Lv *Advances in Difference Equations* 2014, 2014:69
http://www.advancesindifferenceequations.com/content/2014/1/69

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Existence results for m-point boundary value problems of nonlinear fractional differential equations with p-Laplacian operator

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Abstract

In this paper, we discuss the existence and multiplicity of positive solutions to m-point boundary value problems of nonlinear fractional differential equations with p-Laplacian operator

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + \varphi_p(\lambda)f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^{\gamma}u(1) = \sum_{i=1}^{m-2}\xi_i D_{0+}^{\gamma}u(\eta_i), \qquad D_{0+}^{\alpha}u(0) = 0, \end{cases}$$

where D_{0+}^{α}, D_{0+}^{β} and D_{0+}^{γ} are the standard Riemann-Liouville fractional derivatives with $1 < \alpha \le 2$, $0 < \beta, \gamma \le 1$, $0 \le \alpha - \beta - 1$, $\lambda \in (0,+\infty)$, $0 < \xi_i, \eta_i < 1$, $i = 1, 2, \dots, m-2$, $\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1} < 1$, $0 \le \alpha - \gamma - 1$, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, and $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. Our results are based on the monotone iterative technique and the theory of the fixed point index in a cone. Furthermore, two examples are also given to illustrate the results.

Keywords: fractional differential equation; m-point boundary value problems; p-Laplacian operator

1 Introduction

Fractional differential equations arise in various areas of science and engineering. Due to their applications, fractional differential equations have gained considerable attention (see, *e.g.*, [1–26] and the references therein).

Recently, there have been some papers dealing with the existence of solutions for nonlinear fractional differential equations with p-Laplacian operator. In [1], Wang *et al.* investigated the following boundary value problem for fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \qquad u(1) = au(\xi), \qquad D_{0+}^{\alpha}u(0) = 0, \qquad D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta), \end{cases}$$

where D_{0+}^{α}, D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives, $1 < \alpha, \beta \le 2$, $0 \le a, b \le 1$, $0 < \xi, \eta < 1$, $f(t,u) \in C[(0,1) \times (0,+\infty), [0,+\infty)]$.

In [2], Chai studied the existence of positive solutions of the following fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) + \sigma D_{0+}^{\gamma}u(1) = 0, \qquad D_{0+}^{\alpha}u(0) = 0, \end{cases}$$

where D_{0+}^{α}, D_{0+}^{β} and D_{0+}^{γ} are the standard Riemann-Liouville fractional derivatives with $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $0 \leq \alpha - \gamma - 1$, the constant σ is a positive number, $f(t,u) \in C(I \times \mathbb{R}_+, \mathbb{R}_+)$.

In [3], Chen and Liu studied the following fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}x(t))) = f(t,x(t)), \quad t \in [0,1], \\ x(0) = -x(1), \qquad D_{0+}^{\alpha}x(0) = -D_{0+}^{\alpha}x(1), \end{cases}$$

where $0 < \alpha, \beta \leq 1, 1 < \alpha + \beta \leq 2, D_{0+}^{\alpha}, D_{0+}^{\beta}$ are Caputo fractional derivatives, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

In [4], Lu *et al.* studied the following fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) = f(t,u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = u'(1) = 0, \qquad D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3, 1 < \beta \leq 2, D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, and $f(t,u) \in C([0,1] \times [0,+\infty), [0,+\infty))$.

On the other hand, in [5], Bai studied an eigenvalue interval of the following fractional boundary problem:

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) + \lambda h(t)f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, ${}^{c}D_{0+}^{\alpha}$ is the standard Caputo fractional derivative, $\lambda > 0$.

In [6], Zhang *et al.* studied the following singular eigenvalue problem for a higher order fractional differential equation:

$$\begin{cases} -D^{\alpha}x(t) = \lambda f(x(t), D^{\mu_1}x(t), D^{\mu_2}x(t), \ldots, D^{\mu_{n-1}}x(t)), \quad 0 < t < 1, \\ x(0) = 0, \qquad D^{\mu_i}x(0) = 0, \qquad D^{\mu}x(1) = \sum_{j=1}^{p-2} a_j D^{\mu}x(\xi_j), \quad 1 \leq i \leq n-1, \end{cases}$$

where $n \geq 3$, $n-1 < \alpha \leq n$, $n-l-1 < \alpha - \mu_l < n-1$ for $l = 1,2,\ldots,n-2$, and $\mu - \mu_{n-1} > 0$, $\alpha - \mu_{n-1} \leq 2$, $\alpha - \mu > 1$. D_{0+}^{α} is the standard Riemann-Liouville fractional derivative.

Moreover, in recent years, we have done some work on fractional differential equations [7–9]. In [7], we considered the following m-point boundary value problem for fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^{\beta}u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\beta}u(\eta_i), \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $n = [\alpha] + 1$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 \le \alpha - \beta - 1$, $0 < \xi_i, \eta_i < 1$, $i = 1,2,\dots,m-2$, and $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} < 1$.

