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Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials

Dae San Kim[1], Taekyun Kim[2*], Jong-Jin Seo[3] and Takao Komatsu[4]

*Correspondence: tkkim@kw.ac.kr
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract
In this paper, we consider Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.
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1 Introduction

In this paper, we consider the polynomials $T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)$ whose generating function is given by

$$\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\frac{t^n}{n!},\tag{1}$$

where $r\in\mathbb{Z}_{>\nvdash}$, $k\in\mathbb{Z}$, $a_1,\ldots,a_r\neq 0$, $\lambda_1,\ldots,\lambda_r\neq 1$ and

$$\mathrm{Li}_k(x)=\sum_{m=1}^{\infty}\frac{x^m}{m^k}$$

is the kth polylogarithm function. $T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)$ will be called Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials. When $x=0$, $T_n^{(r,k)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)=T_n^{(r,k)}(0|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)$ will be called Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type numbers.

Recall that, for every integer k, the poly-Bernoulli polynomials $B_n^{(k)}(x)$ are defined by the generating function as follows:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}\tag{2}$$

([1], *cf.* [2]). Also, as a natural generalization of higher-order Frobenius-Euler polynomials, Barnes' multiple Frobenius-Euler polynomials $H_n^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)$ are defined by the generating function as follows:

$$\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\frac{t^n}{n!},\tag{3}$$

where $a_1, \ldots, a_r \neq 0$. Note that the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

(see, *e.g.*, [3]).

In this paper, we consider Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{4}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\left\langle f(t)|x^n\right\rangle = a_n \quad (n \geq 0). \tag{5}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{6}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a *delta series*; if $O(f(t)) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$ for $n,k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\left\langle f(t)g(t)|p(x)\right\rangle = \left\langle f(t)|g(t)p(x)\right\rangle = \left\langle g(t)|f(t)p(x)\right\rangle \tag{7}$$

and

$$f(t)=\sum_{k=0}^{\infty}\langle f(t)|x^k\rangle\frac{t^k}{k!},\qquad p(x)=\sum_{k=0}^{\infty}\langle t^k|p(x)\rangle\frac{x^k}{k!} \tag{8}$$

[4, Theorem 2.2.5]. Thus, by (8), we get

$$t^k p(x)=p^{(k)}(x)=\frac{d^k p(x)}{dx^k}\quad\text{and}\quad e^{yt}p(x)=p(x+y). \tag{9}$$

Sheffer sequences are characterized in the generating function [4, Theorem 2.3.4].

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t),f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)}=\sum_{k=0}^{\infty}\frac{s_k(y)}{k!}t^k\quad(y\in\mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x)\sim(g(t),f(t))$, we have the following equations [4, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t)s_n(x)=ns_{n-1}(x)\quad(n\geq 0), \tag{10}$$

$$s_n(x)=\sum_{j=0}^{n}\frac{1}{j!}\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j|x^n\rangle x^j, \tag{11}$$

$$s_n(x+y)=\sum_{j=0}^{n}\binom{n}{j}s_j(x)p_{n-j}(y), \tag{12}$$

where $p_n(x)=g(t)s_n(x)$.

Assume that $p_n(x)\sim(1,f(t))$ and $q_n(x)\sim(1,g(t))$. Then the transfer formula [4, Corollary 3.8.2] is given by

$$q_n(x)=x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x)\quad(n\geq 1).$$

For $s_n(x)\sim(g(t),f(t))$ and $r_n(x)\sim(h(t),l(t))$, assume that

$$s_n(x)=\sum_{m=0}^{n}C_{n,m}r_m(x)\quad(n\geq 0).$$

Then we have [4, p.132]

$$C_{n,m}=\frac{1}{m!}\left\langle\frac{h(\bar{f}(t))}{g(\bar{f}(t))}l(\bar{f}(t))^m\middle|x^n\right\rangle. \tag{13}$$

3 Main results

We now note that $B_n^{(k)}(x)$, $H_n^{(r)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)$ and $T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)$ are the Appell sequences for

$$g_k(t)=\frac{1-e^{-t}}{\operatorname{Li}_k(1-e^{-t})},\qquad g_r(t)=\prod_{j=1}^r\left(\frac{e^{a_jt}-\lambda_j}{1-\lambda_j}\right),$$
$$g_{r,k}(t)=\prod_{j=1}^r\left(\frac{e^{a_jt}-\lambda_j}{1-\lambda_j}\right)\frac{1-e^{-t}}{\operatorname{Li}_k(1-e^{-t})}.$$

