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On the oscillation and asymptotic behavior for a kind of fractional differential equations

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Abstract

In this paper, we discuss the oscillations of the fractional order differential equation $D_a^\alpha x(t) + q(t)f(x(t)) = 0$, $t \in [a, +\infty)$, $a > 0$, where q is a positive real-valued function and f is a continuous functional; D_a^α denotes the Riemann-Liouville differential operator of order α, $0 < \alpha \leq 1$. We use the Riccati transformation technique to obtain some sufficient conditions which guarantee that every solution of the equation is oscillatory or the limit inferior converges to zero. Two examples are given to show the applications of our main results.

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Keywords: oscillation; fractional differential equation; Riemann-Liouville differential operator

1 Introduction

The theory of fractional calculus goes back to Leibniz's note in his list to L'Hospital [1], dated 30 September 1695, in which the meaning of the derivative of order 1/2 is discussed. After that in pure mathematics field the foundation of the fractional differential equations had been established. However, in recent years, many researchers found that the fractional differential equations are more accurate in describing some practical models, *e.g.* polymers. Today it has been used widely in physics, electrochemistry, control theory, and electromagnetic fields [2–7]. Furthermore, the fractional calculus can also provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a 'memory' term in the model [8–13]. Since these studies there has been much research actively concerned with the fractional differential equations and many useful achievements have been obtained [14–18].

From the 1960s, a lot of books and theses about the oscillatory behavior for first, second, and higher order differential equations are presented, see [19–21]. The study of the oscillatory problem with a view on fractional differential equation is just being initiated. As a new cross-cutting area, recently some attention has been paid to oscillations of fractional differential equations [22–29].

In 2012, Chen *et al.* [22] studied the oscillatory behavior of the following fractional differential equation:

$$\left[r(t)\left(D_-^\alpha y\right)^\eta(t)\right]' - q(t)f\left(\int_t^\infty (v-t)^{-\alpha} y(v)\,dv\right) = 0 \quad \text{for } t > 0,$$

where $D_-^\alpha y$ denotes the Liouville right-sided fractional derivative of order α with the form

$$(D_-^\alpha y)(t) := -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty (v-t)^{-\alpha}y(v)\,dv \quad \text{for } t \in \mathbb{R}_+ := (0,\infty).$$

By the Riccati transformation technique the authors obtained some sufficient conditions, which guarantee that every solution of the equation is oscillatory.

Using the same method, in 2013, Chen [23] studied oscillatory behavior of the fractional differential equation of the form

$$(D_-^{1+\alpha} y)(t) - p(t)(D_-^\alpha y)(t) + q(t)f\left(\int_t^\infty (v-t)^{-\alpha}y(v)\,dv\right) = 0 \quad \text{for } t > 0,$$

where $D_-^\alpha y$ is the Liouville right-sided fractional derivative of order $\alpha \in (0,1)$ of y.

Zhang [24] considered the oscillation of the nonlinear fractional differential equation with damping term,

$$\left[a(t)\left(D_-^\alpha x(t)\right)^\gamma\right]' + p(t)\left(D_-^\alpha x(t)\right)^\gamma - q(t)f\left(\int_t^\infty (\xi-t)^{-\alpha}x(\xi)\,d\xi\right) = 0, \quad t \in [t_0,\infty),$$

where $D_-^\alpha x(t)$ denotes the Liouville right-sided fractional derivative of order α of x. Using a generalized Riccati function and the inequality technique, he established some new oscillation criteria.

Han *et al.* [25] considered the oscillation for a class of fractional differential equations,

$$\left[r(t)g\left((D_-^\alpha y)(t)\right)\right]' - p(t)f\left(\int_t^\infty (s-t)^{-\alpha}y(s)\,ds\right) = 0 \quad \text{for } t > 0,$$

where $0 < \alpha < 1$ is a real number, $D_-^\alpha y$ is the Liouville right-sided fractional derivative of order α of y. By a generalized Riccati transformation technique, oscillation criteria for the nonlinear fractional differential equation are obtained.

