Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Positive periodic solutions in shifts $\delta_\pm$ for a nonlinear first-order functional dynamic equation on time scales

Erbil Çetin*

*Correspondence:
erbil.cetin@ege.edu.tr
Department of Mathematics, Ege University, Bornova, Izmir, 35100, Turkey

Abstract

Let $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_\mathbb{T}$. We consider the existence of positive periodic solutions in shifts $\delta_\pm$ for the nonlinear functional dynamic equation of the form

$$x^\Delta(t) = -a(t)x^\sigma(t) + \lambda b(t)f\big(t, x\big(h(t)\big)\big), \quad t \in \mathbb{T}$$

using the cone theory techniques. We extend and unify periodic differential, difference, h-difference and q-difference equations and more by a new periodicity concept on time scales.

MSC: Primary 34N05; 39A12; secondary 35B10

Keywords: periodic time scale; periodic solution; shift operator; time scale

1 Introduction

Functional differential equations include many mathematical ecological and population models, such as the Lasota-Wazewska model [1–6], Nicholson's blowflies model [1, 4, 7–10], the model for blood cell production [1, 4, 9, 11] *etc.* Particularly, since the periodic variation of the environment plays an important role in many biological and ecological systems, many researchers have been interested in studying the existence of periodic solutions of the above models. Chow [12], Freedman and Wy [13], Hadeler and Tomiuk [14], Kuang [8], Wang [15], Weng and Sun [16] and many others studied the existence of at least one and at least two positive periodic solutions of nonlinear first-order differential equations using the fixed point theorem of cone expansion and the cone compression method, the upper and lower solution method and iterative technique [17]. On the other hand, it has been observed that very few papers exist in the literature on the existence of at least three and the nonexistence of a nonnegative periodic solution for first-order differential equations. For example, see [1, 15, 18].

In fact, both continuous and discrete systems are very important in implementation and application. Therefore, the study of dynamic equations on time scales, which unifies differential, difference, h-difference, q-differences equations and more, has received much attention; see [19–23]. The theory of dynamic equations on times-scales was introduced by Stefan Hilger in 1988 [24]. There are only a few results concerning periodic solutions of dynamic equations on time scales such as in [20, 25]. In these papers, all periodic time scales must be unbounded above and below, but there are many time scales that do not satisfy this condition such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$. Adıvar introduced a new periodicity concept in [26] with the aid of shift operators $\delta_\pm$. With the new periodicity concept, the time scale need not be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. There are only few existence results related with the new periodicity; see [19].

Let $\mathbb{T}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$. We are concerned with the existence, multiplicity and nonexistence of periodic solutions in shifts $\delta_\pm$ for the nonlinear dynamic equation

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + \lambda b(t)f\bigl(t, x\bigl(h(t)\bigr)\bigr), \quad t \in \mathbb{T}, \tag{1}$$

where $a \in C(\mathbb{T}, (0,\infty))$ is Δ-periodic in shifts $\delta_\pm$ with period T and $a \in \mathcal{R}$, $\lambda > 0$ is a positive parameter, $b \in C(\mathbb{T}, [0,\infty))$ is Δ-periodic in shifts with period T, $\int_{t_0}^{\delta_+^T(t_0)} b(s)\Delta s > 0$, $h \in C(\mathbb{T}, [0,\infty))$ is periodic in shifts with period T and $f \in C(\mathbb{T} \times (0,\infty), (0,\infty))$ is periodic in shifts $\delta_\pm$ with period T with respect to the first variable and $T \in [P, \infty)_{\mathbb{T}}$.

Hereafter, we use the notation $[a,b]_{\mathbb{T}}$ to indicate the time scale interval $[a,b] \cap \mathbb{T}$. The intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ and $(a,b)_{\mathbb{T}}$ are similarly defined.

In this study, we shall show that the number of positive periodic solutions in shifts $\delta_\pm$ of (1) can be determined by the asymptotic behaviors of the quotient of $\frac{f(t,x)}{x}$ at zero and infinity. We shall organize this paper as follows. In Section 2, we state some facts about exponential function on time scales, the new periodicity concept for time scales and some important theorems which will be needed to show the existence and nonexistence of periodic solutions in shifts $\delta_\pm$. Besides these, in Section 2, we give some lemmas about the exponential function and the graininess function with shift operators. We also present some lemmas to be used later. Finally, we state our main results and give their proofs in Section 3 by using the Krasnosel'skiĭ fixed point theorem.

2 Preliminaries

In this section, we mention some definitions, lemmas and theorems from calculus on time scales which can be found in [18, 27]. Next, we state some definitions, lemmas and theorems about the shift operators and the new periodicity concept for time scales which can be found in [26].

Definition 2.1 [27] A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$, while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{\varphi \in \mathcal{R} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $\varphi \in \mathcal{R}$ and $\mu(t) > 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_{\varphi}(t,s) = \exp\left(\int_s^t \zeta_{\mu(r)}\bigl(\varphi(r)\bigr)\Delta r\right), \tag{2}$$

where $\zeta_{\mu(s)}$ is the cylinder transformation given by

$$\zeta_{\mu(r)}\bigl(\varphi(r)\bigr) := \begin{cases} \frac{1}{\mu(r)}\operatorname{Log}(1 + \mu(r)\varphi(r)) & \text{if } \mu(r) > 0; \\ \varphi(r) & \text{if } \mu(r) = 0. \end{cases} \tag{3}$$

Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma ([27], Theorem 2.36).

