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New identities and relations derived from the generalized Bernoulli polynomials, Euler polynomials and Genocchi polynomials

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Abstract

In this article, we give some identities for the q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials and recurrence relations between these polynomials in (Mahmudov in Discrete Dyn. Nat. Soc. 2012:169348, 2012; Mahmudov in Adv. Differ. Equ. 2013:1, 2013).

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1 Introduction, definitions and notations

In the usual notations, let $B_n(x)$ and $E_n(x)$ denote, respectively, the classical Bernoulli and Euler polynomials of degree n in x, defined by the generating functions

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{xt}, \quad |t| < 2\pi$$

and

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t + 1}e^{xt}, \quad |t| < \pi.$$

Also, let

$$B_n := B_n(0) \quad \text{and} \quad E_n := E_n(0),$$

where B_n and E_n are, respectively, the Bernoulli and Euler numbers of order n.

Carlitz first extended the classical Bernoulli polynomials and numbers, Euler polynomials and numbers [1]. There are numerous recent investigations on this subject by many authors. Cheon [2], Kurt [3], Luo [4], Luo and Srivastava [5], Srivastava *et al.* [6, 7], Tremblay *et al.* [8], and Mahmudov [9, 10].

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers.

The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1, \qquad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad n \in \mathbb{N}, a \in \mathbb{C},$$

respectively, where $[0]_q! = 1$, $n \in \mathbb{N}$, $a \in \mathbb{C}$. The q-polynomials coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q:q)_n}{(q:q)_{n-k}(q:q)_k},$$

where $(q:q)_n = (1-q)\cdots(1-q^n)_n$.

The q-analogue of the function $(x+y)_q^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k.$$

The q-binomial formula is known as

$$(n:q)_a = (1-a)_q^n = \prod_{j=0}^{n-1}\left(1-q^j a\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} (-1)^k a^k.$$

The q-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, |z| < \frac{1}{|1-q|}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty}\left(1+(1-q)q^k z\right), \quad 0 < |q| < 1, z \in \mathbb{C}.$$

From these forms, we easily see that $e_q(z)E_q(-z) = 1$. Moreover, $D_q e_q(z) = e_q(z)$, $D_q E_q(z) = E_q(qz)$, where D_q is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, 0 \neq z \in \mathbb{C}.$$

The above q-standard notation can be found in [10].

Mahmudov defined and studied properties of the following generalized q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ of order α as follows [10].

Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$ and $0 < |q| < 1$. The q-Bernoulli numbers $\mathcal{B}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1}\right)^{\alpha}, \quad |t| < 2\pi, \tag{1}$$

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx)E_q(ty), \quad |t| < 2\pi. \tag{2}$$

The q-Euler numbers $\mathcal{E}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t)+1}\right)^{\alpha}, \quad |t| < \pi, \tag{3}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t)+1}\right)^{\alpha} e_q(tx)E_q(ty), \quad |t| < \pi. \tag{4}$$

The q-Genocchi numbers $\mathcal{G}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t)+1}\right)^{\alpha}, \quad |t| < \pi, \tag{5}$$

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t)+1}\right)^{\alpha} e_q(tx)E_q(ty), \quad |t| < \pi. \tag{6}$$

It is obvious that

$$\mathcal{B}_{n,q}^{(\alpha)} = \mathcal{B}_{n,q}^{(\alpha)}(0,0), \qquad \lim_{q\to 1^-} \mathcal{B}_{n,q}^{(\alpha)}(x,y) = \mathcal{B}_{n}^{(\alpha)}(x+y), \qquad \lim_{q\to 1^-} \mathcal{B}_{n,q}^{(\alpha)} = \mathcal{B}_{n}^{(\alpha)},$$
$$\mathcal{E}_{n,q}^{(\alpha)} = \mathcal{E}_{n,q}^{(\alpha)}(0,0), \qquad \lim_{q\to 1^-} \mathcal{E}_{n,q}^{(\alpha)}(x,y) = \mathcal{E}_{n}^{(\alpha)}(x+y), \qquad \lim_{q\to 1^-} \mathcal{E}_{n,q}^{(\alpha)} = \mathcal{E}_{n}^{(\alpha)}$$

and

$$\mathcal{G}_{n,q}^{(\alpha)} = \mathcal{G}_{n,q}^{(\alpha)}(0,0), \qquad \lim_{q\to 1^-} \mathcal{G}_{n,q}^{(\alpha)}(x,y) = \mathcal{G}_{n}^{(\alpha)}(x+y), \qquad \lim_{q\to 1^-} \mathcal{G}_{n,q}^{(\alpha)} = \mathcal{G}_{n}^{(\alpha)}.$$

From (2), (4) and (6), it is easy to check that

$$\mathcal{B}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{B}_{n-k,q}(x,0)\mathcal{B}_{k,q}^{(\alpha-1)}(0,y),$$
$$\mathcal{E}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{E}_{n-k,q}(x,0)\mathcal{E}_{k,q}^{(\alpha-1)}(0,y)$$

and

$$\mathcal{G}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{G}_{n-k,q}(x,0)\mathcal{G}_{k,q}^{(\alpha-1)}(0,y).$$

In this work, we give a different form of the analogue of the Srivastava-Pintér addition theorem.

