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On a generalization of statistical cluster and limit points

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Abstract

This paper is concerned with the notions of statistical limit and cluster points defined by Fridy. Following the concept of a Δ-density for a subset of a time scale, we established a generalization of these notions which are called Δ-limit and Δ-cluster points for a function defined on a time scale $\mathbb{T}$.

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1 Introduction

The theory of time scales was first constructed by Hilger in his PhD thesis in [1]. Measure theory on time scales has been introduced in [2], then further studies were made in [3] and [4]. Deniz and Ufuktepe defined the Lebesgue-Stieltjes Δ and ∇-measures and by using these measures, they defined an integral which is adaptable to a time scale, specifically the Lebesgue-Stieltjes Δ-integral, in [5]. In the light of these studies, let us introduce some time scale and measure theoretic notations.

The time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. In fact $\mathbb{T}$ is a complete metric space with the usual metric. Throughout this paper we consider a time scale $\mathbb{T}$ with the topology inherited from the real numbers with the standard topology.

We define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, for each $t \in \mathbb{T}$, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

For $a, b \in \mathbb{T}$ with $a \leq b$ we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals are defined similarly.

Let $\mathcal{S}$ be a semi-ring of left-closed and right-open intervals and m^* be a Carathéodory extension of the Lebesgue set function m which is defined by $m([a, b)) = b - a$, associated with the family $\mathcal{S}$ in the time scale $\mathbb{T}$ as in the real case. Also let $\mathcal{M}(m^*)$ be the σ-algebra of all m^*-measurable sets. Recall that $\mathcal{M}(m^*)$ consisting of such a subset E has the property that $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ for all $A \subset \mathbb{T}$. It is well known that the restriction of m^* to $\mathcal{M}(m^*)$, which we denote by μ_Δ, is a countably additive measure on $\mathcal{M}(m^*)$.

This measure is called a Lebesgue Δ-measure. The measurable subsets of $\mathbb{T}$ is called Δ-measurable and a function $f : \mathbb{T} \to \mathbb{R}$ is called a measurable function, if $f^{-1}(O) \in \mathcal{M}(m^*)$ for every open subset O of $\mathbb{R}$. From [2] we know that if $a, b \in \mathbb{T}$ and $a \leq b$, then

$$\mu_\Delta\bigl([a,b)\bigr) = b - a, \qquad \mu_\Delta\bigl((a,b)\bigr) = b - \sigma(a).$$

If $a, b \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ and $a \leq b$, then

$$\mu_\Delta\bigl((a,b]\bigr) = \sigma(b) - \sigma(a), \qquad \mu_\Delta\bigl([a,b]\bigr) = \sigma(b) - a.$$

The theory of statistical convergence has been introduced in [6]. Fridy made progress with the concept of a statistically Cauchy sequence in [7] and proved that it is equivalent to statistical convergence. Besides, in [8] the notion of the statistical limit point is defined. The central purpose of the present paper is to extend the notions of statistical limit point and statistical cluster point for functions defined on a time scale and valued on real numbers. It is natural to attempt to do this by adapting the 'Δ-density', which is defined in [9]. Let us recall the definitions of Δ-density, Δ-convergence, and the Δ-Cauchy property, and some results necessary for our purpose. Throughout this paper let us take all time scales unbounded from above and having a minimum point.

Let A be a Δ-measurable subset of $\mathbb{T}$ and $a = \min \mathbb{T}$, the Δ-density of A in $\mathbb{T}$ is defined by

$$\delta_\Delta(A) = \lim_{s \to \infty} \frac{\mu_\Delta(A(s))}{\sigma(s) - a}$$

(if this limit exists) where $A(s) = \{t \in A : t \leq s\}$. The Δ-density function can be consider as a probabilistic finite additive measure on the algebra of a subset of $\mathbb{T}$ which has a Δ-density. We will denote this space by $\mathcal{M}_d$. Obviously a subset of $\mathbb{T}$ which has a zero Δ-density is an element of $\mathcal{M}_d$. This collection is denoted by $\mathcal{M}_d^0$ and it has the ring structure. By using the Δ-density we obtained the following new type of convergence, which is a generalization of the definitions of natural statistical convergence and statistical Cauchy sequences. Let us recall the Δ-convergence and Δ-Cauchy definitions for functions:

A measurable function f is Δ-convergent to the number L provided that for each $\varepsilon > 0$

$$\delta_\Delta\bigl(\bigl\{t \in \mathbb{T} : \bigl|f(t) - L\bigr| \geq \varepsilon\bigr\}\bigr) = 0.$$

