

and it contradicts with (5). This means that sets $\mathcal{R}_{x_0}(t_1)$ and $X$ are not disjoint, so system (3) is $\Omega$-controllable to an open and convex target set $X$ in time $t_1 \in \mathbb{T}$ from $x(t_0) = x_0$. □

**Example 6** Let us consider the control system

$$x^{\Delta}(t) = -x(t) + u(t) \tag{6}$$

with the initial state $x(t_0) = x_0 \in \mathbb{R}_+$ and piecewise constant controls $u \in \mathbb{R}$.

- If $\mathbb{T} = \mathbb{R}$, $t_0 = 0$ and $\Omega = [\frac{1}{100}, \frac{1}{2}]$, then for a positive $\lambda$ one has

$$I(x_0, 0, t_1, \lambda) = \lambda e^t x_0 + \int_0^{t_1} \max_{u\in\Omega}\bigl(\lambda e^t u\bigr)\, ds - \inf_{x\in[-1,1]} (\lambda x) = \lambda\left(e^t x_0 + \frac{1}{2} + 1\right) > 0.$$

Note also that in this case

$$\psi(t_1, 0, x_0, u) = x_0 e^{t_1} + \int_0^{t_1} u e^{t_1 - s}\, ds = (x_0 + u)e^{t_1} - u.$$

Since $u \in \Omega$ is a piecewise constant control, then it is easy to see that $\psi(t_1, 0, x_0, u) \in [-1, 1] = X$, so system (6) is controllable to $X$.

- If $\mathbb{T} = \overline{q^Z} = \{(\frac{1}{2})^k : k \in Z_+\} \cup \{0\}$, $t_0 = 1$ and $\Omega = [10^{-3}; 0,15]$, then for simplicity of computation, let us take $t_1 = \frac{1}{4}$ and $x_0 = 0$. Since

$$e_{-1}\bigl(q^k, 1\bigr) = \prod_{s\in[1,t_1)} \left(1 + \frac{1}{2}s\right),$$

then for a positive $\lambda$ one has

$$I\left(0,1,\frac{1}{4},\lambda\right) = \sum_{s\in\{1,\frac{1}{2},\frac{1}{4}\}} \max_{u\in\Omega}\left(\lambda e_{-1}\left(t,\frac{1}{2}s\right)u\right) - \inf_{x\in[-1,1]}(\lambda x) = \frac{449}{128}\lambda > 0.$$

Since $\psi(\frac{1}{4},1,x_0,u) = \frac{321}{64}u \leq \frac{321}{64}\cdot 0,15 \in [-1,1]$, hence the given system is $\Omega$-controllable to the target set $[-1,1]$.

- If $\mathbb{T} = \mathbb{Z}$, then $e_{-1}(t,0) = (1-1)^t = 0$, so for any $t_1$ as well $I(x_0,0,t_1,\lambda) = 0$ as $\psi(t_1,0,x_0,u) = 0$. Hence we cannot conclude anything about $\Omega$-controllability. Such a case follows from the fact that on the time scale $\mathbb{T} = \mathbb{Z}$ system (6) is not regressive,[a] *i.e.*, $1+\mu(t)(-1) = 1-1 = 0$.

## 4 Approximation of the time scale cost function by real functions

Since the matrix exponential function $e_{A(t)}(t_1,t_0)$ on time scale can be difficult to compute, it can be estimated by a real exponential one.

To this aim, let us recall that $h_n : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ are polynomials defined recursively for all $t, t_0 \in \mathbb{T}$ as follows (see [13]):

$$h_0(t,t_0) \equiv 1, \qquad h_{k+1}(t,t_0) = \int_{t_0}^{t} h_k(\tau,t_0)\Delta\tau. \tag{7}$$

**Proposition 7** *Let $t \in [t_0, \sup\mathbb{T})_{\mathbb{T}}$. Suppose that $A \in \mathbb{R}^{n\times n}$ is a constant matrix. Then*

$$e_A(t,t_0) = \sum_{n=0}^{\infty} A^n h_n(t,t_0).$$

*Proof* It is the same as the proof given in [18] for $e_{p(t)}(t,t_0)$ such that $p(t_0) = p_0 = A$. □

Let $n \in \mathbb{N}_0$. In [19] it was shown that

$$h_n(t,t_0) \leq \frac{(t-t_0)^n}{n!} \tag{8}$$

for all $t \in [t_0, \sup\mathbb{T})_{\mathbb{T}}$.