So,

$$B_n^{(k)}(x)\sim\left(\frac{1-e^{-t}}{\operatorname{Li}_k(1-e^{-t})},t\right),\tag{14}$$

$$H_n^{(r)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\sim\left(\prod_{j=1}^r\left(\frac{e^{a_jt}-\lambda_j}{1-\lambda_j}\right),t\right),\tag{15}$$

$$T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\sim\left(\prod_{j=1}^r\left(\frac{e^{a_jt}-\lambda_j}{1-\lambda_j}\right)\frac{1-e^{-t}}{\operatorname{Li}_k(1-e^{-t})},t\right).\tag{16}$$

In particular, we have

$$tB_n^{(k)}(x)=\frac{d}{dx}B_n^{(k)}(x)=nB_{n-1}^{(k)}(x),\tag{17}$$

$$\begin{aligned}tH_n^{(r)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)&=\frac{d}{dx}H_n^{(r)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\\&=nH_{n-1}^{(r)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r),\end{aligned}\tag{18}$$

$$\begin{aligned}tT_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)&=\frac{d}{dx}T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\\&=nT_{n-1}^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r).\end{aligned}\tag{19}$$

Notice that

$$\frac{d}{dx}\operatorname{Li}_k(x)=\frac{1}{x}\operatorname{Li}_{k-1}(x).$$

3.1 Explicit expressions

Write $H_n^{(r)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r):=H_n^{(r)}(0|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)$. Let $(n)_j=n(n-1)\cdots(n-j+1)$ $(j\geq 1)$ with $(n)_0=1$.

Theorem 1

$$T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)$$
$$=\sum_{l=0}^n\binom{n}{l}B_l^{(k)}(x)H_{n-l}^{(r)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\tag{20}$$
$$=\sum_{l=0}^n\binom{n}{l}B_{n-l}^{(k)}H_l^{(r)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\tag{21}$$

$$= \sum_{l=0}^{n}\sum_{m=0}^{n}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{n}{l}\frac{1}{(m+1)^k}H_{n-l}^{(r)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)(x-j)^l \tag{22}$$

$$= \sum_{l=0}^{n}\left(\sum_{j=l}^{n}\sum_{m=0}^{n-j}(-1)^{n-m-j}\binom{n}{j}\binom{j}{l}\right.$$

$$\left.\times \frac{m!}{(m+1)^k}S_2(n-j,m)H_{j-l}^{(r)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\right)x^l \tag{23}$$

$$= \sum_{j=0}^{n}\binom{n}{j}T_{n-j}^{(r,k)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)x^j. \tag{24}$$

Proof By (1), (2) and (3), we have

$$\begin{aligned}
T_n^{(r,k)}(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) &= \left\langle \sum_{i=0}^{\infty}T_i^{(r,k)}(y|a_1,\dots,a_r,\lambda_1,\dots,\lambda_r)\frac{t^i}{i!}\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\middle| \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\middle| \sum_{l=0}^{\infty}B_l^{(k)}(y)\frac{t^l}{l!}x^n\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}B_l^{(k)}(y)\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\middle| x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}B_l^{(k)}(y)\left\langle \sum_{i=0}^{\infty}H_i^{(r)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\frac{t^i}{i!}\middle| x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}B_l^{(k)}(y)H_{n-l}^{(r)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r).
\end{aligned}$$

So, we get (20).

We also have

$$\begin{aligned}
T_n^{(r,k)}(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) &= \left\langle \sum_{i=0}^{\infty}T_i^{(r,k)}(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\frac{t^i}{i!}\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle| x^n\right\rangle \\
&= \left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle| \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)e^{yt}x^n\right\rangle \\
&= \left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle| \sum_{l=0}^{\infty}H_l^{(r)}(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\frac{t^l}{l!}x^n\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{n} \binom{n}{l} H_l^{(r)}(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) \left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} H_l^{(r)}(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) \left\langle \sum_{i=0}^{\infty} B_i^{(k)} \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} H_l^{(r)}(y|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) B_{n-l}^{(k)}.
\end{aligned}
$$

Thus, we get (21).