Combining our work, in this paper, we discuss the existence of positive solutions for the following fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + \varphi_p(\lambda)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^{\gamma}u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\gamma}u(\eta_i), \qquad D_{0+}^{\alpha}u(0) = 0, \end{cases} \tag{1.1}$$

where D_{0+}^{α}, D_{0+}^{β} and D_{0+}^{γ} are the standard Riemann-Liouville fractional derivatives with $1 < \alpha \le 2$, $0 < \beta, \gamma \le 1$, $0 \le \alpha - \beta - 1$, $\lambda \in (0,+\infty)$, $0 < \xi_i, \eta_i < 1$, $i = 1,2,\dots,m-2$, $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} < 1$, $0 \le \alpha - \gamma - 1$, $f \in C([0,1] \times [0,+\infty),[0,+\infty))$, and $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

Our work presented in this paper has the following features. Firstly, to the best of the author's knowledge, there are few results on the existence of solutions for nonlinear fractional p-Laplacian differential equations with m-point boundary value problems. Secondly, we transform (1.1) into an equivalent integral equation and discuss the eigenvalue interval for the existence of multiplicity of positive solutions. The paper is organized as follows. In Section 2, we present some background materials and preliminaries. Section 3 deals with some existence results. In Section 4, two examples are given to illustrate the results.

2 Background materials and preliminaries

Definition 2.1 ([10, 11]) The fractional integral of order α with the lower limit t_0 for a function f is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s)\,ds, \quad t > t_0, \alpha > 0,$$

where Γ is the gamma function.

Definition 2.2 ([10, 11]) The Riemann-Liouville derivative of order α with the lower limit t_0 for a function f is

$$D_{t_0}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} (t-s)^{n-\alpha-1} f(s)\,ds, \quad t > t_0, \alpha > 0, n = [\alpha] + 1.$$

Lemma 2.1 ([12]) *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N} \quad \textit{for some } C_i \in \mathbb{R}, i = 1,2,\dots,N,$$

where N is the smallest integer greater than or equal to α.

Lemma 2.2 ([7]) *Let $y \in C[0,1]$. Then the fractional differential equation*

$$\begin{cases} D_{0+}^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, 1 < \alpha \le 2, \\ u(0) = 0, \qquad D_{0+}^{\beta}u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\beta}u(\eta_i) \end{cases}$$

has a unique solution which is given by

$$u(t) = \int_0^1 G(t,s)y(s)\,ds,$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s),$$

in which

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{1}{A\Gamma(\alpha)}[\sum_{0\le s\le \eta_i}(\xi_i {\eta_i}^{\alpha-\beta-1} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \xi_i t^{\alpha-1}(\eta_i - s)^{\alpha-\beta-1})], & t \in [0,1], \\ \frac{1}{A\Gamma(\alpha)}(\sum_{\eta_i \le s \le 1} \xi_i {\eta_i}^{\alpha-\beta-1} t^{\alpha-1}(1-s)^{\alpha-\beta-1}), & t \in [0,1], \end{cases}$$

where

$$A = 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}.$$

Lemma 2.3 ([7]) *If* $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} < 1$, *then the function* $G(t,s)$ *in Lemma* 2.2 *satisfies the following conditions*:

(i) $G(t,s) > 0$, *for* $s, t \in (0,1)$,

(ii) $G(t,s) \le G_*(s,s)$, *for* $s, t \in [0,1]$,

where

$$G_*(s,s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta-1} + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}.$$

Lemma 2.4 $G(t,s)$ *in* [7] *has the following property*:

$$G(t,s) \ge t^{\alpha-1} G(1,s).$$

Proof For $0 \le s \le t \le 1$, we conclude that

$$\begin{aligned} & t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ & \quad = t^{\alpha-1}\left[(1-s)^{\alpha-\beta-1} - \left(1 - \frac{s}{t}\right)^{\alpha-1}\right] \\ & \quad \ge t^{\alpha-1}\left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right]. \end{aligned}$$

Thus

$$G_1(t,s) \ge t^{\alpha-1} G_1(1,s).$$

It is obvious that

$$G_2(t,s) \ge t^{\alpha-1} G_2(1,s).$$

Therefore

$$G(t,s) \geq t^{\alpha-1}G(1,s). \qquad \square$$

Lemma 2.5 *Let $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, then BVP* (1.1) *has a unique solution*

$$u(t) = \lambda \int_0^1 G(t,s)\varphi_p^{-1}\bigl(I_{0+}^{\beta} f\bigl(s,u(s)\bigr)\bigr)\,ds.$$

Proof Let $w = D_{0+}^{\alpha} u$, $v = \varphi_p(w)$. From (1.1), we have

$$\begin{cases} D_{0+}^{\beta} v(t) + \varphi_p(\lambda) f(t,u(t)) = 0, & 0 < t < 1, \\ v(0) = 0. \end{cases}$$

