Zhang *Advances in Difference Equations* (2015) 2015:25
DOI 10.1186/s13662-015-0359-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Note on wandering domains in the dynamics of solutions of certain difference equations

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Abstract

In this note we study the value distribution of solutions of certain difference equations analogous to differential equations, the finite order solutions of which do not have wandering domains. Meanwhile, the nonexistence of wandering domains of solutions with finite order of these difference equations is proved. Thus the nonexistence of wandering domains of solutions of these difference and differential equations is similar in some extent.

MSC: Primary 30D05; 37F10; secondary 37F50

Keywords: Nevanlinna theory; difference equation; wandering domain; iterate

1 Introduction and main results

Let f be a nonlinear meromorphic function, the Fatou set $F(f)$ is the set of points $z \in \mathbb{C}$ such that iterates of f, $(f^n)_{n\in\mathbb{N}}$, form a normal family in some neighborhood of z. The complement of $F(f)$ is called the Julia set $J(f)$ of f. The Fatou set is open and completely invariant. If U is a component of $F(f)$, then $f^n(U)$ lies in some component U_n of $F(f)$. If $U_n \neq U_m$ for all $n \neq m$, then U is called a wandering domain of f. Otherwise U is called pre-periodic and $U_n = U$ for some $n \in \mathbb{N}$, then U is called periodic. An introduction to iteration theory can be found in [1].

Sullivan [2] proved that rational functions do not have wandering domains. However, transcendental meromorphic functions may have wandering domains (for example, see [2–6]), while many classes of meromorphic functions do not have wandering domains (for example, see [3, 7–12]). In [13], the nonexistence of wandering domains is proved by Wang for a meromorphic function f of finite order satisfying some first order nonlinear differential equations, see the following two theorems.

Theorem A *Let $q(z)$ be a rational function, $p(z)$ be a polynomial and $m, n \in \mathbb{N}$, $t \in \mathbb{N} \cup \{0\}$, $a \in \mathbb{C}\backslash\{0\}$. Suppose that f is a meromorphic solution of the differential equation*

$$(f')^n = q(z)p(f)(f' - a)^t (f - z)^m. \tag{1}$$

Then f does not have wandering domains.

Theorem B *Let $q(z)$ be a rational function, $p(z)$, $Q(z)$ be two polynomials and $m, n \in \mathbb{N}$. Suppose that f is a meromorphic function of finite order satisfying the differential equa-*

tion

$$\left(f'\right)^n = q(z)p(f)e^{Q(z)}(f-z)^m. \tag{2}$$

Then f does not have wandering domains.

We also assume that the readers are familiar with basic Nevanlinna's value distribution theory and its standard notations such as $m(r,f)$, $N(r,f)$, $T(r,f)$. $S(r,f)$ denotes any term satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside some exceptional set of finite measure; see [14, 15] as references for Nevanlinna theory. Also, we use the notations $\sigma(f)$, $\lambda(f)$ to denote the order of f, exponent of convergence of zeros of f, respectively, as usual. Halburd and Korhonen [16, 17], Chiang and Feng [18] established a version of Nevanlinna theory based on difference operators independently. After that many difference equations analogous to differential equations have been studied.

In this note, we study the value distribution and dynamical properties of the solutions of difference equations which are analogous to differential equations (1) and (2). We use f_c to denote the shift $f(z+c)$ of $f(z)$, where c is a nonzero constant.

A general form difference equations analogue of differential equation (1) is as follows:

$$(f_c)^n = P(z,f)(f_c - a)^t, \tag{3}$$

where a is a nonzero constant, $P(z,f)$ is a polynomial in f with degree p, the coefficients of $P(z,f)$ are small functions of f.

We obtain the following results with regard to equation (3).

Theorem 1 *Let $f(z)$ be a finite order meromorphic solution of* (3), *then* $\max\{t,n\} \geq p \geq n-t$.

Theorem 2 *If equation* (3) *admits a finite order meromorphic solution f, then f is rational.*

The following example shows that there are rational solutions satisfying equation (3).

Example Suppose that $n = t$, $c = 1$, then $f = a + \frac{1}{z}$ satisfying equation (3), where $P(z,f) = (\frac{az^2+(a+1)z}{az+1})^n f^n$.

By Sullivan's no existence of wandering domains for rational function, we obtain the following dynamical property for the finite order solutions of (3), which is similar to the dynamical property of solutions of equation (1).

Corollary 1 *The finite order meromorphic solutions of* (3) *do not have wandering domains.*

We also consider the difference equation analogous to differential equation (2). Its general form is as follows:

$$f(z+c)^n = q(z)e^{Q(z)}P(z,f), \tag{4}$$

where $q(z)$, $Q(z)$ are nonconstant polynomials and $P(z,f)$ is a polynomial in f with polynomials as coefficients. Replace $f(z+c)$ by $f(z)$ in (4), the equation can be written as

$$f(z)^n = q(z)e^{Q(z)}P\big(z,f(z-c)\big). \tag{5}$$

Without loss of generality, we only consider that (6) below is enough. Firstly, we study the growth of the finite order solutions of it.

