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Maximum principle for controlled fractional Fokker-Planck equations

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Abstract

In this paper, we obtain a maximum principle for controlled fractional Fokker-Planck equations. We prove the well-posedness of a stochastic differential equation driven by an α-stable process. We give some estimates of the solutions by fractional calculus. A linear-quadratic example is given at the end of the paper.

Keywords: α-stable subordinator; maximum principle; stochastic optimal control problem; well-posedness; Riemann-Liouville derivative

1 Introduction

The real world is full of uncertainty; using stochastic models one may gain real benefits. Thus, stochastic differential equations driven by Brownian motions have been studied extensively. In spite of many obvious advantages, some models based on Brownian diffusion usually fail to provide a satisfactory description of many dynamical processes. We illustrate this by some practical phenomena as follows: long-range correlations, lack of scale invariance, discontinuity of the trajectories and so on [1, 2]. To capture such anomalous properties of physical systems, one introduces the fractional Fokker-Planck equations.

Recently, Magdziarz [3] and Lv *et al.* [4] obtained the stochastic representation on the fractional Fokker-Planck equation with time and space dependent drift and diffusion coefficients. They found that the corresponding stochastic process is driven by an inverse α-stable subordinator and Brownian motion. The fractional Fokker-Planck equation can be described by the following stochastic process (see [4]):

$$dx(t) = f\big(x(t)\big)\,dS_\alpha(t) + g\big(x(t)\big)\,dB\big(S_\alpha(t)\big),$$

with initial value $x(0) = \xi$. The above stochastic process is driven by the inverse α-stable subordinator and Brownian motion. Here, the inverse α-stable subordinator $S_\alpha(t)$ is independent of $B(\tau)$. We explain $S_\alpha(t)$ in Section 2.

In order to make the relevant decisions (controls) based on the most updated information, the decision makers (controllers) must select an optimal decision among all possible ones to achieve the best expected result related to their goals. Such optimization problems are called stochastic optimal control problems. The range of stochastic optimal control problems covers a variety of physical, biological, economic, and management systems.

Generally, one solves the optimal control problem by the Pontryagin maximum principle. Starting with [5–8], backward stochastic differential equations have been used to describe the necessary conditions that the optimal control must satisfy. We also refer to [9–11] and the references therein for some other works. In this paper, α-stable processes involve some fractional calculations. We use fractional derivatives (of Riemann-Liouville type) to prove the well-posedness of the equations and give some estimates.

In this paper, we consider an optimal control problem for fractional Fokker-Planck equations. We examine this issue because it has a very wide range of physical applications. For instance, surface growth, transport of fluid in porous media [12], two-dimensional rotating flow [13], diffusion on fractals [14], or even in multidisciplinary areas such as in analyzing the behavior of CTAM micelles dissolved in salted water [15] or econophysics [16].

This paper is organized as follows. We begin with the well-posedness of the stochastic differential equations driven by α-stable process by Picard iteration, then we give some estimates of the solution for the controlled fractional Fokker-Planck equation in Section 2. In Section 3, we establish necessary and sufficient conditions for optimal pairs. A linear-quadratic optimal control problem is proposed in Section 4, a Riccati differential equation is derived, and the explicit expression of the optimal control is obtained. The conclusion is in Section 5.

2 Preliminaries

2.1 Statement of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\mathcal{F}_t$. The controlled stochastic system is described as follows:

$$\begin{cases} dx(t) = b(t, x(t), u(t))\, dS_\alpha(t) + \sigma(t, x(t), u(t))\, dB(S_\alpha(t)), \\ x(0) = \xi, \quad t \in [0, T], \end{cases} \tag{1}$$

where $b(t, x(t), u(t)) : [0, T] \times \mathbb{R}^n \times \mathcal{U}[0, T] \to \mathbb{R}^n$, $\sigma(t, x(t), u(t)) : [0, T] \times \mathbb{R}^n \times \mathcal{U}[0, T] \to \mathbb{R}^n$ are given functionals, ξ is the initial value, $u(t)$ is the control process, and $x(t)$ is the corresponding state process. The inverse α-stable subordinator is defined in the following way:

$$S_\alpha(t) = \inf\bigl\{\tau > 0 : U_\alpha(\tau) > t\bigr\},$$

where $U_\alpha(\tau)$ is a strictly increasing α-stable Lévy process. U_α is a pure-jump process whose Laplace transform is given by $\mathbb{E}(e^{-kU_\alpha(\tau)}) = e^{-\tau k^\alpha}$, $0 < \alpha < 1$. For every jump of $U_\alpha(\tau)$, there is a corresponding flat period of its inverse $S_\alpha(t)$.

The space of admissible controls is defined as

$$\mathcal{U}[0, T] \triangleq \biggl\{u : [0, T] \times \Omega \to \mathbb{R}^n \Big| u \text{ is } \mathcal{F}_t\text{-adapted stochastic process and} \\ E\biggl(\int_0^T \bigl|u(t)\bigr|^2\, dt\biggr) < +\infty\biggr\}.$$

The cost functional is

$$J\bigl(u(t)\bigr) = E\biggl\{\int_0^T l\bigl(t, x(t), u(t)\bigr)\, dS_\alpha(t) + h\bigl(x(T)\bigr)\biggr\}, \tag{2}$$

where $l(t,x(t),u(t)):[0,T]\times\mathbb{R}^n\times\mathcal{U}[0,T]\to\mathbb{R}^n$ and $h(t):\mathbb{R}^n\to\mathbb{R}^n$ are given continuously differentiable functionals. We introduce the following basic assumptions which will be assumed throughout the paper.