Lemma 2.1 *Let* $p, q \in \mathcal{R}$. *Then*

i. $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
ii. $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
iii. $\frac{1}{e_p(t,s)} = e_{\ominus}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
iv. $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
v. $e_p(t,s)e_p(s,r) = e_p(t,r)$;
vi. $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
vii. $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;
viii. $(\frac{1}{e_p(\cdot,s)})^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

The following definitions, lemmas, corollaries and examples are about the shift operators and the new periodicity concept for time scales which can be found in [26].

Definition 2.2 [26] Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_\pm : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:

(P.1) The functions $\delta_\pm$ are strictly increasing with respect to their second arguments, *i.e.*, if

$$(T_0,t),(T_0,u) \in \mathcal{D}_\pm := \{(s,t) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_\mp(s,t) \in \mathbb{T}^*\},$$

then

$$T_0 \le t < u \quad \text{implies} \quad \delta_\pm(T_0,t) < \delta_\pm(T_0,u);$$

(P.2) If $(T_1,u),(T_2,u) \in \mathcal{D}_-$ with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$, and if $(T_1,u),(T_2,u) \in D_+$ with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$;
(P.3) If $t \in [t_0,\infty)_{\mathbb{T}}$, then $(t,t_0) \in D_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in D_+$ and $\delta_+(t_0,t) = t$ holds;
(P.4) If $(s,t) \in D_\pm$, then $(s,\delta_\pm(s,t)) \in D_\mp$ and $\delta_\mp(s,\delta_\pm(s,t)) = t$, respectively;
(P.5) If $(s,t) \in D_\pm$ and $(u,\delta_\pm(s,t)) \in D_\mp$, then $(s,\delta_\mp(u,t)) \in D_\pm$ and $\delta_\mp(u,\delta_\pm(s,t)) = \delta_\pm(s,\delta_\mp(u,t))$, respectively.

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be backward and forward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0,\infty)_{\mathbb{T}}$ in $\delta_\pm(s,t)$ is called the shift size. The value $\delta_+(s,t)$ and $\delta_-(s,t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets $\mathcal{D}_\pm$ are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 2.3 [26] (Periodicity in shifts) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shift

$\delta_\pm$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p,t) \in D_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$P := \inf\bigl\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p,t) \in D_\pm, \forall t \in \mathbb{T}^*\bigr\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Example 2.1 [26] The following time scales are periodic in the sense of shift operators given in Definition 2.3.

i. $\mathbb{T}_1 = \{\pm n^2 : n \in \mathbb{Z}\}$, $\delta_\pm(P,t) = \begin{cases} (\sqrt{t} \pm \sqrt{P})^2, & t > 0; \\ \pm P, & t = 0; \\ -(\sqrt{-t} \pm \sqrt{P})^2, & t < 0; \end{cases}$ $P = 1$, $t_0 = 0$.
ii. $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = q$, $t_0 = 1$.
iii. $\mathbb{T}_3 = \overline{\bigcup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1}]}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = 4$, $t_0 = 1$.
iv. $\mathbb{T}_4 = \{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\} \cup \{0,1\}$,

$$\delta_\pm(P,t) = \frac{q^{(\frac{\ln(\frac{t}{1-t}) \pm \ln(\frac{P}{1-P})}{\ln q})}}{1 + q^{(\frac{\ln(\frac{t}{1-t}) \pm \ln(\frac{P}{1-P})}{\ln q})}}, \quad P = \frac{q}{1+q}.$$

Notice that the time scale $\mathbb{T}_4$ in Example 2.1 is bounded above and below and $\mathbb{T}_4^* = \{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\}$.

Remark 2.1 [26] Let $\mathbb{T}$ be a time scale that is periodic in shifts with the period P. Thus, by (P.4) of Definition 2.2, the mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ defined by $\delta_+^P(t) = \delta_+(P,t)$ is surjective. On the other hand, by (P.1) of Definition 2.2, shift operators $\delta_\pm$ are strictly increasing in their second arguments. That is, the mapping $\delta_+^P(t) = \delta_+(P,t)$ is injective. Hence, δ_+^P is an invertible mapping with the inverse $(\delta_+^P)^{-1} = \delta_-^P$ defined by $\delta_-^P(t) := \delta_-(P,t)$.

We assume that $\mathbb{T}$ is a periodic time scale in shift $\delta_\pm$ with period P. The operators $\delta_\pm^P : \mathbb{T}^* \to \mathbb{T}^*$ are commutative with the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ given by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. That is, $(\delta_\pm^P \circ \sigma)(t) = (\sigma \circ \delta_\pm^P)(t)$ for all $t \in \mathbb{T}^*$.