Qi and Huang [26] studied the oscillation behavior of the equation of the form

$$\left(a(t)\left[r(t)D_-^\alpha x(t)\right]'\right)' + p(t)\left[r(t)D_-^\alpha x(t)\right]' - q(t)\int_t^\infty (\xi-t)^{-\alpha}x(\xi)\,d\xi = 0, \quad t \in [t_0,\infty),$$

where $D_-^\alpha x(t)$ also denotes the Liouville right-sided fractional derivative and some sufficient conditions for the oscillation of the equation have been given.

The above works on the oscillation are all concerned with fractional equations with Liouville right-sided fractional derivative by the Riccati transformation technique.

We notice that very little attention is paid to oscillations of fractional differential equations with a Riemann-Liouville derivative. For work studying the oscillatory behavior of fractional differential equations with the Riemann-Liouville derivative we refer to [27, 28], and [29].

In 2012, Grace *et al.* [27] studied the oscillation theory for fractional differential equations by considering equations of the form

$$D_a^q x + f_1(t,x) = v(t) + f_2(t,x), \qquad \lim_{ta+} J_a^{1-q}x(t) = b_1,$$

under the conditions

$$xf_i(t,x) > 0 \quad \text{for } i = 1,2, x \neq 0 \text{ and } t \geq a$$

and

$$\left|f_1(t,x)\right| > p_1(t)|x|^\beta \quad \text{and} \quad \left|f_2(t,x)\right| > p_2(t)|x|^\gamma \quad \text{for } x \neq 0 \text{ and } t \geq a,$$

where D_a^q denotes the Riemann-Liouville differential operator of order q with $0 < q \leq 1$, and the operator J_a^p is the Rieman-Liouville fractional integral operator. The authors obtained some new oscillation criteria by reducing the fractional differential equation to the equivalent Volterra fractional integral equation and by applying the inequality technique.

Marian [28] presented the oscillatory behavior of forced nonlinear fractional difference equations of the form

$$\Delta^\alpha x(t) + f_1\big(t, x(t+\alpha)\big) = v(t) + f_2\big(t, x(t+\alpha)\big), \quad t \in N_0, 0 < \alpha \leq 1, \Delta^{\alpha-1}x(t)|_{t=0} = x_0,$$

where Δ^α is a Riemann-Liouville like discrete fractional difference operator of order α, and some oscillation criteria are established by the same method in [27].

In 2013, Chen *et al.* [29] improved and extended some work in [27] by considering the forced oscillation of the fractional differential equation

$$D_a^q x + f_1(t,x) = v(t) + f_2(t,x), \qquad \lim_{ta+} J_a^{1-q} x(t) = b_1,$$

with the conditions

$$D_a^{q-k} x(a) = b_k \quad (k = 1, 2, \ldots, m-1)$$

and

$$\lim_{ta+} I_a^{m-q} x(t) = b_m,$$

where D_a^q denotes the Riemann-Liouville or Caputo differential operator of order q with $m-1 < q \leq m$, $m \geq 1$, and the operator I_a^{m-q} is the Rieman-Liouville fractional integral operator. The authors obtained some new oscillation criteria by the same method as [27].

Motivated by above work, in this paper we will extend some oscillation results from integer differential equations to the fractional differential equation

$$D_a^\alpha x(t) + q(t) f\big(x(t)\big) = 0, \quad t \in [a, +\infty), a > 0, \tag{1.1}$$

where D_a^α denotes the standard Riemann-Liouville differential operator of order α with $0 < \alpha \leq 1$, q is a positive real-valued function, f is a continuous functional defined on $[0, +\infty) \to [0, +\infty)$ satisfying

$$\frac{f(x)}{I^{2-\alpha} x} \geq K > 0, \tag{1.2}$$

and $I^{2-\alpha}$ denotes the Riemann-Liouville integral operator.

We will use the method of the Riccati transformation technique to study the oscillatory behavior of the fractional differential equation (1.1). To the best of our knowledge, there is not any result on the oscillation of the fractional differential equation involving the Riemann-Liouville derivative by the method of the Riccati transformation technique.

A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros on $[a, +\infty)$ and otherwise it is non-oscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

The paper is organized as follows. In the next section, we present some basic definitions of the fractional differential and integral operators, and provide some necessary lemmas. In Section 3, we mainly use the Riccati transformation technique to get some sufficient conditions which guarantee that every solution of (1.1) is oscillatory or the limit inferior converges to zero. Our results are essential new. Finally we provide some examples to show applications of our criteria.