(H1) b,σ,l,g are continuously differentiable with respect to x. There exists a constant $L_1>0$ such that, for $\varphi(t,x,u)=b(t,x,u),\sigma(t,x,u)$, we have:
1. $|\varphi(t,x,u)-\varphi(t,\hat{x},\hat{u})|\le L_1(|x-\hat{x}|+|u-\hat{u}|)$, $\forall t\in[0,T]$, $x,\hat{x}\in\mathbb{R}^n$, $u,\hat{u}\in\mathcal{U}[0,T]$.
2. $|\varphi(t,x)|\le C(1+|x|)$, $x\in\mathbb{R}^n$, $t\in[0,T]$.

(H2) The maps b,σ,l,h are C^2 in x with bounded (denoted by M) derivatives. There exists a constant $L_2>0$ such that for $\varphi(t,x,u)=b(t,x,u),\sigma(t,x,u)$, we have

$$\begin{aligned}&\bigl|\varphi_x(t,x,u)-\varphi_x(t,\hat{x},\hat{u})\bigr|\le L_2\bigl(|x-\hat{x}|+|u-\hat{u}|\bigr),\\&\forall t\in[0,T],x,\hat{x}\in\mathbb{R}^n,u,\hat{u}\in\mathcal{U}[0,T].\end{aligned}$$

Then we can pose the following optimal control problem.

Problem (A) Find a pair $(x^*(t),u^*(t))\in\mathbb{R}^n\times\mathcal{U}[0,T]$ such that

$$J\bigl(u^*(t)\bigr)=\inf_{u(t)\in\mathcal{U}[0,T]}J\bigl(u(t)\bigr). \tag{3}$$

Now, we introduce the variational equation of (1),

$$\begin{cases}d\hat{x}(t)=(b_x(t)\hat{x}(t)+b_u(t)\hat{u}(t))\,dS_\alpha(t)+(\sigma_x(t)\hat{x}(t)\\ \qquad\quad+\sigma_u(t)\hat{u}(t))\,dB(S_\alpha(t)),\\ \hat{x}(t)=0,\quad t\in[0,T],\end{cases} \tag{4}$$

and the adjoint equation of (1), respectively,

$$\begin{cases}dy(t)=-[b_x(t,x(t),u(t))y(t)+l_x(t,x(t),u(t))\\ \qquad\quad+\sigma_x(t,x(t),u(t))z(t)]\,dS_\alpha(t)\\ \qquad\quad+z(t,x(t),u(t))\,dB(S_\alpha(t)),\\ y(T)=h_x(x(T)),\quad t\in[0,T].\end{cases} \tag{5}$$

The Hamiltonian of our optimal control problem is obtained as follows:

$$H(t,x,u,y,z)=l\bigl(t,x(t),u(t)\bigr)+b\bigl(t,x(t),u(t)\bigr)y(t)+\sigma\bigl(t,x(t),u(t)\bigr)z(t). \tag{6}$$

2.2 Well-posedness of the problem

To obtain our results of maximum principle, we need the following results.

Proposition 2.1 (Itô formula; see [17, Theorem 2.4]) *Suppose that $x(\cdot)$ has a stochastic differential*

$$dx=F\,dS_\alpha+G\,dB(S_\alpha)$$

for $F\in\mathbb{L}^1(0,T)$, $G\in\mathbb{L}^2(0,T)$. Assume $u:\mathbb{R}\times[0,T]\to\mathbb{R}$ is continuous and that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous. Set

$$Y(t):=u\bigl(x(t),t\bigr).$$

Then Y has the stochastic differential equation

$$dY = \frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\big(F\,dS_\alpha + G\,dB(S_\alpha)\big) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2\,dS_\alpha, \quad 0 < \alpha < 1. \tag{7}$$

Lemma 2.1 (See [4]) *Let $S_\alpha(t)$ be the inverse α-stable subordinator, $g(t)$ be an integrable function. Then*

$$E\left[\int_0^t g\big(S_\alpha(\tau)\big)\,dS_\alpha(\tau)\right] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}E\big[g\big(S_\alpha(\tau)\big)\big]\,d\tau.$$

Lemma 2.2 (See [4]) *The following equation holds for any continuous function $f(t)$*:

$$E\left[\int_0^t f(\tau)g\big(S_\alpha(\tau)\big)\,dS_\alpha(\tau)\right] = \int_0^t f(\tau)D_\tau^{1-\alpha}E\big[g\big(S_\alpha(\tau)\big)\big]\,d\tau.$$

Here the operator $D_t^{1-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-s)^{\alpha-1}f(s)\,ds$ is the fractional derivative of Riemann-Liouville type. Especially, the derivative of a constant C need not be zero $D_t^{1-\alpha}C = \frac{t^{\alpha-1}}{\Gamma(\alpha)}C$.

Remark 2.1 We get $\int_0^t 1\,dS_\alpha(t) = \int_0^t D_t^{1-\alpha}\,dt = \frac{t^\alpha}{\alpha\Gamma(\alpha)}$. It is bounded when $\alpha \in (0,1)$. We set $\frac{t^\alpha}{\alpha\Gamma(\alpha)} < P$.