In [5] we obtained that

$$
\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (x-j)^n.
$$

So,

$$
\begin{aligned}
&T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) \\
&\quad = \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n \\
&\quad = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j} \right) (x-j)^n \\
&\quad = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)(x-j)^l \\
&\quad = \sum_{l=0}^{n} \sum_{m=0}^{n} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n}{l} \frac{1}{(m+1)^k} H_{n-l}^{(r)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)(x-j)^l,
\end{aligned}
$$

which is identity (22).

In [5] we obtained that

$$
\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n = \sum_{j=0}^{n} \left(\sum_{m=0}^{n-j} \frac{(-1)^{n-m-j}}{(m+1)^k} \binom{n}{j} m! S_2(n-j,m) \right) x^j,
$$

where $S_2(l,m)$ are the Stirling numbers of the second kind, defined by

$$
\left(e^t-1\right)^m = m! \sum_{l=m}^{\infty} S_2(l,m) \frac{t^l}{l!}.
$$

Thus,

$$
\begin{aligned}
&T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) \\
&\quad = \sum_{j=0}^{n} \left(\sum_{m=0}^{n-j} \frac{(-1)^{n-m-j}}{(m+1)^k} \binom{n}{j} m! S_2(n-j,m) \right) \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j} \right) x^j
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\frac{(-1)^{n-m-j}}{(m+1)^k}\binom{n}{j}m!S_2(n-j,m)\right)H_j^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\\
&= \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\frac{(-1)^{n-m-j}}{(m+1)^k}\binom{n}{j}m!S_2(n-j,m)\right)\sum_{l=0}^{j}\binom{j}{l}H_{j-l}^{(r)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)x^l\\
&= \sum_{l=0}^{n}\left(\sum_{j=l}^{n}\sum_{m=0}^{n-j}(-1)^{n-m-j}\binom{n}{j}\binom{j}{l}\frac{m!}{(m+1)^k}S_2(n-j,m)H_{j-l}^{(r)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right)x^l,
\end{aligned}
$$

which is identity (23).

By (11) with (16), we have

$$
\begin{aligned}
\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j \middle| x^n\right\rangle &= \left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^j \middle| x^n\right\rangle\\
&= (n)_j\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| x^{n-j}\right\rangle\\
&= (n)_j\left\langle \sum_{i=0}^{\infty}T_i^{(r,k)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\frac{t^i}{i!} \middle| x^{n-j}\right\rangle\\
&= (n)_jT_{n-j}^{(r,k)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).
\end{aligned}
$$

Thus, we get (24). □

3.2 Sheffer identity

Theorem 2

$$
T_n^{(r,k)}(x+y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) = \sum_{j=0}^{n}\binom{n}{j}T_j^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)y^{n-j}. \tag{25}
$$

Proof By (16) with

$$
\begin{aligned}
p_n(x) &= \prod_{j=1}^{r}\left(\frac{e^{a_jt}-\lambda_j}{1-\lambda_j}\right)\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\\
&= x^n \sim (1,t),
\end{aligned}
$$

using (12), we have (25). □

3.3 Recurrence

Theorem 3

$$
\begin{aligned}
T_{n+1}^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) &= xT_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\\
&\quad - \sum_{j=1}^{r}\frac{a_j}{1-\lambda_j}T_n^{(r+1,k)}(x+a_j|a_1,\ldots,a_r,a_j;\lambda_1,\ldots,\lambda_r,\lambda_j)\\
&\quad - \frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}B_{n+1-l}
\end{aligned}
$$

$$\times \bigl(T_l^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) - T_l^{(r,k-1)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\bigr), \tag{26}$$

where B_n is the nth ordinary Bernoulli number.