Theorem 3 *Let $q(z)$, $Q(z)$ be nonconstant polynomials and $P(z,f)$ be a polynomial in f with degree p, the coefficients of $P(z,f)$ are also polynomials. If $n > p$ and f is a finite order entire solution of the difference equation*

$$f(z)^n = q(z)e^{Q(z)}P\bigl(z,f(z+c)\bigr), \tag{6}$$

then $\sigma(f) = \deg Q(z)$.

In the following, we shall show the properties of the solutions of the following difference equation (7), which is a special case of (6).

Theorem 4 *Let $m,n \geq 2$ be integers and $n > m$, $m|n$, let $c \in \mathbb{C}\backslash\{0\}$, and let $q(z)$, $Q(z)$ be polynomials such that $Q(z)$ is not a constant and $q(z) \not\equiv 0$. If f is a finite order entire solution of the difference equation*

$$f(z)^n = q(z)e^{Q(z)}f(z+c)^m, \tag{7}$$

then the solution f is of the form $f = e^{\alpha(z)}$, where $\alpha(z)$ is a nonconstant polynomial.

Some ideas of this theorem are from [19]. Recall the following theorem about the nonexistence of wandering domains for a class of entire functions, which is due to Baker [4].

Theorem C *Let P and Q be polynomials with Q nonconstant, then*

$$f(z) = \int_0^z P(t)e^{Q(t)}\,dt \tag{8}$$

has no wandering domains. Particularly, the form $f = P_1 e^{Q_1}$ for polynomials P_1, Q_1 is a special case of (8).

Combining Theorem 4 and Theorem C, obviously, we have the corollary below.

Corollary 2 *Under the hypothesis of Theorem* 4, *every finite order entire solution of* (7) *has no wandering domains.*

2 Preliminary lemmas

The following lemma introduced by Laine and Yang [20] is an analogue of findings of Mohonko and Mohonko [21] on differential equations.

Lemma 1 ([20]) *Let $w(z)$ be a transcendental meromorphic solution of finite order of the difference equation*

$$P(z,w) = 0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$ and its shift. If $P(z,a)\not\equiv 0$ for a slowly moving target function a, that is, $T(r,a)=S(r,w)$, then

$$m\left(r,\frac{1}{w-a}\right)=S(r,w).$$

Lemma 2 ([18]) *Let $f(z)$ be a meromorphic function with order $\sigma=\sigma(f)$, $\sigma<+\infty$, and let c be a fixed nonzero complex number, then for each $\varepsilon>0$, we have*

$$T\bigl(r,f(z+c)\bigr)=T(r,f)+O\bigl(r^{\sigma-1+\varepsilon}\bigr)+O(\log r).$$

The following result is due to Valiron and Mohonko, one can find the proof in Laine's book [22, p.29].

Lemma 3 *Let f be a meromorphic function. Then, for all irreducible rational functions in f,*

$$R\bigl(z,f(z)\bigr)=\frac{\sum_{j=0}^{p}a_j(z)f(z)^j}{\sum_{j=0}^{q}b_j(z)f(z)^j} \tag{9}$$

with meromorphic coefficients $a_j(z)$, $b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies

$$T\bigl(r,R\bigl(z,f(z)\bigr)\bigr)=dT(r,f)+O\bigl(\Psi(r)\bigr), \tag{10}$$

where $d=\max\{p,q\}$ and

$$\Psi(r)=\max_{i,j}\bigl\{T(r,a_j),T(r,b_j)\bigr\}.$$

In the particular case when

$$T(r,a_j)=S(r,f),\quad j=0,1,\ldots,p,$$
$$T(r,b_j)=S(r,f),\quad j=0,1,\ldots,q,$$

we have $T(r,R(z,f(z)))=dT(r,f)+S(r,f)$.

3 Proof of theorems

Proof of Theorem 1 By the assumption of $\sigma(f)<\infty$, Lemma 3 and (3), we have

$$T\bigl(r,P(z,f)\bigr)\le T\left(r,\frac{f_c^n}{(f_c-a)^t}\right)=\max\{t,n\}T(r,f_c)+S(r,f_c). \tag{11}$$

Then, by Lemma 2 and Lemma 3, we get that

$$pT(r,f)\le\max\{t,n\}T(r,f)+S(r,f). \tag{12}$$

Thus, we obtain $\max\{t,n\}\ge p$.