Theorem 2.1 *Let b and σ be measurable functions satisfying* (H1) *and* (H2), $T > 0$ *and T be independent of $X(0)$. Then the stochastic differential equation*

$$dX(t) = b\big(t, X(t)\big)\,dS_\alpha(t) + \sigma\big(t, X(t)\big)\,dB\big(S_\alpha(t)\big), \quad t \in [0,T] \tag{8}$$

has a unique solution $X(t)$.

Proof Define $Y^{(0)}(t) = X(0)$ and $Y^{(k)}(t) = Y^{(k)}(t)(\omega)$. We consider the equation

$$Y^{(k+1)}(t) = X(0) + \int_0^t b\big(s, Y^{(k)}(s)\big)\,dS_\alpha(s) + \int_0^t \sigma\big(s, Y^{(k)}(s)\big)\,dB\big(S_\alpha(s)\big). \tag{9}$$

Then, for $k \geq 1$, $t \leq T$, we have

$$\begin{aligned}
&E\big\|Y^{(k+1)}(t) - Y^{(k)}(t)\big\|^2 \\
&\quad = E\bigg\|\int_0^t \big(b\big(s, X^{(k)}(t)\big) - b\big(s, X^{(k-1)}(t)\big)\big)\,dS_\alpha(s) \\
&\qquad + \int_0^t \big(\sigma\big(s, X^{(k)}(t)\big) - \sigma\big(s, X^{(k-1)}(t)\big)\big)\,dB\big(S_\alpha(s)\big)\bigg\|^2 \\
&\quad \leq 2\frac{t^\alpha}{\alpha\Gamma(\alpha)}E\int_0^t \big\|\big(b\big(s, X^{(k)}(t)\big) - b\big(s, X^{(k-1)}(t)\big)\big)\big\|^2\,dS_\alpha(s) \\
&\qquad + 2E\int_0^t \big\|\big(\sigma\big(s, X^{(k)}(t)\big) - \sigma\big(s, X^{(k-1)}(t)\big)\big)\big\|^2\,dS_\alpha(s) \\
&\quad \leq 2(P+1)\frac{t^\alpha}{\alpha\Gamma(\alpha)}L^2E\int_0^t \big\|X^{(k)}(t) - X^{(k-1)}(t)\big\|^2\,dS_\alpha(t)
\end{aligned}$$

and

$$\begin{aligned} E\|Y^{(1)}(t)-Y^{(0)}(t)\|^2 &\le \frac{4t^\alpha}{\alpha\Gamma(\alpha)}\big(1+E|X_0|^2\big)\frac{t^\alpha}{\alpha\Gamma(\alpha)}+\frac{4t^\alpha}{\alpha\Gamma(\alpha)}\big(1+E|X_0|^2\big) \\ &\le \frac{4t^\alpha}{\alpha\Gamma(\alpha)}\big(1+E|X_0|^2\big)\left(\frac{t^\alpha}{\alpha\Gamma(\alpha)}+1\right) \\ &\le A_1 t, \end{aligned}$$

where the constant A_1 depends on L, P, and $E|X_0|^2$. Hence we obtain

$$E\|Y^{(k+1)}(t)-Y^{(k)}(t)\|^2 \le \big(2(P+1)PL^2\big)^k (A_1 t)^k \le (A_2 t)^k.$$

Here the constant A_2 depends on L, P, and $E|X_0|^2$. We set $A_2 t < \frac{1}{2}$, $m \ge n \ge 0$. Then

$$\begin{aligned} \|Y^{(m)}(t)-Y^{(n)}(t)\|_{L^2(0,T)} &= \left\|\sum_{k=n}^{m-1} Y^{(k+1)}(t)-Y^{(k)}(t)\right\|_{L^2(0,T)} \\ &\le \sum_{k=n}^{m-1}\left(E\left[\int_0^T \big|Y^{(k+1)}(t)-Y^{(k)}(t)\big|^2\,dS_\alpha(t)\right]^{\frac{1}{2}}\right) \\ &\le \sum_{k=n}^{m-1}\left(\int_0^T (A_2 t)^k\,dS_\alpha(t)\right)^{\frac{1}{2}} \\ &\le \sum_{k=n}^{m-1}\big(P(A_2 t)^k\big)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $m,n\to\infty$. Therefore $\{Y^{(n)}(t)\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(0,T)$. Hence $Y^{(n)}(t)_{n=0}^\infty$ is convergent in $L^2(0,T)$. Define

$$X(t) := \lim_{n\to\infty} Y^{(n)}(t).$$

Next, we prove that $X(t)$ satisfies (8). For all n and $t\in[0,T]$, we have

$$Y^{(n+1)}(t) = X(0) + \int_0^t b\big(s, Y^{(n)}(s)\big)\,dS_\alpha(s) + \int_0^t \sigma\big(s, Y^{(n)}(s)\big)\,dB\big(S_\alpha(s)\big).$$

Then we get

$$\int_0^t b\big(s, Y^{(n)}(s)\big)\,dS_\alpha(s) \to \int_0^t b\big(s, X(s)\big)\,dS_\alpha(s) \quad \text{as } n\to\infty.$$

Also

$$\int_0^t \sigma\big(s, Y^{(n)}(s)\big)\,dB\big(S_\alpha(s)\big) \to \int_0^t \sigma\big(s, X(s)\big)\,dB\big(S_\alpha(s)\big) \quad \text{as } n\to\infty.$$

We conclude that for all $t\in[0,T]$ we have

$$X(t) = X(0) + \int_0^t b\big(s, X(s)\big)\,dS_\alpha(s) + \int_0^t \sigma\big(s, X(s)\big)\,dB\big(S_\alpha(s)\big).$$

That is, $X(t)$ satisfies (8).