Proof By applying

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x)$$

[4, Corollary 3.7.2] with (16), we get

$$T_{n+1}^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) = \left(x - \frac{g'_{r,k}(t)}{g_{r,k}(t)}\right)T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).$$

Now,

$$\begin{aligned}\frac{g'_{r,k}(t)}{g_{r,k}(t)} &= \bigl(\ln g_{r,k}(t)\bigr)' \\ &= \left(\sum_{j=1}^r \ln\bigl(e^{a_jt}-\lambda_j\bigr) - \sum_{j=1}^r \ln(1-\lambda_j) + \ln\bigl(1-e^{-t}\bigr) - \ln \mathrm{Li}_k\bigl(1-e^{-t}\bigr)\right)' \\ &= \sum_{j=1}^r \frac{a_je^{a_jt}}{e^{a_jt}-\lambda_j} + \frac{e^{-t}}{1-e^{-t}}\left(1 - \frac{\mathrm{Li}_{k-1}(1-e^{-t})}{\mathrm{Li}_k(1-e^{-t})}\right) \\ &= \sum_{j=1}^r \frac{a_je^{a_jt}}{e^{a_jt}-\lambda_j} + \frac{t}{e^t-1}\frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{t\,\mathrm{Li}_k(1-e^{-t})}.\end{aligned}$$

Since

$$T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) = \prod_{i=1}^r\left(\frac{1-\lambda_i}{e^{a_it}-\lambda_i}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}x^n,$$

we have

$$\begin{aligned}&T_{n+1}^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \\ &\quad = xT_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \\ &\qquad - \sum_{j=1}^r \frac{a_je^{a_jt}}{1-\lambda_j}\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\prod_{i=1}^r\left(\frac{1-\lambda_i}{e^{a_it}-\lambda_i}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}x^n \\ &\qquad - \frac{t}{e^t-1}\prod_{i=1}^r\left(\frac{1-\lambda_i}{e^{a_it}-\lambda_i}\right)\frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{t(1-e^{-t})}x^n.\end{aligned}$$

Since

$$\frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{1-e^{-t}} = \left(\frac{1}{2^k} - \frac{1}{2^{k-1}}\right)t + \cdots$$

is a delta series, we get

$$\frac{\mathrm{Li}_k(1-e^{-t})-\mathrm{Li}_{k-1}(1-e^{-t})}{t(1-e^{-t})}x^n=\frac{1}{n+1}\frac{\mathrm{Li}_k(1-e^{-t})-\mathrm{Li}_{k-1}(1-e^{-t})}{1-e^{-t}}x^{n+1}.$$

Therefore, by

$$\frac{t}{e^t-1}x^{n+1}=B_{n+1}(x)=\sum_{l=0}^{n+1}\binom{n+1}{l}B_{n+1-l}x^l,$$

we obtain

$$\begin{aligned}
&T_{n+1}^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\\
&\quad=xT_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)-\sum_{j=1}^{r}\frac{a_j}{1-\lambda_j}T_n^{(r+1,k)}(x+a_j|a_1,\ldots,a_r,a_j;\lambda_1,\ldots,\lambda_r,\lambda_j)\\
&\qquad-\frac{1}{n+1}\prod_{i=1}^{r}\left(\frac{1-\lambda_i}{e^{a_it}-\lambda_i}\right)\frac{\mathrm{Li}_k(1-e^{-t})-\mathrm{Li}_{k-1}(1-e^{-t})}{1-e^{-t}}\frac{t}{e^t-1}x^{n+1}\\
&\quad=xT_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)-\sum_{j=1}^{r}\frac{a_j}{1-\lambda_j}T_n^{(r+1,k)}(x+a_j|a_1,\ldots,a_r,a_j;\lambda_1,\ldots,\lambda_r,\lambda_j)\\
&\qquad-\frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}B_{n+1-l}\\
&\qquad\times\left(T_l^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)-T_l^{(r,k-1)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right),
\end{aligned}$$

which is identity (26). □

3.4 A more relation

Theorem 4

$$\begin{aligned}
&T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\\
&\quad=xT_{n-1}^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\\
&\qquad-\sum_{j=1}^{r}\frac{a_j}{1-\lambda_j}T_{n-1}^{(r+1,k)}(x+a_j|a_1,\ldots,a_r,a_j;\lambda_1,\ldots,\lambda_r,\lambda_j)\\
&\qquad-\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}B_{n-l}\bigl(T_l^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\\
&\qquad-T_l^{(r,k-1)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\bigr).
\end{aligned}\tag{27}$$