Lemma 2.2 [26] *The mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ preserves the structure of the points in $\mathbb{T}^*$. That is,*

$$\sigma(t) = t \quad \textit{implies} \quad \sigma\bigl(\delta_+(P,t)\bigr) = \delta_+(P,t) \quad \textit{and}$$
$$\sigma(t) > t \quad \textit{implies} \quad \sigma\bigl(\delta_+(P,t)\bigr) > \delta_+(P,t).$$

Corollary 2.1 [26] $\delta_+(P,\sigma(t)) = \sigma(\delta_+(P,t))$ *and* $\delta_-(P,\sigma(t)) = \sigma(\delta_-(P,t))$ *for all* $t \in \mathbb{T}^*$.

Definition 2.4 [26] (Periodic function in shift $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T,t) \in D_\pm \quad \text{and} \quad f\bigl(\delta_\pm^T(t)\bigr) = f(t) \quad \text{for all } t \in \mathbb{T}^*, \tag{4}$$

where $\delta_\pm^T := \delta_\pm(T,t)$. The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ such that (4) holds is called the period of f.

Definition 2.5 [26] (Δ-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists $T \in [P,\infty)_{\mathbb{T}^*}$ such that

$$(T,t) \in D_\pm \quad \text{for all } t \in \mathbb{T}^*, \tag{5}$$

$$\text{the shifts } \delta_\pm^T \text{ are } \Delta\text{-differentiable with rd-continuous derivatives} \tag{6}$$

and

$$f\big(\delta_\pm^T(t)\big)\delta_\pm^{\Delta T} = f(t) \quad \text{for all } t \in \mathbb{T}^*, \tag{7}$$

where $\delta_\pm^T := \delta_\pm(T,t)$. The smallest number $T \in [P,\infty)_{\mathbb{T}^*}$ such that (5)-(7) hold is called the period of f.

Notice that Definition 2.4 and Definition 2.5 give the classic periodicity definition on time scales whenever $\delta_\pm^T := t \pm T$ are the shifts satisfying the assumptions of Definition 2.4 and Definition 2.5.

Now, we give a theorem which is the substitution rule on periodic time scales in shifts $\delta_\pm$ which can be found in [26].

Theorem 2.1 *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with period $P \in [t_0,\infty)_{\mathbb{T}^*}$, and let f be a Δ-periodic function in shifts $\delta_\pm$ with the period $T \in [P,\infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{\mathrm{rd}}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_\pm^T(t_0)}^{\delta_\pm^T(t)} f(s)\Delta s. \tag{8}$$

We give some interesting properties of the exponential functions $e_p(t,t_0)$ and shift operators on time scales which can be found in [19].

Lemma 2.3 *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^T$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P,\infty)_{\mathbb{T}^*}$. Then the graininess function $\mu : \mathbb{T} \to [0,\infty)$ satisfies*

$$\mu\big(\delta_\pm^T(t)\big) = \delta_\pm^{\Delta T}(t)\mu(t).$$

Lemma 2.4 *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^T$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P,\infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period T. Then*

i. $e_p(\delta_\pm^T(t), \delta_\pm^T(t_0)) = e_p(t,t_0)$ *for* $t, t_0 \in \mathbb{T}^*$,
ii. $e_p(\delta_\pm^T(t), \sigma(\delta_\pm^T(s))) = e_p(t,\sigma(s)) = \frac{e_p(t,s)}{1+\mu(t)p(t)}$ *for* $t, s \in \mathbb{T}^*$.

We define

$$P_T = \big\{x \in C(\mathbb{T},\mathbb{R}) : x\big(\delta_+^T(t)\big) = x(t)\big\},$$

where $\mathcal{C}(\mathbb{T},\mathbb{R})$ is the space of all real-valued continuous functions endowed with the norm

$$\|x\| = \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \left|x(t)\right|,$$

then P_T is a Banach space.

Lemma 2.5 [19] *Let $x \in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

Lemma 2.6 *$x(t) \in P_T$ is a solution of* (1) *if and only if*

$$x(t) = \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f\bigl(s,x\bigl(h(s)\bigr)\bigr)\Delta s,$$

where $G(t,s) = \frac{e_{a(s)}(s,t)}{e_{a(t)}(\delta_+^T(t),t)-1}$ is the Green's function.

Proof Let $x(t) \in P_T$ be a solution of (1). We can rewrite equation (1) as

$$x^\Delta(t) + a(t)x^\sigma(t) = \lambda b(t)f\bigl(t,x\bigl(h(t)\bigr)\bigr).$$

Multiply both sides of the above equation by $e_{a(t)}(t,t_0)$ and then integrate from t to $\delta_+^T(t)$ to obtain

$$\int_t^{\delta_+^T(t)} \bigl[x(s)e_{a(s)}(s,t_0)\bigr]^\Delta \Delta s = \lambda \int_t^{\delta_+^T(t)} e_{a(s)}(s,t_0)b(s)f\bigl(s,x\bigl(h(s)\bigr)\bigr)\Delta s.$$