Now we prove uniqueness. Let $X_1(t)$ and $X_2(t)$ be solutions of (8) with the same initial values. Then

$$\begin{aligned} E\|X_1(t)-X_2(t)\|^2 &= E\bigg\| \int_0^t \big(b(s,X_1(s))-b(s,X_2(s))\big)\,dS_\alpha(s) \\ &\quad + \int_0^t \big(\sigma(s,X_1(s))-\sigma(s,X_2(s))\,dB(S_\alpha(s))\big)\bigg\|^2 \\ &\le 2\frac{t^\alpha}{\alpha\Gamma(\alpha)}E\int_0^t \big\|\big(b(s,X_1(s))-b(s,X_2(t))\big)\big\|^2\,dS_\alpha(s) \\ &\quad + 2E\int_0^t \big\|\big(\sigma(s,X_1(s))-\sigma(s,X_2(s))\big)\big\|^2\,dS_\alpha(s) \\ &\le 2(P+1)\frac{t^\alpha}{\alpha\Gamma(\alpha)}L^2E\int_0^t \|X_1(s)-X_2(s)\|^2\,dS_\alpha(s). \end{aligned}$$

From Lemmas 2.1 and 2.2, we get

$$\begin{aligned} &E\|X_1(t)-X_2(t)\|^2 \\ &\quad\le 2(P+1)\frac{t^\alpha}{\alpha\Gamma(\alpha)}L^2E\int_0^t \|X_1(s)-X_2(s)\|^2\frac{s^{\alpha-1}}{\Gamma(\alpha)}\,ds \\ &\quad\le 2(P+1)PL^2CE\int_0^t \|X_1(s)-X_2(s)\|^2\,ds. \end{aligned}$$

By the Gronwall inequality, we conclude that

$$E\|X_1(t)-X_2(t)\|^2 = 0 \quad \text{for all } t\in[0,T].$$

The uniqueness is proved. □

2.3 Some estimates of the solution

Let u^* and v be two admissible controls. For any $\varepsilon\in\mathbb{R}$, we denote $u^\varepsilon = u^* + \varepsilon(v-u^*)$. Corresponding to u^ε and u^*, there are two solutions $x^\varepsilon(\cdot)$ and $x^*(\cdot)$ to (1). That is,

$$\begin{aligned} x^*(t) &= \xi + \int_0^t b\big(s,x^*(s),u^*(s)\big)\,dS_\alpha(s) + \int_0^t \sigma\big(s,x^*(s),u^*(s)\big)\,dB\big(S_\alpha(s)\big), \\ x^\varepsilon(t) &= \xi + \int_0^t b\big(s,x^\varepsilon(s),u^\varepsilon(s)\big)\,dS_\alpha(s) + \int_0^t \sigma\big(s,x^\varepsilon(s),u^\varepsilon(s)\big)\,dB\big(S_\alpha(s)\big). \end{aligned}$$

Theorem 2.2 *Let* (H1)-(H2) *hold. Then, for any* $K\ge 1$,

$$\sup_{t\in[0,T]} E\big|x^\varepsilon(t)-x^*(t)\big|^2 = O\big(\varepsilon^2\big), \tag{10}$$

$$\sup_{t\in[0,T]} E|\hat{x}|^2 = O\big(\varepsilon^2\big), \tag{11}$$

$$\sup_{t\in[0,T]} E\big|x^\varepsilon(t)-x^*(t)-\hat{x}(t)\big|^2 = O\big(\varepsilon^2\big). \tag{12}$$

Proof We have

$$
\begin{aligned}
&\sup_{t\in[0,T]} E\bigl|x^{\varepsilon}(t)-x^{*}(t)\bigr|^{2}\\
&\quad=\sup_{t\in[0,T]} E\biggl|\int_{0}^{t}\bigl(b\bigl(s,x^{\varepsilon}(s),u^{\varepsilon}(s)\bigr)-b\bigl(s,x^{*}(s),u^{*}(s)\bigr)\bigr)\,dS_{\alpha}(s)\\
&\qquad+\int_{0}^{t}\bigl(\sigma\bigl(s,x^{\varepsilon}(s),u^{\varepsilon}(s)\bigr)-\sigma\bigl(s,x^{*}(s),u^{*}(s)\bigr)\bigr)\,dB\bigl(S_{\alpha}(s)\bigr)\biggr|^{2}\\
&\quad\le\sup_{t\in[0,T]} 2E\biggl\{\frac{t^{\alpha}}{\alpha\Gamma(\alpha)}\biggl|\int_{0}^{t}\bigl(b\bigl(s,x^{\varepsilon}(s),u^{\varepsilon}(s)\bigr)-b\bigl(s,x^{*}(s),u^{*}(s)\bigr)\bigr)^{2}\,dS_{\alpha}(s)\biggr|\\
&\qquad+\biggl|\int_{0}^{t}\bigl(\sigma\bigl(s,x^{\varepsilon}(s),u^{\varepsilon}(s)\bigr)-\sigma\bigl(s,x^{*}(s),u^{*}(s)\bigr)\bigr)^{2}\,dS_{\alpha}(s)\biggr|\biggr\}\\
&\quad\le\sup_{t\in[0,T]} 4\frac{t^{\alpha}}{\alpha\Gamma(\alpha)}L^{2}(P+1)\biggl\{\int_{0}^{T}E\bigl(\bigl|x^{\varepsilon}(s)-x^{*}(s)\bigr|^{2}\bigr)\,dS_{\alpha}(s)\\
&\qquad+\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}\varepsilon^{2}E\bigl(v-u^{*}\bigr)^{2}\biggr\}.
\end{aligned}
$$