Proof For $n\geq 1$, we have

$$\begin{aligned}
T_n^{(r,k)}(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)&=\left\langle\sum_{l=0}^{\infty}T_l^{(r,k)}(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\frac{t^l}{l!}\middle|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|x^n\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \partial_t \left(\prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \left(\partial_t \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \right) e^{yt} \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \left(\partial_t e^{yt} \right) \middle| x^{n-1} \right\rangle \\
&= y T_{n-1}^{(r,k)}(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \\
&\quad + \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \left(\partial_t \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \right) e^{yt} \middle| x^{n-1} \right\rangle.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\partial_t \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) &= \prod_{j=1}^{r} (1-\lambda_j) \partial_t \left(\frac{1}{\prod_{j=1}^{r} (e^{a_j t} - \lambda_j)} \right) \\
&= \prod_{j=1}^{r} (1-\lambda_j) \frac{-(\prod_{j=1}^{r} (e^{a_j t} - \lambda_j))'}{(\prod_{j=1}^{r} (e^{a_j t} - \lambda_j))^2} \\
&= -\prod_{j=1}^{r} (1-\lambda_j) \frac{\sum_{j=1}^{r} a_j e^{a_j t} \prod_{i \neq j} (e^{a_i t} - \lambda_i)}{(\prod_{j=1}^{r} (e^{a_j t} - \lambda_j))^2} \\
&= -\sum_{j=1}^{r} \frac{a_j e^{a_j t}}{e^{a_j t} - \lambda_j} \prod_{i=1}^{r} \left(\frac{1-\lambda_i}{e^{a_i t} - \lambda_i} \right) \\
&= -\sum_{j=1}^{r} \frac{a_j e^{a_j t}}{1-\lambda_j} \frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \prod_{i=1}^{r} \left(\frac{1-\lambda_i}{e^{a_i t} - \lambda_i} \right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-1} \right\rangle \\
&\quad = -\sum_{j=1}^{r} \frac{a_j}{1-\lambda_j} \left\langle \frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \prod_{i=1}^{r} \left(\frac{1-\lambda_i}{e^{a_i t} - \lambda_i} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{(y+a_j)t} \middle| x^{n-1} \right\rangle \\
&\quad = -\sum_{j=1}^{r} \frac{a_j}{1-\lambda_j} T_{n-1}^{(r+1,k)}(y+a_j|a_1,\ldots,a_r,a_j;\lambda_1,\ldots,\lambda_r,\lambda_j).
\end{aligned}
$$

Since

$$\partial_t\left(\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\right)=\frac{e^{-t}(\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t}))}{(1-e^{-t})^2}$$
$$=\frac{t}{e^t-1}\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{t(1-e^{-t})}$$

and the fact that

$$\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\left(\frac{1}{2^{k-1}}-\frac{1}{2^k}\right)t+\cdots$$

is a delta series, we have

$$\left\langle\prod_{j=1}^r\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\left(\partial_t\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\right)e^{yt}\middle|x^{n-1}\right\rangle$$
$$=\left\langle\prod_{j=1}^r\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{t}{e^t-1}\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{t(1-e^{-t})}e^{yt}\middle|x^{n-1}\right\rangle$$
$$=\left\langle\prod_{j=1}^r\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{t}{e^t-1}\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t}}{1-e^{-t}}e^{yt}\middle|\frac{x^n}{n}\right\rangle$$
$$=\frac{1}{n}\left\langle\prod_{j=1}^r\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|\frac{t}{e^t-1}x^n\right\rangle$$
$$=\frac{1}{n}\sum_{l=0}^n\binom{n}{l}B_{n-l}\left\langle\prod_{j=1}^r\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|x^l\right\rangle$$
$$=\frac{1}{n}\sum_{l=0}^n\binom{n}{l}B_{n-l}\left(T_l^{(r,k-1)}(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)-T_l^{(r,k)}(y|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right).$$

Therefore, we obtain the desired result. □

Remark After n is replaced by $n+1$, identity (27) becomes the recurrence formula (26).