A measurable function f is Δ-Cauchy provided that for every $\varepsilon > 0$ there exists a number $t_0 \in \mathbb{T}$ such that

$$\delta_\Delta\bigl(\bigl\{t \in \mathbb{T} : \bigl|f(t) - f(t_0)\bigr| \geq \varepsilon\bigr\}\bigr) = 0.$$

Note that these definitions coincide with the notions of statistical convergence and of statistical Cauchy definitions for the real sequences whenever $\mathbb{T}$ is taken as the natural numbers. The reason of this fact is that the notions of Δ-density and natural density coincide in the case $\mathbb{T} = \mathbb{N}$. In [9] we showed that these definitions are equivalent for a measurable function. Recall the main result in [9].

Theorem 1.1 *Let $f : \mathbb{T} \to \mathbb{R}$ be a measurable function. The following statements are equivalent*:

(i) *A function f is Δ-convergent*,

(ii) *a function f is Δ-Cauchy*,

(iii) *for a function f there exists a measurable and convergent function $g : \mathbb{T} \to \mathbb{R}$ such that $f(t) = g(t)$ for Δ-a.a. t.*

The notation of Δ-a.a. t. for a property means that the set of elements for which the property does not hold is a set of Δ-density zero.

For further information as regards statistically convergence in a time scale, see [10, 11], and [12].

2 Δ-Limit point, Δ-cluster point

In the present section we investigate the Δ-limit point and Δ-cluster point concepts for a function defined on a time scale $\mathbb{T}$. The results of this section coincide with the statistical limit point and the statistical cluster point in the case $\mathbb{T} = \mathbb{N}$. In other words, these new notions give us a progression of generalizations of results for statistical limit points and statistical cluster points, introduced by Fridy in [8].

Definition 2.1 (Δ-Limit point) A real number L is called a Δ-limit point of a function $f : \mathbb{T} \to \mathbb{R}$ if there exists a subset K of $\mathbb{T}$ with a non-zero Δ-density or if it does not have a Δ-density such that $f(t) \to L$ whenever $t \to \infty$ in K.

Note that in Definition 2.1 the measurable set K may have a positive Δ-density or may not have even a Δ-density. For describing this situation we will use the Δ-*non-thin subset* notation. This notation can be considered as a modified Fridy non-thin term defined for subsequences. Detailed information as regards the classical thin or non-thin concepts can be found in [8]. We proceed with the next definition.

Definition 2.2 (Δ-Cluster point) A real number L is called a Δ-cluster point of a measurable function $f : \mathbb{T} \to \mathbb{R}$ if for all $\varepsilon > 0$ the set $\{t \in \mathbb{T} : |f(t) - L| < \varepsilon\}$ is a Δ-non-thin set.

We denote the set of Δ-limit points and Δ-cluster points of f by Λ_f and Γ_f, respectively.

Definition 2.3 (Δ-Boundedness) A measurable function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-bounded if there exists a real number r such that $\delta_\Delta(\{t \in \mathbb{T} : |f(t)| \le r\}) = 1$.

Definition 2.4 (Δ-Monotone increasing) A function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-monotone increasing if there exists a subset K of $\mathbb{T}$ with $\delta_\Delta(K) = 1$ such that f is monotone on K. That is, for each pair $t_1, t_2 \in K$, $t_1 < t_2$ implies $f(t_1) \le f(t_2)$.

Proposition 2.5 *Let $f : \mathbb{T} \to \mathbb{R}$ be a measurable function, then $\Lambda_f \subset \Gamma_f$.*

Proof Let $L \in \Lambda_f$. Then there exists a Δ-non-thin set $K \subset \mathbb{T}$ such that

$$\lim_{\substack{s \to \infty \\ s \in K}} f(s) = L$$

and

$$\limsup_{s\to\infty} \frac{\mu_\Delta(K(s))}{\sigma(s)-a} = d > 0.$$

Let an arbitrary $\varepsilon > 0$ be given. The set $\{t \in K : |f(t)-L| \geq \varepsilon\}$ is measurable and bounded. From

$$\{t \in \mathbb{T} : |f(t)-L| < \varepsilon\} \supset K - \{t \in K : |f(t)-L| \geq \varepsilon\},$$

we have

$$\frac{\mu_\Delta(\{t \in \mathbb{T} : |f(t)-L| < \varepsilon\}(s))}{\sigma(s)-a} \geq \frac{\mu_\Delta(K(s))}{\sigma(s)-a} - \frac{O(1)}{\sigma(s)-a}.$$

Finally we have

$$\limsup_{s\to\infty} \frac{\mu_\Delta(\{t \in \mathbb{T} : |f(t)-L| < \varepsilon\}(s))}{\sigma(s)-a} \geq d > 0,$$

which means the set $\{t \in \mathbb{T} : |f(t)-L| < \varepsilon\}$ is a Δ-non-thin set; $L \in \Gamma_f$. □

We will proceed some special cases of the above concepts. We also should emphasize that the sets Λ_f and Γ_f are not equal in general. Details are in the following example.