From Lemmas 2.1 and 2.2 and the Gronwall inequality, we get

$$
\begin{aligned}
&\sup_{t\in[0,T]} E\bigl|x^{\varepsilon}(t)-x^{*}(t)\bigr|^{2}\\
&\quad\le 4\frac{t^{\alpha}}{\alpha\Gamma(\alpha)}L^{2}(P+1)\biggl\{\int_{0}^{T}E\bigl(\bigl|x^{\varepsilon}(s)-x^{*}(s)\bigr|^{2}\bigr)\frac{s^{\alpha-1}}{\Gamma(\alpha)}\,ds+\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}\varepsilon^{2}E\bigl(v-u^{*}\bigr)^{2}\biggr\}\\
&\quad\le C_{P,L}\varepsilon^{2},
\end{aligned}
$$

where $C_{P,L}$ is a constant that depends on P, L. This proves (10). Similarly, we can prove (11).

We set $\eta(t)=x^{\varepsilon}(t)-x^{*}(t)-\hat{x}(t)$. Then

$$
\begin{aligned}
\bigl|\eta(t)\bigr|^{2}={}&\biggl|\int_{0}^{T}\biggl\{\int_{0}^{1}b_{x}\bigl(s,x^{*}(s)+\theta\bigl(x^{\varepsilon}(s)-x^{*}(s)\bigr),u^{\varepsilon}(s)\bigr)\,d\theta\bigl(x^{\varepsilon}(s)-x^{*}(s)\bigr)\\
&+\int_{0}^{1}b_{u}\bigl(s,x^{*}(s),u^{*}(s)+\theta\bigl(u^{\varepsilon}(s)-u^{*}(s)\bigr)\bigr)\,d\theta\bigl(u^{\varepsilon}(s)-u^{*}(s)\bigr)\\
&-b_{x}(s)\hat{x}(s)-b_{u}(s)\hat{u}(s)\biggr\}\,dS_{\alpha}(s)\\
&+\int_{0}^{T}\biggl\{\int_{0}^{1}\sigma_{x}\bigl(s,x^{*}(s)+\theta\bigl(x^{\varepsilon}(s)-x^{*}(s)\bigr),u^{\varepsilon}\bigr)\,d\theta\bigl(x^{\varepsilon}(s)-x^{*}(s)\bigr)\\
&+\int_{0}^{1}\sigma_{u}\bigl(s,x^{*},u^{*}(s)+\theta\bigl(u^{\varepsilon}(s)-u^{*}(s)\bigr)\bigr)\,d\theta\bigl(u^{\varepsilon}(s)-u^{*}(s)\bigr)\\
&-\sigma_{x}(s)\hat{x}(s)-b_{u}(s)\hat{u}(s)\biggr\}\,dB\bigl(S_{\alpha}(s)\bigr)\biggr|^{2}\\
={}&\biggl|\int_{0}^{T}\biggl\{\int_{0}^{1}b_{x}\bigl(s,x^{*}(s)+\theta\bigl(x^{\varepsilon}(s)-x^{*}(s)\bigr),u^{\varepsilon}\bigr)\,d\theta\,\eta(s)\\
&+\biggl[\int_{0}^{1}b_{x}\bigl(s,x^{*}(s)+\theta\bigl(x^{\varepsilon}(s)-x^{*}(s)\bigr),u^{\varepsilon}(s)\bigr)\,d\theta-b_{x}(s)\biggr]\hat{x}(s)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^1 b_u\bigl(s,x^*,u^*(s)+\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr)\bigr)\,d\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr) \\
&- b_u(s)\hat{u}(s)\biggr\}\,dS_\alpha(s) \\
&+ \int_0^T\biggl\{\int_0^1 \sigma_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\,d\theta\eta(s) \\
&+ \biggl[\int_0^1 \sigma_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\,d\theta-\sigma_x(s)\biggr]\hat{x}(s) \\
&+ \int_0^1 \sigma_u\bigl(s,x^*,u^*(s)+\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr)\bigr)\,d\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr) \\
&- \sigma_u(s)\hat{u}(s)\biggr\}\,dBS_\alpha(s)\biggr|^2 \\
\le{}& 2\biggl|\int_0^T\biggl\{\int_0^1 b_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\,d\theta\eta(s) \\
&+ \biggl[\int_0^1 b_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\,d\theta-b_x(s)\biggr]\hat{x}(s) \\
&+ \int_0^1 b_u\bigl(s,x^*,u^*(s)+\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr)\bigr)\,d\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr) \\
&- b_u(s)\hat{u}(s)\biggr\}^2\,dS_\alpha(s)\biggr| \\
&+ 2\biggl|\int_0^T\biggl\{\int_0^1 \sigma_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\,d\theta\eta(s) \\
&+ \biggl[\int_0^1 \sigma_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\,d\theta-\sigma_x(s)\biggr]\hat{x}(s) \\
&+ \int_0^1 \sigma_u\bigl(s,x^*,u^*(s)+\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr)\bigr)\,d\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr) \\
&- \sigma_u(s)\hat{u}(s)\biggr\}^2\,dS_\alpha(s)\biggr| \\
\le{}& 8\int_0^T\biggl(\int_0^1 b_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\biggr)^2\,d\theta\eta^2(s) \\
&+ \biggl[\int_0^1 b_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\,d\theta-b_x(s)\biggr]^2\hat{x}^2(s) \\
&+ \int_0^1 b_u\bigl(s,x^*,u^*(s)+\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr)\bigr)^2\,d\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr)^2 \\
&- b_u(s)^2\hat{u}(s)^2\,dS_\alpha(s) \\
&+ 8\int_0^T\biggl(\int_0^1 \sigma_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\biggr)^2\,d\theta\eta^2(s) \\
&+ \biggl[\int_0^1 \sigma_x\bigl(s,x^*(s)+\theta\bigl(x^\varepsilon(s)-x^*(s)\bigr),u^\varepsilon\bigr)\,d\theta-b_x(s)\biggr]^2\hat{x}^2(s) \\
&+ \int_0^1 \sigma_u\bigl(s,x^*,u^*(s)+\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr)\bigr)^2\,d\theta\bigl(u^\varepsilon(s)-u^*(s)\bigr)^2 \\
&- \sigma_u(s)^2\hat{u}(s)^2\,dS_\alpha(s).
\end{aligned}
\tag{13}
$$