2 Some preliminary lemmas

The operator D_a^α with $0 < \alpha < 1$ defined by

$$D_a^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} x(s)\,ds \tag{2.1}$$

is called the Riemann-Liouville derivative operator. The operator I_a^α defined by

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\,ds \tag{2.2}$$

is called the Riemann-Liouville integral operator. Using the integral operator I_a^α we can define D_a^α as

$$D_a^\alpha x(t) := \frac{d}{dt} I_a^{1-\alpha} x(t). \tag{2.3}$$

In general, if $n \geq 1$ is an integer and $n - 1 < \alpha \leq n$, then

$$D_a^\alpha x(t) := \frac{d^n}{dt^n} I_a^{n-\alpha} x(t). \tag{2.4}$$

The integral operator has the following properties, which will be used in the next lemma:

$$\text{(i)}\quad I_a^\alpha I_a^\beta f(t) = I_a^{\alpha+\beta} f(t), \qquad D_a^\alpha I_a^\alpha f(t) = f(t), \quad \alpha > 0, \beta > 0, f \in L(0,1); \tag{2.5}$$

$$\text{(ii)}\quad \Gamma(z+1) = z\Gamma(z). \tag{2.6}$$

The Riemann-Liouville integral operator also has a general relationship between $I_a^{2-\alpha}$ and $I_a^{1-\alpha}$ like

$$\begin{aligned}\left(I_a^{2-\alpha} x(t)\right)' &= \left(\frac{1}{\Gamma(2-\alpha)} \int_a^t (t-s)^{1-\alpha} x(s)\,ds\right)' \\ &= \frac{1-\alpha}{\Gamma(2-\alpha)} \int_a^t (t-s)^{-\alpha} x(s)\,ds + \frac{1}{\Gamma(2-\alpha)} (t-t)^{1-\alpha} x(t)\end{aligned}$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} x(s)\, ds$$
$$= \bigl(I_a^{1-\alpha} x\bigr)(t).$$

Lemma 2.1 [3] *Let $\alpha > 0$. Assume for x: $D_{0^+}^{\alpha} x \in L(0,1)$. Then the following equality holds*:

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α.

For more details on the Riemann-Liouville type fractional operators, see for example [2–7].

Before stating our main results, we begin with the following lemmas which are crucial in the proofs of the main results.

Lemma 2.2 *Suppose that x is an eventually positive solution of* (1.1) *and*

$$\liminf_{t\to\infty} x(t) = l \neq 0. \tag{2.7}$$

Then there is a sufficiently large $t_2 \in [a, +\infty)$ such that

$$I_a^{1-\alpha} x(t) > 0 \quad \textit{for } t \in [t_2, +\infty).$$

Proof Let x be an eventually positive solution of (1.1), which means that there exists a $t_0 \in [a, +\infty)$ such that $x(t) > 0$ for $t \in [t_0, +\infty)$.

From the condition (2.7) we can find a $t_1 > t_0$ and a constant l' with $0 < l' < l$ such that $x(t) > l'$ for $t \in [t_1, +\infty)$. So we can divide $I_a^{1-\alpha} x$ into three parts

$$\begin{aligned} I_a^{1-\alpha} x &= \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{1}{(t-s)^{\alpha}} x(s)\, ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^{t_0} \frac{1}{(t-s)^{\alpha}} x(s)\, ds + \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_1} \frac{1}{(t-s)^{\alpha}} x(s)\, ds \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t} \frac{1}{(t-s)^{\alpha}} x(s)\, ds. \end{aligned}$$

From Lemma 2.1 we know that if $D_a^{\alpha} x(t)$ exists, and this means $(t-s)^{-\alpha} x(s) \in L[a,t]$ for any $t \in [a, +\infty)$, especially $(t_0 - s)^{-\alpha} x(s) \in L[a, t_0]$.