Example (i) Let the time scale be $\mathbb{T} = \mathbb{N}$. This case is called the discrete case and it is easy to see that all definitions above coincide with the definition of a limit point and cluster point in the classical statistically convergence theory.

(ii) Let $q > 1$ be a fixed integer and $\mathbb{T} = \{q^m : m \in \mathbb{N}\}$. Consider the sequence of natural numbers (k_n) such that $k_{n+1} - k_n > 1$. If we take the subset $K = \{q^{k_n} :\in \mathbb{N}\}$ then we can easily show that K is a Δ-non-thin set. Let $k_n \leq k < k_{n+1}$ and $t = q^k$, then we have

$$\frac{\mu_\Delta(K(t))}{\sigma(t)-q} = \frac{q-1}{q(q^k-1)} \sum_{i=1}^{n} q^{k_i}.$$

If one can take the limit $t \to \infty$ in the above equality we can conclude that the limit does not exist. That means K can be considered as a Δ-non-thin set.

(iii) Let us consider the continuous case $\mathbb{T} = [0,\infty)$ and the function $f : \mathbb{T} = [0,\infty) \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0, & t = 0, \\ \frac{1}{n+1}, & t \in \bigcup_{k=0}^{\infty}((2k+1)2^n - 1, (2k+1)2^n], \end{cases}$$

for $n \in \mathbb{N}$. Since for each $n = 0, 1, 2, \ldots$, we have

$$\delta_\Delta\left(\left\{t \in \mathbb{T} : f(t) = \frac{1}{n+1}\right\}\right) = \frac{1}{2^{n+1}} > 0,$$

and we have $1/(n+1) \in \Lambda_f$. Moreover,

$$\delta_\Delta\left(\left\{t \in \mathbb{T} : f(t) \geq \frac{1}{n+1}\right\}\right) = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}} = 1 - \frac{1}{2^{n+1}}$$

implies that

$$\delta_\Delta\left(\left\{t\in\mathbb{T}:f(t)<\frac{1}{n+1}\right\}\right)=1-\left(1-\frac{1}{2^{n+1}}\right)=\frac{1}{2^{n+1}}. \tag{2.1}$$

This means that $0\in\Gamma_f$. The set of Δ-cluster points is exactly the set $\{1,1/2,1/3,\dots\}\cup\{0\}$. Now we shall show that $0\notin\Lambda_f$. For our purpose, we can consider a measurable subset $A\subset\mathbb{T}$ with

$$\lim_{\substack{t\to\infty\\ t\in A}} f(t)=0.$$

We claim that $\delta_\Delta(A)=0$. Let $\varepsilon>0$ be given. There exists a natural number m such that $2^{-m-1}<\varepsilon/3$. Then there exists $s_1\in\mathbb{T}$ such that

$$\frac{\mu_\Delta(\{t\in A:f(t)\geq\frac{1}{m+1}\}(s))}{s}<\frac{\varepsilon}{3} \tag{2.2}$$

holds for all $s>s_1$. On the other hand, from (2.1), $s_2\in\mathbb{T}$ can be chosen such that

$$\begin{aligned}\frac{\mu_\Delta(\{t\in A:f(t)<\frac{1}{m+1}\}(s))}{s}&\leq\frac{\mu_\Delta(\{t\in\mathbb{T}:f(t)<\frac{1}{m+1}\}(s))}{s}\\&<\frac{\varepsilon}{3}+\frac{1}{2^{m+1}}\\&<\frac{2\varepsilon}{3}\end{aligned} \tag{2.3}$$

holds for all $s>s_2$. Let us define $s_0=\max\{s_1,s_2\}$. From (2.2) and (2.3) we have

$$\begin{aligned}\frac{\mu_\Delta(A(s))}{s}&=\frac{\mu_\Delta(\{t\in A:f(t)\geq\frac{1}{m+1}\}(s))}{s}+\frac{\mu_\Delta(\{t\in A:f(t)<\frac{1}{m+1}\}(s))}{s}\\&<\frac{\varepsilon}{3}+\frac{2\varepsilon}{3}=\varepsilon\end{aligned}$$

for all $s>s_0$.