More precisely, we prove

$$\begin{aligned} \mathcal{G}_{n,q}(x,y) &= y\mathcal{G}_{n-1,q}(x,qy) + x\mathcal{G}_{n-1,q}(x,y) \\ &\quad + \frac{1}{[n]_q}\left\{\mathcal{G}_{n,q}(x,y) - \frac{1}{2}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{G}_{k,q}(x,y)\mathcal{G}_{n-k,q}(1,0)\right\}, \end{aligned}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{G}_{k,q}(x,y) + \mathcal{G}_{n,q}(x,y) = 2[n]_q(x+y)_q^{n-1},$$

$$\begin{aligned}
&\mathcal{G}_{n,q}^{(\alpha)}(x,y) \\
&\quad = \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \left\{ \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{G}_{j,q}^{(\alpha)}(x,0) m^{j-k} + \mathcal{G}_{k,q}^{(\alpha)}(x,0) \right\} \mathcal{G}_{n+1-k,q}(0,my) m^{k-n} \\
&\quad = \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \left\{ \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{G}_{j,q}^{(\alpha)}(0,y) m^{j-k} + \mathcal{G}_{k+1,q}^{(\alpha)}(0,y) \right\} \\
&\qquad \times \mathcal{G}_{n+1-k,q}(mx,0) m^{k-n},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{G}_{n,q}^{(\alpha)}(x,y) \\
&\quad = \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \left\{ \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{G}_{j,q}^{(\alpha)}(x,0) m^{j-n} - \mathcal{G}_{k,q}^{(\alpha)}(x,0) \right\} \mathcal{B}_{n+1-k,q}(0,my) m^{k-n},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}_{n,q}^{(\alpha)}(x,y) \\
&\quad = \frac{1}{2} \sum_{r=0}^{n+1} \begin{bmatrix} n+1 \\ r \end{bmatrix}_q \frac{1}{[n+1]_q} \left(\sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \mathcal{B}_{k,q}^{(\alpha)}(x,0) m^{k-r} + \mathcal{B}_{r,q}^{(\alpha)}(x,0) \right) \\
&\qquad \times \mathcal{G}_{n+1-r,q}(0,my) m^{r-n}.
\end{aligned}$$

2 Main theorems

Proposition 2.1 *The generalized q-Bernoulli and q-Euler polynomials satisfy the following relations*:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{B}_{k,q}^{(\alpha)}(x,0) \mathcal{B}_{n-k,q}^{(-\alpha)} = x^n, \tag{7}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{B}_{k,q}^{(\alpha)}(0,y) \mathcal{B}_{n-k,q}^{(-\alpha)} = q^{\frac{n(n-1)}{2}} y^n, \tag{8}$$

$$\mathcal{B}_{n,q}^{(\alpha)}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{B}_{n-l,q}^{(\alpha)}(0,y) \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q \mathcal{E}_{k,q}^{(\alpha)}(x,0) \mathcal{E}_{l-k,q}^{(-\alpha)}(0,0), \tag{9}$$

$$\mathcal{E}_{n,q}^{(\alpha)}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{E}_{n-l,q}^{(\alpha)}(0,y) \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q \mathcal{E}_{k,q}^{(\alpha)}(x,0) \mathcal{B}_{l-k,q}^{(-\alpha)}(0,0). \tag{10}$$

Proposition 2.2 *For $x, y, z \in \mathbb{C}$, the following relations hold true*:

$$\mathcal{G}_{n,q}^{(\alpha)}(x+z,y) = \sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix}_q \mathcal{G}_{n-p,q}^{(\alpha)}(0,y) \sum_{r=0}^{p} \begin{bmatrix} p \\ r \end{bmatrix}_q x^r z^{p-r}, \tag{11}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{G}_{k,q}^{(\alpha)}(x,y) \mathcal{G}_{n-k,q}^{(-\alpha)}(0,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} q^{\frac{(n-k)(n-k-1)}{2}} = (x+y)_q^n. \tag{12}$$