Also we get $|(t_0 - s)^{-\alpha} x(s)| \in L[a, t_0]$. Therefore we can take $M = \frac{1}{\Gamma(1-\alpha)} \int_a^{t_0} |\frac{1}{(t-s)^{\alpha}} x(s)|\, ds$. Then we have

$$\begin{aligned} I_a^{1-\alpha} x &\geq -M + \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_1} \frac{1}{(t-s)^{\alpha}} x(s)\, ds + \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t} \frac{1}{(t-s)^{\alpha}} l'\, ds \\ &\geq -M + \frac{l'(t-t_1)^{1-\alpha}}{\Gamma(2-\alpha)}. \end{aligned}$$

Obviously there exists a sufficient large $t_2 \in [t_1, +\infty)$ such that $\frac{l'(t-t_1)^{1-\alpha}}{\Gamma(2-\alpha)} > M$. So $I_a^{1-\alpha} x(t) > 0$ for $t \in [t_2, +\infty)$. The proof is complete. □

Lemma 2.3 [30] *If X and Y are nonnegative, then*

$$X^{\lambda} - \lambda XY^{\lambda-1} + (\lambda - 1)Y^{\lambda} \geq 0, \quad \text{when } \lambda > 1 \tag{2.8}$$

and

$$X^{\lambda} - \lambda XY^{\lambda-1} - (1 - \lambda)Y^{\lambda} \leq 0, \quad \text{when } 0 < \lambda < 1, \tag{2.9}$$

where equality holds if and only if $X = Y$.

3 Main results

Theorem 3.1 *If there exists a positive function $\sigma \in C^1(0, +\infty)$ and a sufficiently large $t_2 \geq a$ such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[K\sigma(s)q(s) - \frac{(\sigma'_+(s))^2}{4\sigma(s)} \right] ds = \infty, \tag{3.1}$$

where $\sigma'_+(s) := \max\{\sigma'(s), 0\}$, then every solution x of (1.1) *is oscillatory or* $\liminf_{t\to\infty} x(t) = 0$.

Proof Assume to the contrary that there exists a non-oscillatory solution x of (1.1). Without loss of generality, we only consider the case when $x(t)$ is eventually positive, since the case when $x(t)$ is eventually negative is similar. Thus there exists $t_0 \in (a, +\infty)$ such that $x(t) > 0$ for $t \in [t_0, +\infty)$. Next we define the 'Riccati' type function w by

$$w(t) = \sigma(t)\frac{I_a^{1-\alpha}x(t)}{I_a^{2-\alpha}x(t)}. \tag{3.2}$$

If $\liminf_{t\to\infty} x(t) \neq 0$, from Lemma 2.2, there exists a $t_1 \in [a, +\infty)$ such that $I_a^{1-\alpha}x(t) > 0$ for $t > t_1$. Furthermore, using the same measure in Lemma 2.2, we can easily obtain the result that there exists a $t_2 \in [a, +\infty)$ such that $I_a^{2-\alpha}x(t) > 0$ for $t > t_2$. So we get $w(t) > 0$ for $t \in [t_2, +\infty)$.

Now differentiating $w(t)$ on $[t_2, \infty)$ we have

$$\begin{aligned} w'(t) &= \sigma'(t)\frac{I_a^{1-\alpha}x(t)}{I_a^{2-\alpha}x(t)} + \sigma(t)\left(\frac{I_a^{1-\alpha}x(t)}{I_a^{2-\alpha}x(t)}\right)' \\ &= \frac{\sigma'(t)}{\sigma(t)}w(t) + \sigma(t)\frac{I_a^{2-\alpha}x(t)(I_a^{1-\alpha}x(t))'}{(I_a^{2-\alpha}x(t))^2} - \sigma(t)\frac{I_a^{1-\alpha}x(t)(I_a^{2-\alpha}x(t))'}{(I_a^{2-\alpha}x(t))^2} \\ &= \frac{\sigma'(t)}{\sigma(t)}w(t) + \sigma(t)\frac{D_a^{\alpha}x(t)}{I_a^{2-\alpha}x(t)} - \sigma(t)\frac{(I_a^{1-\alpha}x(t))^2}{(I_a^{2-\alpha}x(t))^2} \\ &= \frac{\sigma'(t)}{\sigma(t)}w(t) - \sigma(t)\frac{q(t)f(x(t))}{I_a^{2-\alpha}x(t)} - \frac{w^2(t)}{\sigma(t)}. \end{aligned}$$