From Lemmas 2.1 and 2.2 and the Gronwall inequality, we get

$$\begin{aligned}\sup_{t\in[0,T]} E\left|\eta(t)\right|^2 &\le E\int_0^T \left(C_1\eta^2(s)+\varepsilon^2 C_2\right) dS_\alpha(s)\\ &\le \int_0^T C_1 E\eta^2(s)\frac{s^{\alpha-1}}{\Gamma(\alpha)}\,ds+\varepsilon^2 C_2\frac{T^\alpha}{\Gamma(\alpha)\alpha}\\ &\le \varepsilon^2 M_{C_1,C_2}, \end{aligned} \tag{14}$$

where $C_1 = 16(M^2 + L^2C)$, $C_2 = (L^2C - M^2)(v - \hat{u})^2 + M^2(v - u^*)^2$, M_{C_1,C_2} is a constant that depends on C_1, C_2. □

3 The maximum principle

Now, we give the sufficient conditions of Problem (A).

Theorem 3.1 *Let* (H1) *and* (H2) *hold. Let* $(x^*(t), u^*(t))$ *be an admissible pair, and* $(y(t), z(t))$ *satisfies* (5). *Moreover, the Hamiltonian* $H(t)$ *and* $h(t)$ *are convex, and*

$$H\big(t,x^*(t),u^*(t),y(t),z(t),l(t)\big)=\min_{u\in\mathcal{U}[0,T]} H\big(t,x(t),u(t),y(t),z(t),l(t)\big). \tag{15}$$

Then $u^*(t)$ *is an optimal control.*

Proof Fix $u \in \mathcal{U}[0, T]$ with corresponding solution $x = x^{(u)}$. Then

$$J\big(x^*(t),u^*(t)\big)-J\big(x(t),u(t)\big)=I_1+I_2, \tag{16}$$

where

$$\begin{aligned} I_1 &= E\int_0^T l\big(t,x^*(t),u^*(t)\big)-l\big(t,x(t),u(t)\big)\,dS_\alpha(t),\\ I_2 &= E\big(h\big(x^*(T)\big)-h\big(x(T)\big)\big). \end{aligned}$$

By the definition of H, we get

$$\begin{aligned} I_1 = E\bigg[\int_0^T &\big\{H\big(t,x^*(t),u^*(t),y(t),z(t),l(t)\big)-H\big(t,x(t),u(t),y(t),z(t),l(t)\big)\\ &-\big(b\big(t,\hat{x}(t),\hat{u}(t)\big)(t)-b\big(t,x(t),u(t)\big)\big)y(t)\\ &-\big(\sigma\big(t,\hat{x}(t),\hat{u}(t)\big)-\sigma\big(t,x(t),u(t)\big)\big)z(t)\big\}\,dS_\alpha\bigg]. \end{aligned} \tag{17}$$

We use the convexity of $h(t)$ to obtain the inequality

$$\begin{aligned} I_2 &= E\big(h\big(x^*(T)\big)-h\big(x(T)\big)\big)\\ &\le E\big(h_x\big(\hat{x}(T)\big)\big(x^*(T)-x(T)\big)\big)\\ &\le E\big(y(T)\big(x^*(T)-x(T)\big)\big). \end{aligned} \tag{18}$$