3.5 Relations with poly-Bernoulli numbers and Barnes' multiple Bernoulli numbers

Theorem 5

$$\sum_{m=0}^n(-1)^{n-m}\binom{n+1}{m}T_m^{(r,k)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)$$
$$=\sum_{l=0}^n\sum_{m=0}^l(-1)^{l-m}\binom{l}{m}\binom{n+1}{l+1}B_m^{(k-1)}H_{n-l}^{(r)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).\tag{28}$$

Proof We shall compute

$$\left\langle\prod_{j=1}^r\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\mathrm{Li}_k\left(1-e^{-t}\right)\middle|x^{n+1}\right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\mathrm{Li}_k\left(1-e^{-t}\right)\middle| x^{n+1}\right\rangle \\
&\quad=\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle| \left(1-e^{-t}\right)x^{n+1}\right\rangle \\
&\quad=\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle| x^{n+1}-(x-1)^{n+1}\right\rangle \\
&\quad=\sum_{m=0}^{n}\binom{n+1}{m}(-1)^{n-m}\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle| x^{m}\right\rangle \\
&\quad=\sum_{m=0}^{n}\binom{n+1}{m}(-1)^{n-m}T_m^{(r,k)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\mathrm{Li}_k\left(1-e^{-t}\right)\middle| x^{n+1}\right\rangle \\
&\quad=\left\langle \mathrm{Li}_k\left(1-e^{-t}\right)\middle| \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)x^{n+1}\right\rangle \\
&\quad=\left\langle \mathrm{Li}_k\left(1-e^{-t}\right)\middle| H_{n+1}^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right\rangle \\
&\quad=\left\langle \int_0^t \left(\mathrm{Li}_k\left(1-e^{-s}\right)\right)'\,ds\middle| H_{n+1}^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right\rangle \\
&\quad=\left\langle \int_0^t e^{-s}\frac{\mathrm{Li}_{k-1}(1-e^{-s})}{1-e^{-s}}\,ds\middle| H_{n+1}^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right\rangle \\
&\quad=\left\langle \int_0^t \left(\sum_{j=0}^{\infty}\frac{(-s)^j}{j!}\right)\left(\sum_{m=0}^{\infty}\frac{B_m^{(k-1)}}{m!}s^m\right)ds\middle| H_{n+1}^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right\rangle \\
&\quad=\left\langle \sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}B_m^{(k-1)}\right)\frac{1}{l!}\int_0^t s^l\,ds\middle| H_{n+1}^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right\rangle \\
&\quad=\sum_{l=0}^{n}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}\frac{B_m^{(k-1)}}{(l+1)!}\left\langle t^{l+1}\middle| H_{n+1}^{(r)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right\rangle \\
&\quad=\sum_{l=0}^{n}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}\frac{B_m^{(k-1)}}{(l+1)!}(n+1)_{l+1}H_{n-l}^{(r)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \\
&\quad=\sum_{l=0}^{n}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}\binom{n+1}{l+1}B_m^{(k-1)}H_{n-l}^{(r)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).
\end{aligned}$$

Here, $H_{n-l}^{(r)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)=H_{n-l}^{(r)}(0|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)$. Thus, we get (28). $\square$

3.6 Relations with the Stirling numbers of the second kind and the falling factorials

Theorem 6

$$\begin{aligned} &T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) \\ &\quad = \sum_{m=0}^{n}\left(\sum_{l=m}^{n}\binom{n}{l}S_2(l,m)T_{n-l}^{(r,k)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\right)(x)_m. \end{aligned} \tag{29}$$

Proof For (16) and $(x)_n \sim (1, e^t - 1)$, assume that

$$T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) = \sum_{m=0}^{n} C_{n,m}(x)_m.$$

By (13), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \frac{1}{\prod_{j=1}^{r}(\frac{e^{a_j t}-\lambda_j}{1-\lambda_j})\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}}\left(e^t-1\right)^m \middle| x^n\right\rangle \\ &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| \left(e^t-1\right)^m x^n\right\rangle \\ &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| m!\sum_{l=m}^{n} S_2(l,m)\frac{t^l}{l!}x^n\right\rangle \\ &= \sum_{l=m}^{n}\binom{n}{l}S_2(l,m)\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| x^{n-l}\right\rangle \\ &= \sum_{l=m}^{n}\binom{n}{l}S_2(l,m)T_{n-l}^{(r,k)}(a_1,\dots,a_r;\lambda_1,\dots,\lambda_r). \end{aligned}$$

Thus, we get identity (29). □

3.7 Relations with the Stirling numbers of the second kind and the rising factorials

Theorem 7

$$\begin{aligned} &T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) \\ &\quad = \sum_{m=0}^{n}\left(\sum_{l=m}^{n}\binom{n}{l}S_2(l,m)T_{n-l}^{(r,k)}(-m|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r)\right)(x)^{(m)}. \end{aligned} \tag{30}$$