We arrive at

$$\bigl[e_{a(t)}\bigl(\delta_+^T(t),t_0\bigr) - e_{a(t)}(t,t_0)\bigr]x(t) = \lambda \int_t^{\delta_+^T(t)} e_{a(s)}(s,t_0)b(s)f\bigl(s,x\bigl(h(s)\bigr)\bigr)\Delta s.$$

Dividing both sides of the above equation by $e_{a(t)}(t,t_0)$ and using Lemma 2.1, we have

$$x(t)\bigl[e_{a(t)}\bigl(\delta_+^T(t),t\bigr) - 1\bigr] = \lambda \int_t^{\delta_+^T(t)} e_{a(s)}(s,t)b(s)f\bigl(s,x\bigl(h(s)\bigr)\bigr)\Delta s. \tag{9}$$

We get

$$x(t) = \lambda \int_t^{\delta_+^T(t)} \frac{e_{a(s)}(s,t)}{e_{a(t)}(\delta_+^T(t),t) - 1} b(s)f\bigl(s,x\bigl(h(s)\bigr)\bigr)\Delta s.$$

Thus, the proof is complete. □

It is easy to verify that the Green's function $G(t,s)$ satisfies the property

$$0 < \frac{1}{\xi - 1} \le G(t,s) \le \frac{\xi}{\xi - 1} \quad \text{for } s \in \bigl[t,\delta_+^T(t)\bigr]_{\mathbb{T}}, \tag{10}$$

where $\xi = e_{a(t_0)}(\delta_+^T(t_0),t_0)$ and

$$G\bigl(\delta_+^T(t),\delta_+^T(s)\bigr) = G(t,s) \quad \text{for } t \in \mathbb{T}^*, s \in \bigl[t,\delta_+^T(t)\bigr]_{\mathbb{T}}. \tag{11}$$

Define K, a cone in P_T, by

$$K = \left\{ x \in P_T : x(t) \geq \frac{1}{\xi}\|x\|, \forall t \in \left[t_0, \delta_+^T(t_0)\right]_{\mathbb{T}} \right\}$$

and an operator $A_\lambda : K \to P_T$ by

$$(A_\lambda x)(t) = \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f\big(s, x\big(h(s)\big)\big)\Delta s.$$

Define

$$B := \int_{t_0}^{\delta_+^T(t_0)} b(s)\Delta s, \qquad C := \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} a(t). \tag{12}$$

Lemma 2.7 *$A_\lambda(K) \subset K$ and $A_\lambda : K \to K$ is compact and continuous.*

Proof By using Theorem 2.1, for $x \in K$, we have

$$\begin{aligned}
(A_\lambda x)\big(\delta_+^T(t)\big) &= \lambda \int_{\delta_+^T(t)}^{\delta_+^T(\delta_+^T(t))} G\big(\delta_+^T(t), s\big)b(s)f\big(s, x\big(h(s)\big)\big)\Delta s \\
&= \lambda \int_t^{\delta_+^T(t)} G\big(\delta_+^T(t), \delta_+^T(s)\big)b\big(\delta_+^T(s)\big)\delta_+^{T\Delta}(s)f\big(\delta_+^T(s), x\big(h\big(\delta_+^T(s)\big)\big)\big)\Delta s \\
&= \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f\big(s, x\big(h(s)\big)\big)\Delta s \\
&= (A_\lambda x)(t).
\end{aligned}$$

One can show that for $x \in K$, we have

$$\begin{aligned}
A_\lambda x(t) &\geq \lambda \int_t^{\delta_+^T(t)} \frac{1}{\xi - 1} b(s)f\big(s, x\big(h(s)\big)\big)\Delta s \\
&= \frac{1}{\xi}\lambda \int_{t_0}^{\delta_+^T(t_0)} \frac{\xi}{\xi - 1} b(s)f\big(s, x\big(h(s)\big)\big)\Delta s \\
&\geq \frac{1}{\xi}\|A_\lambda x\|.
\end{aligned}$$

Therefore, $A_\lambda(K) \subset K$. We will prove that A_λ is continuous and compact. Firstly, we will consider the continuity of A_λ. Let $x_n \in K$ and $\|x_n - x\| \to 0$ as $n \to \infty$, then $x \in K$ and $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ for any $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. Because of the continuity of f, for any $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$ and $\epsilon^* > 0$, we have

$$\begin{aligned}
\|A_\lambda x_n - A_\lambda x\| &= \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} |A_\lambda x_n - A_\lambda x| \\
&\leq \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)\big|f\big(s, x_n\big(h(s)\big)\big) - f\big(s, x\big(h(s)\big)\big)\big|\Delta s \\
&\leq \lambda\epsilon^* \frac{\xi}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s)\Delta s = \epsilon,
\end{aligned}$$

where $\epsilon^* = [\lambda B \frac{\xi}{\xi-1}]^{-1}\epsilon$. Thus A_λ is continuous on K.

Next, we prove that A_λ is a compact operator. It is equal to proving that A_λ maps bounded sets into relatively compact sets.