Proof The proof of these propositions can be found from (1)-(6). □

Theorem 2.3 *The generalized q-Genocchi polynomials satisfy the following recurrence relation*:

$$\mathcal{G}_{n,q}(x,y) = y\mathcal{G}_{n-1,q}(x,qy) + x\mathcal{G}_{n-1,q}(x,y) + \frac{1}{[n]_q}\left\{\mathcal{G}_{n,q}(x,y) - \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{G}_{k,q}(x,y)\mathcal{G}_{n-k,q}(1,0)\right\}. \tag{13}$$

Proof In (6) for $\alpha = 1$, we take the q-derivative of the generalized q-Genocchi polynomials $\mathcal{G}_{n,q}(x,y)$ according to t. We note that

$$\begin{aligned}\sum_{n=0}^{\infty} D_{q,t}\mathcal{G}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= D_{q,t}\left\{\frac{2t}{e_q(t)+1}e_q(tx)E_q(yt)\right\} \\ &= \frac{2e_q(tx)E_q(yt)}{e_q(t)+1} + \frac{y2te_q(tx)E_q(yt)}{e_q(t)+1} + \frac{x2te_q(tx)E_q(yt)}{e_q(t)+1} \\ &\quad - \frac{2te_q(tx)E_q(yt)}{e_q(t)+1}\frac{e_q(x)}{e_q(t)+1}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_{n+1,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{t}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x,y)\frac{t^n}{[n]_q!} + y\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x,qy)\frac{t^n}{[n]_q!} \\ &\quad + x\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x,y)\frac{t^n}{[n]_q!} - \frac{1}{2t}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(1,0)\frac{t^n}{[n]_q!}.\end{aligned}$$

If we take necessary operation, comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have (13). □

Theorem 2.4 *There is the following relation for the q-Genocchi polynomials*:

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\mathcal{G}_{k,q}^{(\alpha)}(x,0) + \mathcal{G}_{k,q}^{(\alpha)}(x,-1)\right) = 2[n]_q\mathcal{G}_{n-1,q}^{(\alpha-1)}(x,0). \tag{14}$$

Proof From (6) and $e_q(z)E_q(-z) = 1$, we have

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,-1)\frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t)+1}\right)^{\alpha} e_q(tx)\left(1 + E_q(-t)\right)$$

and

$$\sum_{n=0}^{\infty}\left(\mathcal{G}_{n,q}^{(\alpha)}(x,0) + \mathcal{G}_{n,q}^{(\alpha)}(x,-1)\right)\frac{t^n}{[n]_q!} = 2t\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha-1)}(x,0)\frac{t^n}{[n]_q!}.$$

Thus, we obtain

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\mathcal{G}_{k,q}^{(\alpha)}(x,0) + \mathcal{G}_{k,q}^{(\alpha)}(x,-1)\right)\right\}\frac{t^n}{[n]_q!} = 2\sum_{n=1}^{\infty}[n]_q\mathcal{G}_{n-1,q}^{(\alpha-1)}(x,0)\frac{t^n}{[n]_q!}.$$

From this last equality, we have (14). □

Theorem 2.5 *There is the following identity for the q-Genocchi polynomials*:

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{G}_{k,q}(x,y)+\mathcal{G}_{n,q}(x,y)=2[n]_q(x+y)_q^{n-1}. \tag{15}$$

Proof From $e_q(t)E_q(-t)=1$, we write as

$$\begin{aligned}
&\frac{1}{E_q(-t)+1}=1-\frac{1}{e_q(t)+1},\\
&\frac{2te_q(tx)E_q(yt)}{E_q(-t)+1}=2te_q(tx)E_q(yt)-2t\frac{e_q(tx)E_q(yt)}{e_q(t)+1},\\
&\frac{2t}{e_q(t)+1}e_q(tx)E_q(yt)e_q(t)=2te_q(tx)E_q(ty)-\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x,y)\frac{t^n}{[n]_q!},\\
&\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}=2\sum_{n=0}^{\infty}(x,y)_q^n\frac{t^{n+1}}{[n]_q!}-\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}.
\end{aligned}$$

By using the Cauchy product, compression of the results, we have (15). □

Theorem 2.6 *There are the following relationships for the q-Genocchi polynomials*:

$$\mathcal{G}_{n,q}^{(\alpha)}(x,y)=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q\left\{\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q\mathcal{G}_{j,q}^{(\alpha)}(x,0)m^{j-k}+\mathcal{G}_{k,q}^{(\alpha)}(x,0)\right\}\times\mathcal{G}_{n+1-k,q}(0,my)m^{k-n}, \tag{16}$$

$$\mathcal{G}_{n,q}^{(\alpha)}(x,y)=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q\left\{\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q\mathcal{G}_{j,q}^{(\alpha)}(0,y)m^{j-k}+\mathcal{G}_{k+1,q}^{(\alpha)}(0,y)\right\}\times\mathcal{G}_{n+1-k,q}(mx,0)m^{k-n}. \tag{17}$$