Then using condition (1.2) we get the inequality

$$w'(t) \leq \frac{\sigma'_+(t)}{\sigma(t)}w(t) - K\sigma(t)q(t) - \frac{w^2(t)}{\sigma(t)}. \tag{3.3}$$

Now taking

$$\lambda = 2, \qquad X = \frac{1}{\sigma^{\frac{1}{2}}(t)} w(t), \qquad Y = \frac{\sigma'_+(t)}{2\sigma^{\frac{1}{2}}(t)},$$

and using Lemma 2.3 and (3.3) we conclude that

$$w'(t) \leq -K\sigma(t)q(t) + \frac{(\sigma'_+(t))^2}{4\sigma(t)}.$$

Integrating both sides from t_2 to t, and letting $t \to +\infty$, we have

$$\int_{t_2}^{t} \left[K\sigma(s)q(s) - \frac{(\sigma'_+(s))^2}{4\sigma(s)} \right] ds \leq w(t_2) - w(t) < w(t_2).$$

So

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[K\sigma(s)q(s) - \frac{(\sigma'_+(s))^2}{4\sigma(s)} \right] ds \leq w(t_2) < +\infty,$$

which is a contradiction to the condition (3.1) and the proof is complete. □

Corollary 3.1 *Assume that* (1.2) *hold, and there exists a sufficient large* t_2 *such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[Ksq(s) - \frac{1}{4s} \right] ds = \infty. \tag{3.4}$$

Then every solution x *of* (1.1) *is either oscillatory or* $\liminf_{t\to\infty} x(t) = 0$.

Proof This follows from Theorem 3.1 by taking $\sigma(t) = t$. □

Corollary 3.2 *Assume that* (1.2) *hold, and there exists a sufficiently large* t_2 *such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} q(s)\, ds = \infty. \tag{3.5}$$

Then every solution x *of* (1.1) *is either oscillatory or* $\liminf_{t\to\infty} x(t) = 0$.

Proof Taking $\sigma(t) = 1$, then the condition (3.1) in Theorem 3.1 is reduced to (3.5). Hence the result is obtained from Theorem 3.1. □

Theorem 3.2 *Assume that* (1.2) *holds. Also, assume that there exist functions* $H \in C(D, \mathbb{R}^+)$, $\sigma \in C^1(0, +\infty)$ *such that*

$$H(t,t) = 0 \quad \textit{and} \quad H(t,s) > 0 \quad \textit{for } t > s \geq a,$$

where $D = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq a\}$ *and* H *has a nonpositive continuous partial derivative* $H'_s(t,s) := \frac{\partial H(t,s)}{\partial s}$ *on* D *with respect to the second variable. Also assume there exists a nonnegative continuous function* h *defined on* D *and a differentiable positive function* $\sigma(t)$ *satisfying for all* $t \in [a, +\infty)$

$$\frac{\sigma'_+(s)}{\sigma(s)} H(t,s) + H'_s(t,s) = \frac{1}{\sigma(s)} h(t,s) H^{\frac{1}{2}}(t,s), \tag{3.6}$$

where $\sigma_+'(s) := \max\{\sigma'(s), 0\}$. *If these assumptions hold and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[K\sigma(s)q(s)H(t,s) - \frac{h^2(t,s)}{4\sigma(s)}\right] ds = \infty, \tag{3.7}$$

then every solution x *of* (1.1) *is oscillatory or* $\liminf_{t\to\infty} x(t) = 0$.

Proof Suppose x is a non-oscillatory solution of (1.1). We only consider the case that $x(t)$ is eventually positive, since the case that $x(t)$ is eventually negative is similar. Assume that $x(t) > 0$ for all $t \ge t_1$ with large enough t_1.