On the other hand, by (3), Lemma 2 and Lemma 3, we have

$$\begin{aligned} nT(r,f_c) &= T\big(r,(f_c-a)^t P(z,f)\big) \\ &\le tT(r,f_c) + pT(r,f) + S(r,f) \\ &= (t+p)T(r,f) + S(r,f). \end{aligned} \tag{13}$$

Then we have $p \ge n-t$. □

Proof of Theorem 2 Suppose that f is a transcendental meromorphic function of (3) with finite order. Set $Q(z,f_c) = f_c^n - P(z,f)(f_c-a)^t$. Since $Q(z,a) = a^n \not\equiv 0$, by Lemmas 1, 2 and (3), we obtain that

$$m\left(r,\frac{1}{f_c-a}\right) = S(r,f_c) = S(r,f). \tag{14}$$

Additionally, it follows from (3) that a is a Picard value of f_c. By the first main theorem of Nevanlinna theory, this implies that

$$T(r,f_c) = T(r,f_c-a) + O(1) = T\left(r,\frac{1}{f_c-a}\right) + O(1) = S(r,f_c), \tag{15}$$

which is a contradiction. Thus, every finite order meromorphic solution of (3) is rational. □

Proof of Theorem 3 Suppose that f is an entire solution of (6) with $\sigma(f) = \sigma < \infty$. By Lemma 2 and (6), we deduce that

$$\begin{aligned} nT(r,f) &= T\big(r,q(z)e^{Q(z)}P\big(z,f(z+c)\big)\big) \\ &\le T\big(r,e^{Q(z)}\big) + T\big(r,P\big(z,f(z+c)\big)\big) + O(\log r) \\ &\le T\big(r,e^{Q(z)}\big) + pT\big(r,f(z+c)\big) + O(\log r) \\ &\le T\big(r,e^{Q(z)}\big) + pT(r,f) + O\big(r^{\sigma-1+\varepsilon}\big) + O(\log r) \end{aligned} \tag{16}$$

that is,

$$(n-p)T(r,f) \le T\big(r,e^{Q(z)}\big) + O\big(r^{\sigma-1+\varepsilon}\big) + O(\log r).$$

Since $n > p$, this shows $\sigma(f) \le \deg(Q)$. On the other hand, by equation (6), Lemma 2 and the first main theorem of Nevanlinna theory, we obtain

$$\begin{aligned} T\big(r,e^{Q(z)}\big) &\le T\left(r,\frac{f^n}{q(z)P(z,f(z+c))}\right) \\ &\le T\big(r,f^n\big) + T\big(r,q(z)P\big(z,f(z+c)\big)\big) + O(1) \\ &\le nT(r,f) + pT\big(r,f(z+c)\big) + O(\log r) \\ &\le (n+p)T(r,f) + O\big(r^{\sigma-1+\varepsilon}\big) + O(\log r), \end{aligned} \tag{17}$$

which shows that $\deg(Q) \le \sigma(f)$. Hence $\sigma(f) = \deg(Q)$. □

Proof of Theorem 4 By the Hadamard factorization theorem, $f(z)$ can be written as

$$f(z) = T(z)e^{\alpha(z)}, \tag{18}$$

where $T(z)$ and $f(z)$ have the same zeros, if any, and $\alpha(z)$ is a polynomial.

Substituting (18) into (7), we obtain

$$\frac{T^n(z)}{q(z)T(z+c)^m} = e^{Q(z)+m\alpha(z+c)-n\alpha(z)}. \tag{19}$$

If $T(z)$ has infinitely many zeros, then there exists a zero z_0 of $T(z)$ such that none of the points $z_l = z_0 + lc$, $l \in \mathbb{N} \cup \{0\}$, is a zero of $q(z)$. If z_0 is of multiplicity $k \geq 1$, then by (19), $z_0 + c$ is a zero of $T(z)$ of multiplicity of nk/m. Continuing inductively, we deduce that z_l is a zero of $T(z)$ of multiplicity $(n/m)^l k$. Since $n/m \geq 2$, the sequence of zeros (counting multiplicities) is of infinite convergence exponent. This is a contradiction. Hence $T(z)$ has finite zeros, that is, is a polynomial. So, $\lambda(f) = 0 < \sigma(f) = \deg Q(z)$.

Since $T(z)$ is a polynomial, observing the both sides of (19), we know that $\frac{T^n(z)}{q(z)T(z+c)^m}$ must be a constant. Without loss of generality, we set $T^n(z) = q(z)T(z+c)^m$. If $T(z+c)$ has a zero that is not a zero of $T(z)$, we get a contradiction immediately. Hence every zero of $T(z+c)$ must be a zero of $T(z)$, but maybe with different multiplicity. In other words, every distinct zero of $T(z)$ must be a zero of $T(z-c)$. Since $c \neq 0$ and $n > m$, $m|n$, by continuing inductively, $T(z)$ has infinitely many zeros, this is a contradiction. Hence $T(z)$ cannot have any zeros, in which case $T(z)$ and $q(z)$ are constants. By (18), f is of the form $f = e^{\alpha(z)}$, where $\alpha(z)$ is a nonconstant polynomial. □

Competing interests
The author declares that there is no conflict of interests regarding the publication of this article.

Author's contributions
The author carried out the proof and conceived of the study. The author read and approved the final manuscript.

Acknowledgements
The author is very grateful to the editor and the referees for their insightful and constructive comments and suggestions, which have led to an improved version of this paper. This work was supported by the Tianyuan Fund for Mathematics of NSFC (No. 11426035).

Received: 24 October 2014 Accepted: 6 January 2015 Published online: 30 January 2015

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