By Lemma 2.1, we have

$$v(t) = c_1 t^{\beta-1} - I_{0+}^{\beta}\bigl(\varphi_p(\lambda) f\bigl(t,u(t)\bigr)\bigr), \quad 0 < t < 1.$$

It follows from $v(0) = 0$ that

$$v(t) = -I_{0+}^{\beta}\bigl(\varphi_p(\lambda) f\bigl(t,u(t)\bigr)\bigr), \quad 0 < t < 1.$$

Thus, from (1.1) we know that

$$\begin{cases} D_{0+}^{\alpha} u(t) = \varphi_p^{-1}(-I_{0+}^{\beta}(\varphi_p(\lambda) f(t,u(t)))), & 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\gamma} u(\eta_i). \end{cases}$$

By Lemma 2.2, (1.1) has a unique solution

$$u(t) = -\lambda \int_0^1 G(t,s)\varphi_p^{-1}\bigl(-I_{0+}^{\beta} f\bigl(s,u(s)\bigr)\bigr)\,ds.$$

It follows from $f \in C([0,1] \times [0,+\infty), [0,+\infty))$ that

$$-\int_0^1 G(t,s)\varphi_p^{-1}\bigl(-I_{0+}^{\beta} f\bigl(s,u(s)\bigr)\bigr)\,ds = \int_0^1 G(t,s)\varphi_p^{-1}\bigl(I_{0+}^{\beta} f\bigl(s,u(s)\bigr)\bigr)\,ds.$$

Thus

$$u(t) = \lambda \int_0^1 G(t,s)\varphi_p^{-1}\bigl(I_{0+}^{\beta} f\bigl(s,u(s)\bigr)\bigr)\,ds. \qquad \square$$

Lemma 2.6 ([27]) *Let E be a real Banach space, $P \subset E$ be a cone, $\Omega_r = \{u \in P : \|u\| \leq r\}$. Let the operator $T : P \cap \Omega_r \to P$ be completely continuous and satisfy $Tx \neq x$, $\forall x \in \partial\Omega_r$. Then*

(i) *If* $\|Tx\| \leq \|x\|$, $\forall x \in \partial\Omega_r$, *then* $i(T,\Omega_r,P) = 1$,

(ii) *If* $\|Tx\| \geq \|x\|$, $\forall x \in \partial\Omega_r$, *then* $i(T,\Omega_r,P) = 0$.

3 Main results

We consider the Banach space $E = C([0,1],\mathbb{R})$ endowed with the norm defined by $\|u\| = \sup_{0\leq t\leq 1}|u(t)|$. Let $P = \{u \in E | u(t) \geq 0\}$, then P is a cone in E. Define an operator $T : P \to P$ as

$$(Tu)(t) = \lambda \int_0^1 G(t,s)\varphi_p^{-1}\big(I_{0+}^{\beta} f\big(s,u(s)\big)\big)\,ds. \tag{3.1}$$

Then T has a solution if and only if the operator T has a fixed point.

Lemma 3.1 *If $f \in C([0,1] \times [0,+\infty),[0,+\infty))$, then the operator $T : P \to P$ is completely continuous.*

Proof From the continuity and non-negativeness of $G(t,s)$ and $f(t,u(t))$, we know that $T : P \to P$ is continuous.

Let $\Omega \subset P$ be bounded. Then, for all $t \in [0,1]$ and $u \in \Omega$, there exists a positive constant M such that $|f(t,u(t))| \leq M$. Thus,

$$\begin{aligned}
\big|(Tu)(t)\big| &= \left|\lambda \int_0^1 G(t,s)\varphi_p^{-1}\big(I_{0+}^{\beta} f\big(s,u(s)\big)\big)\,ds\right| \\
&\leq \lambda \int_0^1 G_*(s,s)\left(\int_0^s (s-\tau)^{\beta-1}\,d\tau\right)^{q-1} ds \frac{M^{q-1}}{(\Gamma(\beta))^{q-1}} \\
&= \lambda \frac{M^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 G_*(s,s)s^{(q-1)\beta}\,ds \\
&\leq \lambda \frac{M^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 G_*(s,s)\,ds \\
&= \lambda \frac{M^{q-1}L}{(\Gamma(\beta+1))^{q-1}},
\end{aligned}$$

where

$$L = \int_0^1 G_*(s,s)\,ds.$$

This means that $T(\Omega)$ is uniformly bounded.