Proof For (16) and $(x)^{(n)} = x(x+1)\cdots(x+n-1) \sim (1, 1-e^{-t})$, assume that $T_n^{(r,k)}(x|a_1,\dots,a_r;\lambda_1,\dots,\lambda_r) = \sum_{m=0}^{n} C_{n,m}(x)^{(m)}$. By (13), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \frac{1}{\prod_{j=1}^{r}(\frac{e^{a_j t}-\lambda_j}{1-\lambda_j})\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}}\left(1-e^{-t}\right)^m \middle| x^n\right\rangle \\ &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{-mt} \middle| \left(e^t-1\right)^m x^n\right\rangle \end{aligned}$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) \left\langle e^{-mt} \middle| \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^{n-l} \right\rangle$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) \langle e^{-mt} | T_{n-l}^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \rangle$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) T_{n-l}^{(r,k)}(-m|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).$$

Thus, we get identity (30). □

3.8 Relations with higher-order Frobenius-Euler polynomials

Theorem 8

$$T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) = \sum_{m=0}^{n} \left(\frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{l=0}^{s} \binom{s}{l} (-\lambda)^{s-l} T_{n-m}^{(r,k)}(l|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \right) H_m^{(s)}(x|\lambda). \tag{31}$$

Proof For (16) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t - \lambda}{1-\lambda} \right)^s, t \right),$$

assume that $T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\lambda)$. By (13), we have

$$C_{n,m} = \frac{1}{m!} \left\langle \left(\frac{e^t - \lambda}{1-\lambda} \right)^s \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} t^m \middle| x^n \right\rangle$$

$$= \frac{1}{m!(1-\lambda)^s} \sum_{l=0}^{s} \binom{s}{l} (-\lambda)^{s-l} \left\langle e^{lt} \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| t^m x^n \right\rangle$$

$$= \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{l=0}^{s} \binom{s}{l} (-\lambda)^{s-l} \left\langle e^{lt} \middle| \prod_{j=1}^{r} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^{n-m} \right\rangle$$

$$= \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{l=0}^{s} \binom{s}{l} (-\lambda)^{s-l} T_{n-m}^{(r,k)}(l|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).$$

Thus, we get identity (31). □

3.9 Relations with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n^{(r)}(x)}{n!} t^n$$

(see, *e.g.*, [4, Section 2.2]).

Theorem 9

$$T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) = \sum_{m=0}^{n}\binom{n}{m}\left(\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}S_2(l+s,s)T_{n-m-l}^{(r,k)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r)\right)\mathfrak{B}_m^{(s)}(x). \tag{32}$$

Proof For (16) and

$$\mathfrak{B}_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right),$$

assume that $T_n^{(r,k)}(x|a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) = \sum_{m=0}^{n} C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (13), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^t-1}{t}\right)^s \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^m \middle| x^n\right\rangle \\
&= \binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| \left(\frac{e^t-1}{t}\right)^s x^{n-m}\right\rangle \\
&= \binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| \sum_{l=0}^{n-m}\frac{s!}{(l+s)!}S_2(l+s,s)t^l x^{n-m}\right\rangle \\
&= \binom{n}{m}\sum_{l=0}^{n-m}\frac{s!}{(l+s)!}S_2(l+s,s)(n-m)_l\left\langle \prod_{j=1}^{r}\left(\frac{1-\lambda_j}{e^{a_jt}-\lambda_j}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| x^{n-m-l}\right\rangle \\
&= \binom{n}{m}\sum_{l=0}^{n-m}\frac{s!}{(l+s)!}S_2(l+s,s)(n-m)_l T_{n-m-l}^{(r,k)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r) \\
&= \binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}S_2(l+s,s)T_{n-m-l}^{(r,k)}(a_1,\ldots,a_r;\lambda_1,\ldots,\lambda_r).
\end{aligned}$$

Thus, we get identity (32). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Author details

[1]Department of Mathematics, Sogang University, Seoul, 121-741, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea. [3]Department of Applied Mathematics, Pukyong National University, Pusan, 608-739, Republic of Korea. [4]Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan.

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