If $\liminf_{t\to\infty} x(t) \neq 0$, we proceed as in the proof of Theorem 3.1 to see that (3.3) holds. Multiplying each side of (3.3) by $H(t,s)$ and integrating from t_1 to t, we obtain

$$\begin{aligned}\int_{t_1}^{t} K\sigma(s)q(s)H(t,s)\,ds \le{}& -\int_{t_1}^{t} H(t,s)w'(s)\,ds + \int_{t_1}^{t} H(t,s)\frac{\sigma_+'(s)}{\sigma(s)}w(s)\,ds \\ &- \int_{t_1}^{t} H(t,s)\frac{1}{\sigma(s)}w^2(s)\,ds.\end{aligned} \tag{3.8}$$

Then using the integration by parts formula and from (3.6), (3.8) we have

$$\begin{aligned}&\int_{t_1}^{t} K\sigma(s)q(s)H(t,s)\,ds \\ &\quad\le H(t,t_1)w(t_1) + \int_{t_1}^{t} H_s'(t,s)w(s)\,ds \\ &\qquad + \int_{t_1}^{t} H(t,s)\frac{\sigma_+'(s)}{\sigma(s)}w(s)\,ds - \int_{t_1}^{t} H(t,s)\frac{1}{\sigma(s)}w^2(s)\,ds \\ &\quad\le H(t,t_1)w(t_1) + \int_{t_1}^{t} \left\{\left[H_s'(t,s) + H(t,s)\frac{\sigma_+'(s)}{\sigma(s)}\right]w(s) - H(t,s)\frac{1}{\sigma(s)}w^2(s)\right\} ds \\ &\quad= H(t,t_1)w(t_1) + \int_{t_1}^{t} \left\{\left[\frac{1}{\sigma(s)}h(t,s)H^{\frac{1}{2}}(t,s)\right]w(s) - H(t,s)\frac{1}{\sigma(s)}w^2(s)\right\} ds.\end{aligned} \tag{3.9}$$

Taking

$$\lambda = 2, \qquad X = \left(H(t,s)\frac{1}{\sigma(s)}\right)^{\frac{1}{2}} w(s), \qquad Y = \frac{h(t,s)H^{\frac{1}{2}}(t,s)}{2H^{\frac{1}{2}}(t,s)} = \frac{h(t,s)}{2},$$

and using Lemma 2.3 we get

$$\int_{t_1}^{t} K\sigma(s)q(s)H(t,s)\,ds \le H(t,t_1)w(t_1) + \int_{t_1}^{t} \frac{h^2(t,s)}{4\sigma(s)}\,ds.$$

Therefore

$$\frac{1}{H(t,t_1)} \int_{t_1}^{t} \left\{K\sigma(s)q(s)H(t,s) - \frac{h^2(t,s)}{4\sigma(s)}\right\} ds \le w(t_1) < +\infty,$$

which contradicts (3.7). The proof is complete. □

Theorem 3.3 *Assume that* (1.2) *holds. Furthermore assume there is a positive function* $\sigma(t)$ *such that* $\sigma'(t)$ *is continuous on* $(0,+\infty)$ *and a sufficiently large* t_1 *satisfies*

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_1}^{t}(t-s)^m\left[K\sigma(s)q(s)-\frac{(\sigma'_+(s))^2}{4\sigma(s)}\right]ds=\infty, \tag{3.10}$$

where $m>1$. *Then every solution of* (1.1) *is either oscillatory or* $\liminf_{t\to\infty}x(t)=0$.

Proof Suppose x is a non-oscillatory solution of (1.1). We only consider the case that $x(t)$ is eventually positive, since the case that $x(t)$ is eventually negative is similar. Assume that $x(t)>0$ for all $t\geq t_1$ where t_1 is chosen large. If $\liminf_{t\to\infty}x(t)\neq 0$, proceeding as in Theorem 3.1, we get

$$w'(t)\leq -K\sigma(t)q(t)+\frac{(\sigma'_+(t))^2}{4\sigma(t)}.$$

Therefore,

$$\int_{t_1}^{t}(t-s)^m\left[K\sigma(s)q(s)-\frac{(\sigma'_+(s))^2}{4\sigma(s)}\right]ds\leq -\int_{t_1}^{t}(t-s)^m w'(s)\,ds. \tag{3.11}$$

Using the integration by parts formula leads to

$$\begin{aligned}\int_{t_1}^{t}(t-s)^m w'(s)\,ds&=(t-s)^m w(s)\big|_{s=t_1}^{s=t}+\int_{t_1}^{t}m(t-s)^{m-1}w(s)\,ds\\&=-(t-t_1)^m w(t_1)+\int_{t_1}^{t}m(t-s)^{m-1}w(s)\,ds\\&\geq -(t-t_1)^m w(t_1).\end{aligned} \tag{3.12}$$