Proposition 2.6 *Let $f:\mathbb{T}\to\mathbb{R}$ be a measurable function with Δ-$\lim_{t\to\infty}f(t)=L$, then $\Lambda_f=\Gamma_f=\{L\}$.*

Proof Assume that Δ-$\lim_{t\to\infty}f(t)=L$. From Theorem 1.1 there exists a measurable function $g:\mathbb{T}\to\mathbb{R}$ such that $\lim_{t\to\infty}g(t)=L$, and $f(t)=g(t)$ holds for Δ-*a.a.* t. Let us define $K=\{t\in\mathbb{T}:f(t)=g(t)\}$. From the definition of Δ-*a.a.* t., $\delta_\Delta(K)=1$ and

$$\lim_{\substack{t\to\infty\\ t\in K}} f(t)=\lim_{\substack{t\to\infty\\ t\in K}} g(t)=L.$$

That is, $L\in\Lambda_f$. By Proposition 2.5, $L\in\Gamma_f$.

Now we will show that L is a unique element of Γ_f. Let $\varepsilon>0$ be given. Since Δ-$\lim_{t\to\infty}f(t)=L$, we have $K_1=\{t\in\mathbb{T}:|f(t)-L|<\varepsilon/2\}$ with $\delta_\Delta(K_1)=1$. If L' is another element of Γ_f then we can define $K_2 = \{t \in \mathbb{T} : |f(t) - L'| < \varepsilon/2\}$; then

$$\limsup_{s\to\infty} \frac{\mu_\Delta(K_2(s))}{\sigma(s) - a} = d > 0. \tag{2.4}$$

We claim that $K_1 \cap K_2 \neq \emptyset$. Assume that $K_1 \cap K_2 = \emptyset$ then $K_1 \subset (K_2)^c$ and from $\delta_\Delta(K_1) = 1$ we have $\delta_\Delta((K_2)^c) = 1$ and so $\delta_\Delta(K_2) = 0$. But this contradicts (2.4). For each $t_0 \in K_1 \cap K_2$ we have

$$\left|L - L'\right| \le \left|L - f(t_0)\right| + \left|f(t_0) - L'\right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $L = L'$. □

Proposition 2.7 *Let $f : \mathbb{T} \to \mathbb{R}$ be a measurable function; then the set Γ_f is closed.*

Proof Let (L_n) be a sequence in Γ_f such that $L_n \to L$ whenever $n \to \infty$. For a given $\varepsilon > 0$, choose n_0 large enough to make $L - \varepsilon < L_{n_0} < L + \varepsilon$ and choose $\varepsilon' > 0$ such that $(L_{n_0} - \varepsilon', L_{n_0} + \varepsilon') \subset (L - \varepsilon, L + \varepsilon)$. In this case, from

$$\left\{t \in \mathbb{T} : \left|f(t) - L_{n_0}\right| < \varepsilon'\right\} \subset \left\{t \in \mathbb{T} : \left|f(t) - L\right| < \varepsilon\right\}$$

and

$$\limsup_{s\to\infty} \frac{\mu_\Delta(\{t \in \mathbb{T} : |f(t) - L_{n_0}| < \varepsilon'\}(s))}{\sigma(s) - a} = d > 0$$

we have

$$\limsup_{s\to\infty} \frac{\mu_\Delta(\{t \in \mathbb{T} : |f(t) - L| < \varepsilon\}(s))}{\sigma(s) - a} = d > 0.$$

That means the set $\{t \in \mathbb{T} : |f(t) - L| < \varepsilon\}$ is not a set that has zero Δ-density; $L \in \Gamma_f$. □

Theorem 2.8 *Let $f, g : \mathbb{T} \to \mathbb{R}$ be measurable functions. If $f(t) = g(t)$ for Δ-a.a. t. then $\Lambda_f = \Lambda_g$ and $\Gamma_f = \Gamma_g$.*

Proof Let $L \in \Lambda_f$. Then there exists a Δ-non-thin set K such that

$$\lim_{\substack{t\to\infty \\ t\in K}} f(t) = L$$

and

$$\limsup_{t\to\infty} \frac{\mu_\Delta(K(t))}{\sigma(t) - a} > 0.$$

Since $\delta_\Delta(\{t \in \mathbb{T} : f(t) \neq g(t)\}) = 0$ we have $\delta_\Delta(\{t \in K : f(t) \neq g(t)\}) = 0$. This means that the set $\{t \in K : f(t) = g(t)\}$ is not a set that has zero-Δ-density and so $L \in \Lambda_g$. Then we have $\Lambda_f \subset \Lambda_g$. It is easy to see that $\Lambda_g \subset \Lambda_f$ from symmetry. Finally we have $\Lambda_f = \Lambda_g$. The equality $\Gamma_f = \Gamma_g$ can be shown in a similar way. □