Let $S \subset K$ be an arbitrary bounded set in K, then there exists a number $R > 0$ such that $\|x\| < R$ for any $x \in K$. We prove that $\overline{A_\lambda S}$ is compact. In fact, for any $\{x_n\}_{n\in\mathbb{N}} \in K$ and $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$, we have

$$\begin{aligned}\|A_\lambda x_n\| &\leq \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)\bigl|f\bigl(s,x_n\bigl(h(s)\bigr)\bigr)\bigr|\Delta s \\ &\leq \lambda\frac{\xi}{\xi-1}\int_{t_0}^{\delta_+^T(t_0)} b(s)\bigl|f\bigl(s,x_n\bigl(h(s)\bigr)\bigr)\bigr|\Delta s := D\end{aligned}$$

and

$$\begin{aligned}\bigl\|A_\lambda^\Delta x_n\bigr\| &\leq \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\left|a(t)A_\lambda x_n(t) + \lambda\left(a(t) + \frac{a(t)}{1+\mu(t)a(t)}\right)\right. \\ &\quad \left.\times \int_t^{\delta_+^T(t)} G(t,s)b(s)f\bigl(s,x_n\bigl(h(s)\bigr)\bigr)\Delta s + \frac{\lambda b(t)f(t,x_n(h(t)))}{1+\mu(t)a(t)}\right| \\ &\leq C\|A_\lambda x_n\| + 2\lambda\frac{\xi}{\xi-1}C\int_{t_0}^{\delta_+^T(t_0)} b(s)\bigl|f\bigl(s,x_n\bigl(h(s)\bigr)\bigr)\bigr|\Delta s \\ &\quad + \lambda \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\bigl|b(t)\bigr|\bigl|f\bigl(t,x_n\bigl(h(t)\bigr)\bigr)\bigr| \\ &= 3CD + \lambda \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\bigl|b(t)\bigr|\bigl|f\bigl(t,x_n\bigl(h(t)\bigr)\bigr)\bigr| := L,\end{aligned}$$

which imply that $\{A_\lambda x_n\}_{n\in\mathbb{N}}$ and $\{A_\lambda^\Delta x_n\}_{n\in\mathbb{N}}$ are uniformly bounded on $[t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. There exists a subsequence of $\{A_\lambda x_n\}_{n\in\mathbb{N}}$ converging uniformly on $[t_0, \delta_+^T(t_0)]_{\mathbb{T}}$, namely, $\overline{A_\lambda S}$ is compact. The proof is complete. □

Lemma 2.8 *The existence of positive periodic solutions in shifts $\delta_\pm$ of* (1) *is equivalent to the existence of fixed point problem of A_λ in K.*

The proof of Lemma 2.8 is straightforward and hence omitted.

3 Main result

In this section, we consider the existence of one or two positive T-periodic solutions in $\delta_\pm$ of (1). Let us define

$$\underline{f_0} = \lim_{x\to 0^+}\min_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\frac{f(t,x)}{x}, \qquad \overline{f_0} = \lim_{x\to 0^+}\max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\frac{f(t,x)}{x},$$

$$\underline{f_\infty} = \lim_{x\to \infty}\min_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\frac{f(t,x)}{x}, \qquad \overline{f_\infty} = \lim_{x\to \infty}\max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\frac{f(t,x)}{x}.$$

To prove the results, we will use the following theorem which can be found in Krasnosel'skiĭ's book [28].

Theorem 3.1 (Guo-Krasnoselskiĭ fixed point theorem) *Let X be a Banach space, $K \subset X$ be a cone, and suppose that Ω_1 and Ω_2 are open, bounded subsets of X with $0 \in \Omega_1$ and*

$\overline{\Omega_1} \subset \Omega_2$. *Suppose further that* $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ *is a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$

holds. Then A *has a fixed point in* $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 3.2 *If either* $\underline{f_\infty} = \infty$, $\overline{f_0} = 0$ *or* $\underline{f_0} = \infty$, $\overline{f_\infty} = 0$ *holds, then equation* (1) *has a positive* T*-periodic solution* x *in shifts* $\delta_\pm$ *for*

$$\frac{\xi - 1}{\xi B} \leq \lambda \leq \frac{\xi - 1}{B}. \tag{13}$$

Proof At first, in view of $\underline{f_\infty} = \lim_{x\to\infty} \min_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \frac{f(t,x)}{x} = \infty$ uniformly on $[t_0, \delta_+^T(t_0)]_{\mathbb{T}}$, there exists $r > 0$ such that $f(t,x) \geq \mu x$ for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$, $x \geq r$, where $\mu \geq \xi^2$. We define $\Omega_1 = \{x \in P_T : \|x\| < \xi r\}$ and if $x \in \partial\Omega_1 \cap K$, then $\|x\| = \xi r$ and $r \leq x \leq \xi r$ for all t. We get

$$\begin{aligned}
\|A_\lambda x\| &\geq \frac{\lambda}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s) f\bigl(s, x\bigl(h(s)\bigr)\bigr)\Delta s \\
&\geq \frac{\lambda}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s)\mu x\bigl(h(s)\bigr)\Delta s \\
&\geq \frac{\xi r}{B} \int_{t_0}^{\delta_+^T(t_0)} b(s)\Delta s \\
&= \xi r = \|x\|
\end{aligned}$$

and so $\|A_\lambda x\| \geq \|x\|$ for all $x \in K \cap \partial\Omega_1$.