Then from (3.11) we have

$$\int_{t_1}^{t}(t-s)^m\left[K\sigma(s)q(s)-\frac{(\sigma'_+(s))^2}{4\sigma(s)}\right]ds\leq (t-t_1)^m w(t_1),$$

and so

$$\frac{1}{t^m}\int_{t_1}^{t}(t-s)^m\left[K\sigma(s)q(s)-\frac{(\sigma'_+(s))^2}{4\sigma(s)}\right]ds\leq\left(\frac{t-t_1}{t}\right)^m w(t_1).$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_1}^{t}(t-s)^m\left[K\sigma(s)q(s)-\frac{(\sigma'_+(s))^2}{4\sigma(s)}\right]ds\leq w(t_1),$$

which is a contradiction of (3.10). So the proof is complete. □

4 Examples

In this section, we will show applications of our main results.

Example 4.1 Consider the fractional differential equation

$$D_a^\alpha x(t) + \frac{1}{\sqrt{t}} \int_a^t \frac{1}{\Gamma(2-\alpha)} (t-s)^{1-\alpha} x(s)\, ds = 0, \quad t > a > 0, \tag{4.1}$$

where $\alpha \in (0,1)$, D_a^α is the Riemann-Liouville differential operator. In (4.1), $q(t) = \frac{1}{\sqrt{t}}$, $f(x(t)) = \int_a^t \frac{1}{\Gamma(2-\alpha)} (t-s)^{1-\alpha} x(s)\, ds$. Set $K = 1$. Then $\frac{f(x)}{I^{2-\alpha}x} \geq K > 0$. Taking $\sigma(s) = s$, we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \int_{t_1}^t \left[K\sigma(s)q(s) - \frac{(\sigma_+'(s))^2}{4\sigma(s)} \right] ds \\ &\quad = \limsup_{t\to\infty} \int_{t_1}^t \left[\sqrt{s} - \frac{1}{4s} \right] ds \\ &\quad = \infty, \end{aligned}$$

which implies that all conditions in Theorem 3.1 hold. So by Theorem 3.1 every solution of (4.1) is oscillatory or $\liminf_{t\to\infty} x(t) = 0$.

Example 4.2 Consider the fractional differential equation

$$D_a^\alpha x(t) + e^t \int_a^t \frac{1}{\Gamma(2-\alpha)} (t-s)^{1-\alpha} x(s)\, ds = 0, \quad t > a, \tag{4.2}$$

where $\alpha \in (0,1)$, D_a^α is the Riemann-Liouville differential operator. In (4.2), $q(t) = e^t$, $f(x(t)) = \int_a^t \frac{1}{\Gamma(2-\alpha)} (t-s)^{1-\alpha} x(s)\, ds$. Set $K = 1$. Then $\frac{f(x)}{I^{2-\alpha}x} \geq K > 0$. Taking $\sigma(s) = 1$, and $m = 2$ we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^t (t-s)^m \left[K\sigma(s)q(s) - \frac{(\sigma_+'(s))^2}{4\sigma(s)} \right] ds \\ &\quad = \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_1}^t (t-s)^2 e^s\, ds \\ &\quad = \limsup_{t\to\infty} \frac{1}{t^2} \left[(t-s)^2 e^s \big|_{t_1}^t + \int_{t_1}^t 2(t-s)e^s\, ds \right] \\ &\quad = \limsup_{t\to\infty} \frac{1}{t^2} \left[-(t-t_1)^2 e^{t_1} + 2(t-s)e^s \big|_{t_1}^t + 2e^s \big|_{t_1}^t \right] \\ &\quad = \limsup_{t\to\infty} \frac{1}{t^2} \left[-(t-t_1)^2 e^{t_1} - 2(t-t_1)e^{t_1} + 2e^t - 2e^{t_1} \right] \\ &\quad = \infty, \end{aligned}$$

which yields the result that all conditions on Theorem 3.3 hold. Therefore, by Theorem 3.3 every solution of (4.2) is oscillatory or $\liminf_{t\to\infty} x(t) = 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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