Proposition 2.9 *Let $f:\mathbb{T}\to\mathbb{R}$ be a measurable function. If f is bounded on a Δ-non-thin set then $\Gamma_f \neq \emptyset$.*

Proof Assume that f is bounded on K, which is a Δ-non-thin subset of $\mathbb{T}$ and $\Gamma_f = \emptyset$. Then for each $t \in K$ there exists a neighborhood of $f(t)$

$$\mathcal{N}\bigl(f(t)\bigr) = \bigl\{y \in \mathbb{R} : \bigl|y - f(t)\bigr| < \varepsilon(t)\bigr\}$$

such that $\delta_\Delta(f^{-1}(\mathcal{N}(f(t)))) = 0$. Moreover, it is clear that

$$f(K) \subset \bigcup_{t\in K} \mathcal{N}\bigl(f(t)\bigr).$$

If $f(K)$ is not closed then we can consider $\overline{f(K)}$ by adding limit points β_i ($i \in I$) of $f(K)$ and we have

$$\overline{f(K)} \subset \biggl(\bigcup_{t\in K} \mathcal{N}\bigl(f(t)\bigr)\biggr) \cup \biggl(\bigcup_{i\in I} \mathcal{N}(\beta_i)\biggr).$$

We have

$$\mathcal{N}(\beta_i) = \bigl\{y \in \mathbb{R} : |y - \beta_i| < \varepsilon(i)\bigr\}$$

and note that positive real numbers $\varepsilon(i)$ can be chosen such that $\delta_\Delta(f^{-1}(\mathcal{N}(\beta_i))) = 0$. Since $\overline{f(K)}$ is compact, we can find $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ which is a finite subcover of $\{\mathcal{N}(f(t))\}_{t\in K} \cup \{\mathcal{N}(\beta_i)\}_{i\in I}$ such that

$$\overline{f(K)} \subset \bigcup_{k=1}^{n} \mathcal{B}_k.$$

Then we have

$$K \subset f^{-1}\bigl(\overline{f(K)}\bigr) \subset \bigcup_{k=1}^{n} f^{-1}(\mathcal{B}_k).$$

Since $\delta_\Delta(\bigcup_{k=1}^{n} f^{-1}(\mathcal{B}_k)) = 0$ we can conclude that $\delta_\Delta(K) = 0$. But this conclusion contradicts the Δ-non-thin property of K. One can prove everything in a similar way if $f(K)$ is closed. □

The following corollary can be obtained immediately from Proposition 2.9.

Corollary 2.10 *If a measurable function $f:\mathbb{T}\to\mathbb{R}$ is Δ-bounded, then $\Gamma_f \neq \emptyset$.*

Proposition 2.11 *Let $f:\mathbb{T}\to\mathbb{R}$ be a measurable function. If f is Δ-monotone increasing and Δ-bounded, then it is Δ-convergent.*

Proof Since f is Δ-monotone increasing, there exists a subset $K_1 \subset \mathbb{T}$ with Δ-density equal to one, and for every $t_1, t_2 \in K_1$ and $t_1 < t_2$ implies $f(t_1) \leq f(t_2)$. Since f is Δ-bounded there exists a real number A and $K_2 = \{t \in \mathbb{T} : |f(t)| \leq A\}$ with $\delta_\Delta(K_2) = 1$. Setting $K = K_1 \cap K_2$, one has $\delta_\Delta(K) = 1$. We now consider the following set:

$$f(K) = \{f(t) : t \in K\}$$

and let β be the supremum of $f(K)$ over K. For an arbitrary $\varepsilon > 0$ there is a point $t_0 \in K$ that satisfies $\beta - \varepsilon < f(t_0) \leq \beta$. From the monotonicity we can write $\beta - \varepsilon < f(t) < \beta + \varepsilon$ for every $t \in K$ satisfying $t \geq t_0$. The last statement implies that $\Delta\text{-}\lim_{t\to\infty} f(t) = \beta$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
NÖT carried out the measure theoretic studies, participated in the sequence alignment, and drafted the manuscript. MSS carried out the subjects of time scale and statistically convergence. NÖT and MSS conceived of the study, participated in its design and coordination, and helped to draft the manuscript. All authors read and approved the final manuscript.

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