On the other hand, from the continuity of $G(t,s)$ on $[0,1] \times [0,1]$, we see that it is uniformly continuous on $[0,1] \times [0,1]$. Thus, for fixed $s \in [0,1]$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$,

$$\big|G(t_1,s) - G(t_2,s)\big| < \frac{(\Gamma(\beta+1))^{q-1}}{\lambda M^{q-1}}\varepsilon.$$

Hence, for all $u \in \Omega$,

$$\begin{aligned}
&\big|(Tu)(t_2) - (Tu)(t_1)\big| \\
&\quad\leq \lambda \int_0^1 \big|G(t_2,s) - G(t_1,s)\big|\varphi_p^{-1}\big(I_{0+}^{\beta} f\big(s,u(s)\big)\big)\,ds
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda \int_0^1 \left|G(t_2,s) - G(t_1,s)\right| \left(\int_0^s (s-\tau)^{\beta-1}\, d\tau \right)^{q-1} ds \frac{M^{q-1}}{(\Gamma(\beta))^{q-1}} \\
&= \lambda \frac{M^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 \left|G(t_2,s) - G(t_1,s)\right| s^{(q-1)\beta}\, ds \\
&\leq \lambda \frac{M^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 \left|G(t_2,s) - G(t_1,s)\right| ds \\
&= \varepsilon,
\end{aligned}$$

which implies that $T(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, we obtain that $T : P \to P$ is completely continuous. The proof is complete. □

Theorem 3.2 *If $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $f(t,u)$ is nondecreasing in u and $\lambda \in (0,+\infty)$, then BVP* (1.1) *has a minimal positive solution $\overline{v}$ in B_r and a maximal positive solution $\overline{w}$ in B_r. Moreover, $v_m(t) \to \overline{v}(t)$, $w_m(t) \to \overline{w}(t)$ as $m \to \infty$ uniformly on* $[0,1]$, *where*

$$v_m(t) = \lambda \int_0^1 G(t,s)\varphi_p^{-1}\left(I_{0+}^\beta f\left(s, v_{m-1}(s)\right)\right) ds \tag{3.2}$$

and

$$w_m(t) = \lambda \int_0^1 G(t,s)\varphi_p^{-1}\left(I_{0+}^\beta f\left(s, w_{m-1}(s)\right)\right) ds. \tag{3.3}$$

Proof Let

$$B_r = \left\{u \in P : \|u\| \leq r\right\},$$

where

$$r \geq \frac{\lambda {M_1}^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 G_*(s,s)\, ds.$$

Step 1: Problem (1.1) has at least one solution.

For $u \in B_r$, there exists a positive constant M_1 such that $|f(t,u(t))| \leq M_1$,

$$\begin{aligned}
\left|(Tu)(t)\right| &= \left|\lambda \int_0^1 G(t,s)\varphi_p^{-1}\left(I_{0+}^\beta f\left(s,u(s)\right)\right) ds\right| \\
&\leq \frac{\lambda}{(\Gamma(\beta))^{q-1}} \int_0^1 G_*(s,s) \left(\int_0^s (s-\tau)^{\beta-1} f\left(\tau, u(\tau)\right) d\tau \right)^{q-1} ds \\
&\leq \frac{\lambda {M_1}^{q-1}}{(\Gamma(\beta))^{q-1}} \int_0^1 G_*(s,s) \left(\int_0^s (s-\tau)^{\beta-1}\, d\tau \right)^{q-1} ds \\
&= \frac{\lambda {M_1}^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 G_*(s,s) s^{\beta(q-1)}\, ds \\
&\leq \frac{\lambda {M_1}^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 G_*(s,s)\, ds.
\end{aligned}$$

Thus

$$T : B_r \to B_r.$$

By Lemma 3.1, we can see that $T : B_r \to B_r$ is completely continuous. Hence, by means of the Schauder fixed point theorem, the operator T has at least one fixed point, and BVP (1.1) has at least one solution in B_r.

Step 2: BVP (1.1) has a positive solution in B_r, which is a minimal positive solution.

From (3.1) and (3.2), one can see that

$$v_m(t) = (Tv_{m-1})(t), \quad t \in [0,1], m = 1,2,3,\ldots. \tag{3.4}$$

This, together with $f(t,u)$ being nondecreasing in u, yields that

$$0 = v_0(t) \le v_1(t) \le \cdots \le v_m(t) \le \cdots, \quad t \in [0,1].$$

Since T is compact, we obtain that $\{v_m\}$ is a sequentially compact set. Consequently, there exists $\overline{v} \in B_r$ such that $v_m \to \overline{v}$ $(m \to \infty)$.

Let $u(t)$ be any positive solution of BVP (1.1) in B_r. It is obvious that $0 = v_0(t) \le u(t) = (Tu)(t)$.

Thus,

$$v_m(t) \le u(t) \quad (m = 0,1,2,3,\ldots). \tag{3.5}$$

Taking limits as $m \to \infty$ in (3.5), we get $\overline{v}(t) \le u(t)$ for $t \in [0,1]$.

Step 3: BVP (1.1) has a positive solution in B_r, which is a maximal positive solution.

Let $w_0(t) = r$, $t \in [0,1]$ and $w_1(t) = Tw_0(t)$. From $T : B_r \to B_r$, we have $w_1 \in B_r$. Thus

$$0 \le w_1(t) \le r = w_0(t).$$

This, together with $f(t,u)$ being nondecreasing in u, yields that

$$\cdots \le w_m(t) \le \cdots \le w_1(t) \le w_0(t), \quad t \in [0,1].$$

Using a proof similar to that of Step 2, we can show that

$$w_m(t) \to \overline{w}(t) \quad (m \to \infty)$$

and

$$\overline{w}(t) = \int_0^1 G(t,s) f\bigl(s, \overline{w}(s)\bigr)\, ds.$$

Let $u(t)$ be any positive solution of BVP (1.1) in B_r.