Next we use the assumption $\overline{f_0} = \lim_{x\to 0^+} \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \frac{f(t,x)}{x} = 0$ uniformly on $[t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. We can choose $R > \xi r > 0$ large enough such that

$$f(t,x) \leq \eta x, \quad 0 \leq x \leq R, t \in \bigl[t_0, \delta_+^T(t_0)\bigr]_{\mathbb{T}},$$

where $\eta \in (0, \frac{1}{\xi}]$. Then if Ω_2 is the ball in K centered at the origin with radius R and if $x \in K \cap \partial\Omega_2$, then we have

$$\begin{aligned}
\|A_\lambda x\| &\leq \frac{\lambda\xi}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s) f\bigl(s, x\bigl(h(s)\bigr)\bigr)\Delta s \\
&\leq \frac{\lambda\xi}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s)\eta x\bigl(h(s)\bigr)\Delta s \\
&\leq \frac{1}{B} \int_{t_0}^{\delta_+^T(t_0)} b(s) R\Delta s \\
&= R = \|x\|
\end{aligned}$$

and so $\|A_\lambda x\| \geq \|x\|$ for all $x \in K \cap \partial\Omega_2$. Consequently, Theorem 3.1 yields the existence of a positive T-periodic solution $x \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ of (1) in shifts $\delta_\pm$, that is,

$$\xi r < x(t) \leq R, \quad t \in \bigl[t_0, \delta_+^T(t_0)\bigr]_{\mathbb{T}}.$$

Next, let $\underline{f_0} = \infty, \overline{f_\infty} = 0$ hold. In view of $\underline{f_0} = \lim_{x\to 0^+} \min_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \frac{f(t,x)}{x} = \infty$, there exists $r_1 > 0$ such that

$$f(t,x) \geq \overline{\mu}x \quad \text{for } t \in \left[t_0, \delta_+^T(t_0)\right]_{\mathbb{T}}, 0 < x \leq r_1,$$

where $\overline{\mu} \geq \xi^2$. We define $\Omega_1 = \{x \in P_T : \|x\| < r_1\}$ and if $x \in \partial\Omega_1 \cap K$, then $\|x\| = r_1$ and $\frac{r_1}{\xi} \leq x \leq r_1$ for all t. We get

$$\begin{aligned}\|A_\lambda x\| &\geq \frac{\lambda}{\xi - 1}\int_{t_0}^{\delta_+^T(t_0)} b(s)f\bigl(s,x\bigl(h(s)\bigr)\bigr)\Delta s \\ &\geq \frac{\lambda}{\xi - 1}\int_{t_0}^{\delta_+^T(t_0)} b(s)\overline{\mu}x\bigl(h(s)\bigr)\Delta s \\ &\geq \frac{r_1}{B}\int_{t_0}^{\delta_+^T(t_0)} b(s)\Delta s \\ &= r_1 = \|x\|\end{aligned}$$

and so $\|A_\lambda x\| \geq \|x\|$ for all $x \in K \cap \partial\Omega_1$.

Next, since $\overline{f_\infty} = \lim_{x\to\infty} \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \frac{f(t,x)}{x} = 0$, there exists $r_2 > 0$ such that

$$f(t,x) \leq \overline{\eta}x \quad \text{for } x \geq r_2, t \in \left[t_0, \delta_+^T(t_0)\right]_{\mathbb{T}},$$

where $\overline{\eta} \in (0, \frac{1}{\xi}]$. Let $\overline{r_2} \geq \max\{2r_1, \xi r_2\}$ and it follows that $x(t) \geq \frac{1}{\xi}\|x\| \geq r_2$ for $x \in \partial\Omega_2$ and $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$, where $\Omega_2 = \{x \in P_T : \|x\| < \overline{r_2}\}$. For $x \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}\|A_\lambda x\| &\leq \frac{\lambda\xi}{\xi - 1}\int_{t_0}^{\delta_+^T(t_0)} b(s)f\bigl(s,x\bigl(h(s)\bigr)\bigr)\Delta s \\ &\leq \frac{\lambda\xi}{\xi - 1}\int_{t_0}^{\delta_+^T(t_0)} b(s)\overline{\eta}x\bigl(h(s)\bigr)\Delta s \\ &\leq \frac{1}{B}\int_{t_0}^{\delta_+^T(t_0)} b(s)\overline{r_2}\Delta s \\ &= \overline{r_2} = \|x\|,\end{aligned}$$

and again we have $\|A_\lambda x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_2$. It follows from part (ii) of Theorem 3.1 that A_λ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$ and this implies that our given equation (1) has a positive T-periodic solution x in shifts $\delta_\pm$, that is,

$$r_1 < x(t) \leq \overline{r_2}, \quad t \in \left[t_0, \delta_+^T(t_0)\right]_{\mathbb{T}}. \qquad \square$$