For a vector $\lambda \in \Lambda \subseteq \mathbb{R}^n$, let

$$\hat{I}(x_0,t_0,t_1,\lambda) := \lambda^T e^{A(t_1-t_0)}x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega}\left[\lambda^T e^{A(t_1-\sigma(s))}Bu(s)\right]\Delta s - \inf_{x\in X}\lambda x^T. \tag{9}$$

**Proposition 8** *If system* (3) *is stationary one* (*i.e.*, *matrices $A$ and $B$ do not depend on time*) *and $\Omega$-controllable to an open and convex target set $X$ in time $t_1 \in \mathbb{T}$ from $x(t_0) = x_0$, then there exists a vector $\lambda \in \mathbb{R}^n$ such that $\hat{I}(x_0,t_0,t_1,\lambda) > 0$.*

*Proof* Proposition 7 and inequality (8) imply the following:

$$\begin{aligned} 0 < I(x_0,t_0,t_1,\lambda) &= \lambda^T e_{A(t)}(t_1,t_0)x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega}\left[\lambda^T e_{A(t)}\left(t_1,\sigma(s)\right)B(s)u(s)\right]\Delta s - \inf_{x\in X}\lambda x^T \\ &= \lambda^T \sum_{n=0}^{\infty} A^n h_n(t_1,t_0)x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega}\left[\sum_{n=0}^{\infty}\lambda^T A^n h_n\left(t_1,\sigma(s)\right)Bu(s)\right]\Delta s - \inf_{x\in X}\lambda x^T \end{aligned}$$

$$
\begin{aligned}
&\leq \lambda^T \sum_{n=0}^{\infty} A^n \frac{(t_1 - t_0)^n}{n!} x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega} \left[ \sum_{n=0}^{\infty} \lambda^T A^n \frac{(t_1 - \sigma(s))^n}{n!} Bu(s) \right] \Delta s - \inf_{x\in X} \lambda x^T \\
&= \lambda^T e^{A(t_1 - t_0)} x_0 + \int_{t_0}^{t_1} \max_{u\in\Omega} \left[ \lambda^T e^{A(t_1 - \sigma(s))} Bu(s) \right] \Delta s - \inf_{x\in X} \lambda x^T \\
&= \hat{I}(x_0, t_0, t_1, \lambda).
\end{aligned}
$$

Hence the thesis. □

**Proposition 9** *If for any vector* $\lambda \in \mathbb{R}^n$ *such that* $\|\lambda\| = 1$ *the following holds*

$$
\min_{\|\lambda\|=1} \hat{I}(x_0, t_0, t_1, \lambda) > 0,
$$

*then system* (3) *is* $\Omega$*-controllable to the target set* $X$ *in time* $t_1 \in \mathbb{T}$ *from the initial state* $x(t_0) = x_0$.

*Proof* The result follows directly from Theorem 5 and the definition of function $\hat{I}$. □

## 5 Conclusions

In the paper we have considered the problem of steering the state of linear time-varying control systems defined on a nonuniform model on time (*i.e.*, on time scale) to a given target set when the control is subject to specified magnitude constraints. We give necessary and sufficient conditions for the constrained controllability for this class of systems to a convex target set. The obtained results extend classical results, specially results stated in [4], to systems defined on any nonuniform time domains. Since the matrix exponential function on time scale can be difficult to compute, also alternative conditions for the constrained controllability to a convex target set are presented. To this end the approximation of the exponential function on time scale by a real exponential is used. This result makes it possible to use computer software for the control constraint problem on time scales in an easier way.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

The authors have achieved equal contributions to each part of this paper. All authors read and approved the final manuscript.

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#### Endnote

[a] System (3) is regressive if and only if $I + \mu(t)A(t) \neq 0$. In this case there exist trajectories both forward and backward. In control, in most cases only forward trajectories are needed.

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