Obviously,

$$u(t) \le w_0(t).$$

Thus

$$u(t) \le w_m(t). \tag{3.6}$$

Taking limits as $m \to \infty$ in (3.6), we obtain $u(t) \le \overline{w}(t)$ for $t \in [0,1]$.

The proof is complete. □

Define

$$f^0 = \lim_{u\to 0+} \sup_{t\in[0,1]} \frac{f(t,u)}{\varphi_p(l_1\|u\|)}, \qquad f_0 = \lim_{u\to 0+} \inf_{t\in[0,1]} \frac{f(t,u)}{\varphi_p(l_2\|u\|)},$$
$$f^\infty = \lim_{u\to +\infty} \sup_{t\in[0,1]} \frac{f(t,u)}{\varphi_p(l_3\|u\|)}, \qquad f_\infty = \lim_{u\to +\infty} \inf_{t\in[0,1]} \frac{f(t,u)}{\varphi_p(l_4\|u\|)}.$$

Let

$$B = \int_0^1 G_*(s,s)s^{\beta(q-1)}\,ds \quad \text{and} \quad B_1 = \int_0^1 G(1,s)s^{\beta(q-1)}\,ds.$$

Theorem 3.3 *Assume that* $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, *and the following conditions hold*:

(H_1) $f_0 = f_\infty = +\infty$.
(H_2) *There exists a constant* $\rho_1 > 0$ *such that* $f(t,u) \le \varphi_p(l_5\|u\|)$ *for* $t \in [0,1]$, $u \in [0,\rho_1]$.

Then BVP (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \rho_1 < \|u_2\|$$

for

$$\lambda \in \left(\frac{(\Gamma(\beta+1))^{q-1}}{l_2B_1}, \frac{(\Gamma(\beta+1))^{q-1}}{l_5B}\right) \cap \left(\frac{(\Gamma(\beta+1))^{q-1}}{l_4B_1}, \frac{(\Gamma(\beta+1))^{q-1}}{l_5B}\right), \tag{3.7}$$

where

$$l_2B_1 > l_5B \quad \textit{and} \quad l_4B_1 > l_5B.$$

Proof Since

$$f_0 = \lim_{u\to 0+} \inf_{t\in[0,1]} \frac{f(t,u)}{\varphi_p(l_2\|u\|)} = +\infty,$$

there is $\rho_0 \in (0,\rho_1)$ such that

$$f(t,u) \ge \varphi_p\bigl(l_2\|u\|\bigr) \quad \text{for } t \in [0,1], u \in [0,\rho_0].$$

Let

$$\Omega_{\rho_0} = \bigl\{u \in P : \|u\| \le \rho_0\bigr\}.$$

Then, for any $u \in \partial\Omega_{\rho_0}$, it follows from Lemma 2.4 that

$$\begin{aligned}(Tu)(t) &= \lambda \int_0^1 G(t,s)\varphi_p^{-1}\bigl(I_{0+}^{\beta} f\bigl(s,u(s)\bigr)\bigr)\,ds \\ &\geq \lambda \int_0^1 t^{\alpha-1}G(1,s)\varphi_p^{-1}\bigl(I_{0+}^{\beta}\bigl(\varphi_p\bigl(l_2\|u\|\bigr)\bigr)\bigr)\,ds \\ &= \lambda l_2 \int_0^1 t^{\alpha-1}G(1,s)\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\,d\tau\right)^{q-1} ds\|u\| \\ &= \frac{\lambda l_2}{(\Gamma(\beta+1))^{q-1}}\int_0^1 t^{\alpha-1}G(1,s)s^{\beta(q-1)}\,ds\|u\|.\end{aligned}$$

Thus

$$\|Tu\| \geq \frac{\lambda l_2 B_1}{(\Gamma(\beta+1))^{q-1}}\|u\|.$$

This, together with (3.7), yields that

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_{\rho_0}.$$

By Lemma 2.6, we have

$$i(T,\Omega_{\rho_0},P) = 0. \tag{3.8}$$

In view of

$$f_\infty = \lim_{u\to+\infty}\inf_{t\in[0,1]}\frac{f(t,u)}{\varphi_p(l_4\|u\|)} = +\infty,$$

there is ρ_0^*, $\rho_0^* > \rho_1$, such that

$$f(t,u) \geq \varphi_p\bigl(l_4\|u\|\bigr) \quad \text{for } t \in [0,1], u \in \bigl[\rho_0^*,+\infty\bigr).$$