Theorem 3.3 *Let $\underline{f_0} = \underline{f_\infty} = \infty$ hold. Further, assume that there is a constant $R_1 > 0$ such that*

$$f\bigl(t,x(t)\bigr) \leq \frac{R_1}{\xi} \quad \textit{for } x \in [0,R_1], t \in \left[t_0, \delta_+^T(t_0)\right]_{\mathbb{T}}. \tag{14}$$

Then equation (1) *has two positive T-periodic solutions in shifts $\delta_\pm$ for*

$$\frac{\xi - 1}{\xi B} \leq \lambda \leq \frac{\xi - 1}{B}. \tag{15}$$

Proof At first, in view of $\underline{f_0} = \lim_{x\to 0^+} \min_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \frac{f(t,x)}{x} = \infty$, there exists $R_1 > R^* > 0$ such that

$$f(t,x) \geq \epsilon x \quad \text{for } t \in \left[t_0, \delta_+^T(t_0)\right]_{\mathbb{T}}, 0 < x \leq R^*,$$

where $\epsilon \geq \xi^2$. Set $\Omega_1 = \{x \in P_T : \|x\| < R^*\}$. Then, for $x \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|A_\lambda x\| &\geq \frac{\lambda}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s) f\bigl(s, x\bigl(h(s)\bigr)\bigr)\,\Delta s \\
&\geq \frac{\lambda}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s)\epsilon x\bigl(h(s)\bigr)\,\Delta s \\
&\geq \frac{\xi}{B} \int_{t_0}^{\delta_+^T(t_0)} b(s)\frac{R^*}{\xi}\,\Delta s \\
&= R^* = \|x\|,
\end{aligned}$$

which implies

$$\|A_\lambda x\| \geq \|x\| \quad \text{for } x \in K \cap \partial\Omega_1. \tag{16}$$

Next, since $\underline{f_\infty} = \lim_{x\to\infty} \min_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \frac{f(t,x)}{x} = \infty$, then for any $\alpha \geq \xi^2$ there exists $R_* > R_1$ such that $f(t,x) \geq \alpha x$ for $x \geq R_*$. Set $\Omega_2 = \{x \in P_T : \|x\| < R_*\}$. For $x \in K \cap \partial\Omega_2$, since $x \in K$, $x(t) \geq \frac{1}{\xi}\|x\| = \frac{1}{\xi}R_*$, we have

$$\begin{aligned}
\|A_\lambda x\| &\geq \frac{\lambda}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s) f\bigl(s, x\bigl(h(s)\bigr)\bigr)\,\Delta s \\
&\geq \frac{\lambda}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s)\alpha x\bigl(h(s)\bigr)\,\Delta s \\
&\geq \frac{\xi}{B} \int_{t_0}^{\delta_+^T(t_0)} b(s)\frac{R_*}{\xi}\,\Delta s \\
&= R_* = \|x\|,
\end{aligned}$$

which implies

$$\|A_\lambda x\| \geq \|x\| \quad \text{for } x \in K \cap \partial\Omega_2. \tag{17}$$

Finally, let $\Omega_3 = \{x \in P_T : \|x\| < R_1\}$. For $x \in K \cap \partial\Omega_3$, then from (14), we have

$$\begin{aligned}
\|A_\lambda x\| &\leq \frac{\lambda\xi}{\xi - 1} \int_{t_0}^{\delta_+^T(t_0)} b(s) f\bigl(s, x\bigl(h(s)\bigr)\bigr)\,\Delta s \\
&\leq \frac{\xi}{B} \int_{t_0}^{\delta_+^T(t_0)} b(s)\frac{R_1}{\xi}\,\Delta s \\
&= R_1 = \|x\|,
\end{aligned}$$

which implies

$$\|A_\lambda x\| \leq \|x\| \quad \text{for } x \in K \cap \partial\Omega_3. \tag{18}$$

Hence, since $R^* < R_1 < R_*$ and (16), (17) and (18), it follows from Theorem 3.1 that A_λ has a fixed point x_1 in $K \cap (\overline{\Omega_3} \setminus \Omega_1)$ and a fixed point x_2 in $K \cap (\overline{\Omega_2} \setminus \Omega_3)$. Both are positive T-periodic solutions in shifts $\delta_\pm$ of equation (1) and $0 < \|x_1\| < R_1 < \|x_2\|$. The proof is therefore complete. □

Theorem 3.4 *Let* $\overline{f_0} = \overline{f_\infty} = 0$ *hold. There exists a constant* $R_2 > 0$ *such that*

$$f\bigl(t, x(t)\bigr) \geq \xi R_2 \quad \textit{for } x \in \left[\frac{1}{\xi} R_2, R_2\right], t \in \bigl[t_0, \delta_+^T(t_0)\bigr]_{\mathbb{T}}.$$

Then equation (1) *has two positive* T*-periodic solutions in shifts* $\delta_\pm$ *for*

$$\frac{\xi - 1}{\xi B} \leq \lambda \leq \frac{\xi - 1}{B}.$$

Proof It can be proved similarly to the second part of Theorem 3.2 and Theorem 3.3. □

Competing interests
The author declares that they have no competing interests.