Applying the Itô formula to $h_x(\hat{x}(T))(x^*(T)-x(T))$ and taking the expectation, we get

$$\begin{aligned}
&E\big(y(T)\big(x^*(T)-x(T)\big)\big)\\
&\quad= E\big(y(T)\big(x^*(T)-x(T)\big)-y(0)\big(x^*(0)-x(0)\big)\big)\\
&\quad= E\bigg(\int_0^T y(t)\big(b\big(t,x^*(t),u^*(t)\big)-b\big(t,x(t),u(t)\big)\big)\\
&\qquad+z(t)\big(\sigma\big(t,x^*(t),u^*(t)\big)-\sigma\big(t,x(t),u(t)\big)\big)\\
&\qquad-H_x\big(t,x(t),u(t),y(t),z(t),l(t)\big)\big(x^*(t)-x(t)\big)\,dS_\alpha(t)\bigg).
\end{aligned}$$

Substituting the last equation into (16), we obtain

$$\begin{aligned}
&J\big(x^*(t),u^*(t)\big)-J\big(x(t),u(t)\big)\\
&\quad\le E\int_0^T\big[\big(H\big(t,x^*(t),u^*(t),y(t),z(t),l(t)\big)-H\big(t,x(t),u(t),y(t),z(t),l(t)\big)\big)\\
&\qquad-H_x\big(t,x(t),u(t),y(t),z(t),l(t)\big)\big(x^*(t)-x(t)\big)\big]\,dS_\alpha(t).
\end{aligned}$$

Since $H(t)$ is convex, we get

$$J\big(x^*(t),u^*(t)\big)-J\big(x(t),u(t)\big)\le 0.$$

Then $u^*(t)$ is an optimal control. □

Then we give the necessary conditions of the stochastic control problem.

Theorem 3.2 *Assume that b,σ satisfy* (H1) *and* (H2), *$u^*(t)\in\mathcal{U}[0,T]$ is the optimal control of* (1)-(3). *Then $(y(t),z(t))$ is the solution of* (5) *such that*

$$\int_0^T\big\langle H_u\big(t,x^*(t),u^*(t),y(t),z(t)\big),u_1(t)\big\rangle\,dS_\alpha(t)=0. \tag{19}$$

Proof In order to treat the problem, we have

$$\frac{d}{d\varepsilon}J^{u+\varepsilon v}(t)\bigg|_{\varepsilon=0}=E\bigg\{\int_0^T\big(l_x(t)\hat{x}(t)+l_u(t)\hat{u}(t)\big)\,dS_\alpha(t)+h_x\big(x(T)\big)\hat{x}(T)\bigg\}. \tag{20}$$

Let $(y(t),z(t))$ be the solution of (5). Then applying the differential chain rule to $\langle y(t),\hat{x}(t)\rangle$, we have the following duality relation:

$$\begin{aligned}
h_x\big(x(T)\big)\hat{x}(T)&=y(T)\hat{x}(T)-y(0)\hat{x}(0)\\
&=\int_0^T\big[y(t)b_u(t)\hat{u}(t)+z(t)\sigma_u(t)\hat{u}(t)-\hat{x}(t)l_x(t)\big]\,dS_\alpha(t)\\
&\quad+\int_0^T\big[y(t)\sigma_x(t)\hat{x}(t)+y(t)\sigma_u(t)\hat{u}(t)\\
&\quad+\hat{x}(t)z(t)\big]\,dB\big(S_\alpha(t)\big).
\end{aligned} \tag{21}$$

Combining (21) with (16) and by the optimality of $u^*(t)$, we obtain

$$\begin{aligned}\frac{d}{d\varepsilon}J^{u+\varepsilon v}(t)\bigg|_{\varepsilon=0} &= E\int_0^T \big(l_u(t)\hat{u}(t) + y(t)b_u(t)\hat{u}(t) + z(t)\sigma_u(t)\hat{u}(t)\big)\,dS_\alpha(t) \\ &= E\int_0^T \big\langle H_u\big(t, x^*(t), u^*(t), y(t), z(t)\big), \hat{u}(t)\big\rangle\,dS_\alpha(t) = 0. \end{aligned} \tag{22}$$

□

4 Application

In this section, we consider a linear-quadratic (LQ) optimal control problem as follows:

$$\begin{cases} dx(t) = (A(t)x(t) + C(t)u(t))\,dS_\alpha(t) + (D(t)x(t) + E(t)u(t))\,dB(S_\alpha(t)), \\ x(t) = \eta, \quad t \in [0, T], \end{cases} \tag{23}$$

where $A(\cdot)$, $C(\cdot)$, $D(\cdot)$, $E(\cdot)$ are given matrix valued deterministic functions. η is the initial value, the cost functional is

$$\begin{aligned} J\big(x(t), u(t)\big) = \frac{1}{2}E\bigg\{&\int_0^T \big[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\big]\,dS_\alpha(t) \\ &+ x^T(T)S(T)x(T)\bigg\}, \end{aligned} \tag{24}$$

where $Q(t)$, $R(t)$, $S(t)$ are positive-definite matrices. $x^T(t)$ is the transposition of $x(t)$.