Let

$$\Omega_{\rho_0^*} = \bigl\{u \in P : \|u\| \leq \rho_0^*\bigr\}.$$

Then, for any $u \in \partial\Omega_{\rho_0^*}$, it follows from Lemma 2.4 that

$$\begin{aligned}(Tu)(t) &= \lambda \int_0^1 G(t,s)\varphi_p^{-1}\bigl(I_{0+}^{\beta} f\bigl(s,u(s)\bigr)\bigr)\,ds \\ &\geq \lambda \int_0^1 t^{\alpha-1}G(1,s)\varphi_p^{-1}\bigl(I_{0+}^{\beta}\bigl(\varphi_p\bigl(l_4\|u\|\bigr)\bigr)\bigr)\,ds \\ &= \lambda l_4 \int_0^1 t^{\alpha-1}G(1,s)\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\,d\tau\right)^{q-1} ds\|u\| \\ &= \frac{\lambda l_4}{(\Gamma(\beta+1))^{q-1}}\int_0^1 t^{\alpha-1}G(1,s)s^{\beta(q-1)}\,ds\|u\|.\end{aligned}$$

Thus

$$\|Tu\| \geq \frac{\lambda l_4 B_1}{(\Gamma(\beta+1))^{q-1}}\|u\|.$$

This, together with (3.7), yields that

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_{\rho_0^*}.$$

By Lemma 2.6, we have

$$i(T, \Omega_{\rho_0^*}, P) = 0. \tag{3.9}$$

Finally, let $\Omega_{\rho_1} = \{u \in P : \|u\| \leq \rho_1\}$. For any $u \in \partial\Omega_{\rho_1}$, it follows from Lemma 2.3 and (H_2) that

$$\begin{aligned}(Tu)(t) &= \lambda \int_0^1 G(t,s)\varphi_p^{-1}\big(I_{0+}^{\beta} f\big(s,u(s)\big)\big)\,ds \\ &\leq \lambda \int_0^1 G_*(s,s)\varphi_p^{-1}\big(I_{0+}^{\beta}\big(\varphi_p\big(l_5\|u\|\big)\big)\big)\,ds \\ &= \lambda l_5 \int_0^1 G_*(s,s)\bigg(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\,d\tau\bigg)^{q-1} ds\|u\| \\ &= \frac{\lambda l_5}{(\Gamma(\beta+1))^{q-1}}\int_0^1 G_*(s,s)s^{\beta(q-1)}\,ds\|u\|.\end{aligned}$$

Thus

$$\|Tu\| \leq \frac{\lambda l_5 B}{(\Gamma(\beta+1))^{q-1}}\|u\|.$$

This, together with (3.7), yields that

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{\rho_1}.$$

Using Lemma 2.6, we get

$$i(T, \Omega_{\rho_1}, P) = 1. \tag{3.10}$$

From (3.8)-(3.10) and $\rho_0 < \rho_1 < \rho_0^*$, we have

$$i(T, \Omega_{\rho_0^*}\backslash\overline{\Omega}_{\rho_1}, P) = -1, \qquad i(T, \Omega_{\rho_1}\backslash\overline{\Omega}_{\rho_0}, P) = 1.$$

Therefore, T has a fixed point $u_1 \in \Omega_{\rho_1}\backslash\overline{\Omega}_{\rho_0}$ and a fixed point $u_2 \in \Omega_{\rho_0^*}\backslash\overline{\Omega}_{\rho_1}$. Clearly, u_1, u_2 are both positive solutions of BVP (1.1) and $0 < \|u_1\| < \rho_1 < \|u_2\|$. The proof of Theorem 3.3 is completed. □

In a similar way, we can obtain the following result.

Corollary 3.4 *Assume that* $f \in C([0,1] \times [0,+\infty),[0,+\infty))$, *and the following conditions hold*:

(H_1) $f^0 = f^\infty = 0$.
(H_2) *There exists a constant* $\rho_2 > 0$ *such that* $f(t,u) \geq \varphi_p(l_6\|u\|)$ *for* $t \in [0,1]$, $u \in [0,\rho_2]$.

Then BVP (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \rho_2 < \|u_2\|$$

for

$$\lambda \in \left(\frac{(\Gamma(\beta+1))^{q-1}}{l_6 B_1}, \frac{(\Gamma(\beta+1))^{q-1}}{l_3 B}\right) \cap \left(\frac{(\Gamma(\beta+1))^{q-1}}{l_6 B_1}, \frac{(\Gamma(\beta+1))^{q-1}}{l_1 B}\right),$$

where

$$l_6 B_1 > l_3 B \quad \textit{and} \quad l_6 B_1 > l_1 B.$$

4 Examples

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{1}{2}}(\varphi_2(D_{0+}^{\frac{3}{2}}u(t))) + \varphi_2(\lambda)((t+1)\pi\frac{|u(t)|}{1+|u(t)|}) = 0, & 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^{\frac{1}{2}}u(1) = \sum_{i=1}^{2}\xi_i D_{0+}^{\frac{1}{2}}u(\eta_i), \qquad D_{0+}^{\frac{3}{2}}u(0) = 0, \end{cases} \tag{4.1}$$

where

$$\alpha = \frac{3}{2}, \qquad \beta = \frac{1}{2}, \qquad \gamma = \frac{1}{2}, \qquad m = 4, \qquad p = q = 2,$$
$$\xi_1 = \eta_1 = \frac{1}{4}, \qquad \xi_2 = \eta_2 = \frac{1}{2}, \qquad \lambda \in (0,+\infty), \qquad f(t,u) = (t+1)\pi\frac{|u(t)|}{1+|u(t)|}.$$