Received: 15 October 2013 Accepted: 13 February 2014 Published: 26 Feb 2014

References

1. Padhi, S, Srivastava, S: Existence of three periodic solutions for a nonlinear first order functional differential equation. J. Franklin Inst. **346**, 818-829 (2009)
2. Gopalsamy, K, Trofimchuk, SI: Almost periodic solutions of Lasota-Wazewska type delay differential equations. J. Math. Anal. Appl. **237**, 106-123 (1999)
3. Qiuxiang, F, Rong, Y: On the Lasota-Wazewska model with piecewise constant arguments. Acta Math. Sci., Ser. B **26**(2), 371-378 (2006)
4. Jiang, D, Wei, J, Jhang, B: Positive periodic solutions of functional differential equations and population models. Electron. J. Differ. Equ. **2002**, 71 (2002)
5. Luo, Y, Wang, W, Shen, J: Existence of positive periodic solutions for two kinds of neutral functional differential equations. Appl. Math. Lett. **21**(6), 581-587 (2008)
6. Wazewska-Czyzewska, M, Lasota, A: Mathematical problems of the dynamics of a system of red blood cells. Mat. Stosow. **6**, 23-40 (1976) (in Polish)
7. Gurney, WSC, Blathe, SP, Nishet, RM: Nicholson's blowflies revisited. Nature **287**, 17-21 (1980)
8. Kuang, Y: Delay Differential Equations with Applications in Population Dynamics. Academic Press, New York (1993)
9. Wan, A, Jiang, D: A new existence theory for positive periodic solutions to functional differential equations. Comput. Math. Appl. **47**, 1257-1262 (2004)
10. Alzabut, J, Bolat, Y, Abdeljawad, T: Almost periodic dynamics of a discrete Nicholson's blowflies model involving a linear harvesting term. Adv. Differ. Equ. **2012**, 158 (2012)
11. Gopalsamy, K: Stability and Oscillation in Delay Differential Equations of Population Dynamics. Kluwer Academic, Boston (1992)
12. Chow, SN: Existence of positive periodic solutions of autonomous functional differential equations. J. Differ. Equ. **15**, 350-378 (1974)
13. Freedman, HI, Wu, J: Periodic solutions of single-species models with periodic delay. SIAM J. Math. Anal. **23**, 689-701 (1992)
14. Hadeler, KP, Tomiuk, J: Periodic solutions of difference differential equations. Arch. Ration. Mech. Anal. **65**, 87-95 (1977)
15. Wang, H: Positive periodic solutions of functional differential equations. J. Differ. Equ. **202**, 354-366 (2004)
16. Weng, A, Sun, J: Positive periodic solutions of first-order functional difference equations with parameter. J. Comput. Appl. Math. **229**, 327-332 (2009)
17. Graef, JR, Kong, L: Existence of multiple periodic solutions for first order functional differential equations. Math. Comput. Model. **54**, 2962-2968 (2011)
18. Padhi, S, Srivastava, S, Pati, S: Three periodic solutions for a nonlinear first order functional differential equation. Appl. Math. Comput. **216**, 2450-2456 (2010)
19. Cetin, E, Topal, FS: Periodic solutions in shifts $\delta_\pm$ for a nonlinear dynamical equation on time scales. Abstr. Appl. Anal. **2012**, Article ID 707319 (2012)
20. Kaufmann, ER, Raffoul, YN: Periodic solutions for a neutral nonlinear dynamical equation on a time scale. J. Math. Anal. Appl. **319**, 315-325 (2006)
21. Li, Y, Zhu, L, Liu, P: Positive periodic solutions of nonlinear functional difference equations depending on a parameter. Comput. Math. Appl. **48**, 1453-1459 (2004)

22. Agarwal, RP, Zhang, W: Periodic solutions of difference equations with general periodicity. Comput. Math. Appl. **42**, 719-727 (2001)
23. Li, Y, Yang, L: Existence and stability of almost periodic solutions for Nicholson's blowflies model with patch structure and linear harvesting terms on time scales. Asian-Eur. J. Math. **5**, 1250038 (2012)
24. Hilger, S: Ein Masskettenkalkül mit Anwendug auf Zentrumsmanningfaltigkeiten. PhD thesis, Universität Würzburg (1988)
25. Liu, XL, Li, WT: Periodic solution for dynamic equations on time scales. Nonlinear Anal. **67**(5), 1457-1463 (2007)
26. Adıvar, M: A new periodicity concept for time scales. Math. Slovaca **63**(4), 817-828 (2013)
27. Bohner, M, Peterson, A: Dynamic Equations on Time Scales. An Introduction with Applications. Birkhäuser, Boston (2001)
28. Krasnoselsel'skiĭ, MA: Positive Solutions of Operator Equations. Noordhoff, Groningen (1964)

10.1186/1687-1847-2014-76
Cite this article as: Çetin: Positive periodic solutions in shifts $\delta_{\pm}$ for a nonlinear first-order functional dynamic equation on time scales. *Advances in Difference Equations* 2014, 2014:76