Proof Proof of (16), we write

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}&=\left(\frac{2t}{e_q(t)+1}\right)^{\alpha}e_q(tx)E_q(ty)\\
&=\left(\frac{2t}{e_q(t)+1}\right)^{\alpha}e_q(tx)\frac{e_q(\frac{t}{m})+1}{\frac{t}{m}}\frac{\frac{t}{m}}{e_q(\frac{t}{m})+1}\\
&=\frac{m}{t}\left\{\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}+\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\right\}\\
&\quad\times\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q\left\{\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q\mathcal{G}_{j,q}^{(\alpha)}(x,0)m^{j-k}+\mathcal{G}_{k,q}^{(\alpha)}(x,0)\right\}\right.\\
&\quad\left.\times\mathcal{G}_{n+1-k,q}(0,my)m^{k-n}\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have (16). The proof of (17) is similar to that of (16). □

3 Explicit relation between the q-Bernoulli polynomials and q-Genocchi polynomials

In this section, we prove two interesting relations between the q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α and the q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ of order α.

Theorem 3.1 *There is the following relation between q-Genocchi polynomials and q-Bernoulli polynomials*

$$\mathcal{G}_{n,q}^{(\alpha)}(x,y) = \frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \left\{\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{G}_{j,q}^{(\alpha)}(x,0)m^{j-n} - \mathcal{G}_{k,q}^{(\alpha)}(x,0)\right\} \times \mathcal{B}_{n+1-k,q}(0,my)m^{k-n}. \tag{18}$$

Proof From (6), we deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= \left(\frac{2t}{e_q(t)+1}\right)^{\alpha} e_q(tx)E_q(ty) \\
&= \frac{m}{t}\left\{\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathcal{B}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!} \right.\\
&\quad \left. - \sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathcal{B}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\right\} \\
&= \frac{m}{t}\left\{\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!} - \sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\right\} \\
&\quad \times \sum_{n=0}^{\infty}\mathcal{B}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \left\{\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{G}_{j,q}^{(\alpha)}(x,0)m^{j-n} - \mathcal{G}_{k,q}^{(\alpha)}(x,0)\right\}\right.\\
&\quad \left. \times \mathcal{B}_{n+1-k,q}(0,my)m^{k-n}\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have (18). □

Theorem 3.2 *There is the following relation between q-Bernoulli polynomials and q-Genocchi polynomials*:

$$\mathcal{B}_{n,q}^{(\alpha)}(x,y) = \frac{1}{2}\sum_{r=0}^{n+1}\begin{bmatrix} n+1 \\ r \end{bmatrix}_q \frac{1}{[n+1]_q}\left(\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q \mathcal{B}_{k,q}^{(\alpha)}(x,0)m^{k-r} + \mathcal{B}_{r,q}^{(\alpha)}(x,0)\right) \times \mathcal{G}_{n+1-r,q}(0,my)m^{r-n}. \tag{19}$$

Proof From (2), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx)E_q(ty) \\
&= \frac{m}{2t}\left\{\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx)e_q\left(\frac{t}{m}\right)\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1}E_q\left(\frac{t}{m},my\right)\right. \\
&\quad \left. + \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx)\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1}E_q\left(\frac{t}{m},my\right)\right\} \\
&= \frac{m}{2t}\left\{\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!} + \sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\right\} \\
&\quad \times \sum_{n=0}^{\infty}\mathcal{G}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!} \\
&= \frac{m}{2}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\begin{bmatrix} n \\ r \end{bmatrix}_q \left(\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q \mathcal{B}_{k,q}^{(\alpha)}(x,0)m^{k-r} + \mathcal{B}_{r,q}^{(\alpha)}(x,0)\right) \\
&\quad \times \mathcal{G}_{n-r,q}(0,my)m^{r-n}\frac{1}{[n]_q}\frac{t^{n-1}}{[n-1]_q!} \\
&= \frac{m}{2}\sum_{n=1}^{\infty}\left\{\frac{1}{2}\sum_{r=0}^{n+1}\begin{bmatrix} n+1 \\ r \end{bmatrix}_q \frac{1}{[n+1]_q}\right. \\
&\quad \times \left(\sum_{r=0}^{k}\begin{bmatrix} k \\ r \end{bmatrix}_q \mathcal{B}_{k,q}^{(\alpha)}(x,0)m^{k-r} + \mathcal{B}_{r,q}^{(\alpha)}(x,0)\right) \\
&\quad \left. \times \mathcal{G}_{n+1-r,q}(0,my)m^{r-n}\right\}\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have (19). □

Competing interests
The author declares that they have no competing interests.

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