Thus

$$f \in C\big([0,1] \times [0,+\infty),[0,+\infty)\big) \quad \text{and} \quad \big|f(t,u)\big| = \left|(t+1)\pi\frac{|u(t)|}{1+|u(t)|}\right| \leq 2\pi.$$

By computation, we deduce that

$$\sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1} = \xi_i + \xi_2 = \frac{3}{4},$$
$$A = 1 - \sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1} = \frac{1}{4}$$

and

$$\alpha - \gamma - 1 \geq 0.$$

On the other hand,

$$\begin{aligned}\int_0^1 G_*(s,s)\,ds &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\,ds + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1}\int_0^1 (1-s)^{\alpha-\beta-1}\,ds \\ &= \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1}\right)\int_0^1 (1-s)^{\alpha-\beta-1}\,ds \\ &= \frac{1}{\Gamma(\alpha)} + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i \\ &= \frac{1}{\frac{\sqrt{\pi}}{2}} + \frac{1}{\frac{1}{4}\frac{\sqrt{\pi}}{2}}\frac{3}{4} \\ &= \frac{2}{\sqrt{\pi}} + \frac{6}{\sqrt{\pi}} \\ &= \frac{8}{\sqrt{\pi}}.\end{aligned}$$

Take

$$\begin{aligned} r &\geq \frac{\lambda {M_1}^{q-1}}{(\Gamma(\beta+1))^{q-1}}\int_0^1 G_*(s,s)\,ds \\ &= \frac{\lambda 2\pi}{\frac{\sqrt{\pi}}{2}}\frac{8}{\sqrt{\pi}} \\ &= 32\lambda.\end{aligned}$$

Hence, by Theorem 3.2, BVP (4.1) has a minimal positive solution $\overline{v}$ in B_r and a maximal positive solution $\overline{w}$ in B_r.

Example 4.2 Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{1}{2}}(\varphi_{\frac{3}{2}}(D_{0+}^{\frac{3}{2}}u(t))) + \varphi_{\frac{3}{2}}(\lambda)(1+t)(\frac{1}{2}|u(t)|^{\frac{1}{3}} + \frac{1}{2}\|u\|^{\frac{1}{3}} + \|u\|^2) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^{\frac{1}{2}}u(1) = \sum_{i=1}^{2}\xi_i D_{0+}^{\frac{1}{2}}u(\eta_i), \qquad D_{0+}^{\frac{3}{2}}u(0) = 0, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} &\alpha = \frac{3}{2}, \qquad \beta = \frac{1}{2}, \qquad \gamma = \frac{1}{2}, \qquad p = \frac{3}{2}, \qquad q = 3, \qquad m = 4, \\ &\xi_1 = \eta_1 = \frac{1}{4}, \qquad \xi_2 = \eta_2 = \frac{1}{2}, \qquad \alpha - \gamma - 1 = 0, \qquad \alpha - \beta - 1 = 0 \end{aligned}$$

and

$$f(t,u) = (1+t)\left(\frac{1}{2}\left|u(t)\right|^{\frac{1}{3}} + \frac{1}{2}\|u\|^{\frac{1}{3}} + \|u\|^2\right).$$

It follows from Example 4.1 that

$$\sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1} = \xi_i + \xi_2 = \frac{3}{4}, \qquad A = 1 - \sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1} = \frac{1}{4}.$$

By computation, we deduce that

$$
\begin{aligned}
B &= \int_0^1 G_*(s,s)s^{\beta(q-1)}\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}s^{\beta(q-1)}\,ds + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\eta_i^{\alpha-\beta-1}\int_0^1 (1-s)^{\alpha-\beta-1}s^{\beta(q-1)}\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\frac{1}{2}\times(3-1)}\,ds + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\int_0^1 s^{\frac{1}{2}\times(3-1)}\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 s\,ds + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\int_0^1 s\,ds \\
&= \frac{1}{2}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{A\Gamma(\alpha)}\sum_{i=1}^{2}\xi_i\right) \\
&= \frac{1}{2}\left(\frac{2}{\sqrt{\pi}} + \frac{8}{\sqrt{\pi}}\left(\frac{1}{2}+\frac{1}{4}\right)\right) \\
&= \frac{1}{2}\times\frac{8}{\sqrt{\pi}} \\
&= \frac{4}{\sqrt{\pi}}
\end{aligned}
$$