The optimal control of the LQ problem can be stated as follows:

Problem (B) Find a pair $(x_*(t), u_*(t)) \in \mathbb{R}^n \times \mathcal{U}[0, T]$ such that

$$J\big(u_*(t)\big) = \inf_{u(t)\in\mathcal{U}[0,T]} J\big(u(t)\big). \tag{25}$$

We will proceed to a reduction of our Riccati equations. We assume P is a semimartingale with the following decomposition:

$$dP(t) = \Sigma(t)\,dS_\alpha(t) + \Pi(t)\,dB\big(S_\alpha(t)\big), \quad t \in [0, T]. \tag{26}$$

Applying the Itô formula to $d(x^T(t)P(t)x(t))$, we obtain

$$\begin{aligned} &d\big(x^T(t)P(t)x(t)\big) \\ &\quad= \big\{x^T\big(PA + A^TP + \Sigma + D^TPD + \Pi D + D^T\Pi\big)x \\ &\qquad+ 2u^T\big(C^TP + E^TPD + E^T\Pi\big)x + u^TE^TPEu\big\}\,dS_\alpha(t) \\ &\qquad+ \big\{x^TD^TPx + u^TE^TPx + x\Pi x + x^TPDx + x^TPEu\big\}\,dB\big(S_\alpha(t)\big). \end{aligned} \tag{27}$$

We denote

$$\begin{aligned} K &= R + E^TPE, \\ L &= C^TP + E^TPD + E^T\Pi. \end{aligned}$$

Taking expectations on both sides of (27), adding these to (24) and using the square completion technique, we get

$$\begin{aligned}
&J\big(x(t),u(t)\big)\\
&\quad=\frac{1}{2}E\int_s^T\big\{x^T\big(PA+A^TP+\Sigma+D^TPD+\Pi D+D^T\Pi+Q\big)x\\
&\qquad+2u^T\big(C^TP+E^TPD+E^T\Pi\big)x+u^T\big(E^TPE+R\big)u\big\}\,dS_\alpha(t)\\
&\qquad+\frac{1}{2}x^T(s)P(s)x(s)+\frac{1}{2}E\big[x^T(T)\big(S(T)-P(T)\big)x(T)\big]\\
&\quad=\frac{1}{2}E\int_s^T\big\{x^T\big(PA+A^TP+\Sigma+D^TPD+\Pi D+D^T\Pi+Q-L^TK^{-1}L\big)x\\
&\qquad+\big(u+K^{-1}Lx\big)^TK\big(u+K^{-1}Lx\big)\big\}\,dS_\alpha(t)\\
&\qquad+\frac{1}{2}x^T(s)P(s)x(s)+\frac{1}{2}E\big[x^T(T)\big(S(T)-P(T)\big)x(T)\big].
\end{aligned}\tag{28}$$

Now, if (P,Π) satisfies the Riccati equation, *i.e.*

$$\Sigma=-\big(PA+A^TP+D^TPD+\Pi D+D^T\Pi+Q-L^TK^{-1}L\big).$$

We set $P(T)=S(T)$. Then we get the stochastic Riccati equation as follows:

$$\begin{cases}
dP(t)=\{-(P(t)A(t)+A'(t)P(t)+D'(t)P(t)D(t)+\Pi(t)D(t)\\
\qquad\quad+D'(t)\Pi(t)+Q(t))+(C^T(t)P(t)+E^T(t)p(t)D(t)\\
\qquad\quad+E^T(t)\Pi(t))^T(R(t)+E'(t)P(t)E(t))^{-1}(C^T(t)P(t)\\
\qquad\quad+E^T(t)p(t)D(t)+E^T(t)\Pi(t))\}\,dS_\alpha(t)+\Pi(t)\,dB(S_\alpha(t)),\\
P(T)=S(T),\\
K(t)=R(t)+D'(t)P(t)D(t)>0,\quad \mathbb{P}\text{-a.s.},\forall t\in[0,T].
\end{cases}\tag{29}$$

Theorem 4.1 *If the stochastic Riccati equation* (29) *admits a solution, then the stochastic LQ problem* (23)-(24) *is well-posed.*

Proof We know that (P,Π) satisfies the Riccati equation (29) with $K=R+E^TPE>0$. Then

$$\begin{aligned}
&J\big(x(t),u(t)\big)\\
&\quad=\frac{1}{2}E\int_s^T\big(u+K^{-1}Lx\big)^TK\big(u+K^{-1}Lx\big)\,dS_\alpha(t)+\frac{1}{2}x^T(s)P(s)x(s)\\
&\quad\geq\frac{1}{2}x^T(s)P(s)x(s)>-\infty,\quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Therefore, the stochastic LQ problem is well-posed. □

Remark 4.1 We see that if the Riccati equation (29) admits a solution (P,Π), then the optimal feedback control would be

$$\begin{aligned}
u(t)&=-K^{-1}(t)L(t)x(t)\\
&=-\big(R(t)+E^T(t)P(t)E(t)\big)^{-1}\big(C^T(t)P(t)+E^T(t)PD(t)+E^T(t)\Pi(t)\big)x(t).
\end{aligned}$$

5 Conclusion

In this paper, we present some results as regards controlled fractional Fokker-Planck equations. The well-posedness of the system has been proved by Picard iteration. Some estimates of the solution of the controlled system have been given. Because some terms contain α-stable processes, we use the fractional integral to solve the problem. The necessary and sufficient conditions of Pontryagin type for the optimal controls have been proved. As an application, a LQ problem has been shown.

Competing interests
The author declares that they have no competing interests.

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