and

$$
\begin{aligned}
B_1 &= \int_0^1 G(1,s)s^{\beta(q-1)}\,ds \\
&= \int_0^1 G_1(1,s)s^{\beta(q-1)}\,ds + \int_0^1 G_2(1,s)s^{\beta(q-1)}\,ds \\
&= \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 \bigl[1-(1-s)^{\frac{1}{2}}\bigr]s^{\beta(q-1)}\,ds + \frac{1}{A\Gamma(\frac{3}{2})}\int_0^{\frac{1}{4}}\bigl[\xi_1\eta_1^0(1-s)^0 - \xi_1(\eta_1-s)^0\bigr]s^{\beta(q-1)}\,ds \\
&\quad + \frac{1}{A\Gamma(\frac{3}{2})}\int_{\frac{1}{4}}^1 \xi_1\eta_1^0(1-s)^0 s^{\beta(q-1)}\,ds \\
&\quad + \frac{1}{A\Gamma(\frac{3}{2})}\int_0^{\frac{1}{2}}\bigl[\xi_2\eta_2^0(1-s)^0 - \xi_2(\eta_2-s)^0\bigr]s^{\beta(q-1)}\,ds \\
&\quad + \frac{1}{A\Gamma(\frac{3}{2})}\int_{\frac{1}{2}}^1 \xi_2\eta_2^0(1-s)^0 s^{\beta(q-1)}\,ds \\
&= \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 \bigl[s-s(1-s)^{\frac{1}{2}}\bigr]\,ds + \frac{1}{A\Gamma(\frac{3}{2})}\int_{\frac{1}{4}}^1 \xi_1 s\,ds + \frac{1}{A\Gamma(\frac{3}{2})}\int_{\frac{1}{2}}^1 \xi_2 s\,ds \\
&= \frac{2}{\sqrt{\pi}}\int_0^1 \bigl[s-s(1-s)^{\frac{1}{2}}\bigr]\,ds + \frac{2}{\sqrt{\pi}}\int_{\frac{1}{4}}^1 s\,ds + \frac{4}{\sqrt{\pi}}\int_{\frac{1}{2}}^1 s\,ds \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2}-\frac{4}{15}\right) + \frac{2}{\sqrt{\pi}}\frac{1}{2}s^2\Big|_{\frac{1}{4}}^1 + \frac{4}{\sqrt{\pi}}\frac{1}{2}s^2\Big|_{\frac{1}{2}}^1 \\
&= \frac{697}{240\sqrt{\pi}}.
\end{aligned}
$$

Taking

$$\rho_1 = 8, \qquad l_5 = 2{,}178,$$

we have

$$f(t,u) \le (1+1)(2+64) = 132 = \varphi_p\bigl(l_5\|u\|\bigr) = \varphi_{\frac{3}{2}}(2{,}178 \times 8) \quad \text{for } t \in [0,1], u \in [0,\rho_1].$$

Thus, condition (H_2) is satisfied. It is obvious that condition (H_1) holds.

On the other hand, let $l_2 = 4{,}000$, $l_4 = 3{,}600$, we have $l_2B_1 > l_5B$, $l_4B_1 > l_5B$ and

$$\begin{aligned}
\lambda &\in \left(\frac{(\Gamma(\beta+1))^{q-1}}{l_2B_1}, \frac{(\Gamma(\beta+1))^{q-1}}{l_5B}\right) \cap \left(\frac{(\Gamma(\beta+1))^{q-1}}{l_4B_1}, \frac{(\Gamma(\beta+1))^{q-1}}{l_5B}\right) \\
&= \left(\frac{(\Gamma(\beta+1))^{q-1}}{l_4B_1}, \frac{(\Gamma(\beta+1))^{q-1}}{l_5B}\right) \\
&= \left(\frac{(\frac{\sqrt{\pi}}{2})^2}{3{,}600 \times \frac{697}{240\sqrt{\pi}}}, \frac{(\frac{\sqrt{\pi}}{2})^2}{2{,}178 \times \frac{4}{\sqrt{\pi}}}\right) \\
&= \left(\frac{\pi^{\frac{3}{2}}}{41{,}820}, \frac{\pi^{\frac{3}{2}}}{34{,}848}\right).
\end{aligned}$$

Hence, by Theorem 3.3, BVP (4.2) has at least two solutions u_1 and u_2 such that $0 < \|u_1\| < 8 < \|u_2\|$ for

$$\lambda \in \left(\frac{\pi^{\frac{3}{2}}}{41{,}820}, \frac{\pi^{\frac{3}{2}}}{34{,}848}\right).$$

Competing interests

The author declares that he has no competing interests.

Acknowledgements

This research is supported by Henan Province College Youth Backbone Teacher Funds (2011GGJS-213) and the National Natural Science Foundation of China (11271336).

Received: 21 November 2013 Accepted: 6 February 2014 Published: 18 Feb 2014

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10.1186/1687-1847-2014-69
Cite this article as: Lv: Existence results for m-point boundary value problems of nonlinear fractional differential equations with p-Laplacian operator. *Advances in Difference Equations* 2014, 2014:69