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Poly-Cauchy and Peters mixed-type polynomials

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Abstract
The Peters polynomials are a generalization of Boole polynomials. In this paper, we consider Peters and poly-Cauchy mixed-type polynomials and investigate the properties of those polynomials which are derived from umbral calculus. Finally, we give various identities of those polynomials associated with special polynomials.

1 Introduction

The Peters polynomials are defined by the generating function to be

$$\sum_{n=0}^{\infty} S_n(x;\lambda,\mu)\frac{t^n}{n!} = \bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}(1+t)^x \quad \text{(see [1])}. \tag{1}$$

The first few of them are given by

$$S_0(x;\lambda,\mu) = 2^{-\mu}, \qquad S_1(x;\lambda,\mu) = 2^{-(\mu+1)}(2x-\lambda\mu), \qquad \ldots.$$

If $\mu = 1$, then $S_n(x;\lambda) = S_n(x:\lambda,1)$ are called Boole polynomials.

In particular, for $\mu = 1$, $\lambda = 1$, $S_n(x;1,1) = \mathrm{Ch}_n(x)$ are Changhee polynomials which are defined by

$$\sum_{n=0}^{\infty} \mathrm{Ch}_n(x)\frac{t^n}{n!} = \frac{1}{t+2}(1+t)^x \quad \text{(see [2])}.$$

The generating functions for the poly-Cauchy polynomials of the first kind $C_n^{(k)}(x)$ and of the second kind $\hat{C}_n^{(k)}(x)$ are, respectively, given by

$$\mathrm{Lif}_k\bigl(\log(1+t)\bigr)(1+t)^{-x} = \sum_{n=0}^{\infty} C_n^{(k)}(x)\frac{t^n}{n!} \tag{2}$$

and

$$\mathrm{Lif}_k\bigl(-\log(1+t)\bigr)(1+t)^{x} = \sum_{n=0}^{\infty} \hat{C}_n^{(k)}(x)\frac{t^n}{n!}, \tag{3}$$

where $\mathrm{Lif}_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k}$ $(k \in \mathbb{Z})$ (see [3, 4]).

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In this paper, we consider the poly-Cauchy of the first kind and Peters (respectively the poly-Cauchy of the second kind and Peters) mixed-type polynomials as follows:

$$\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(\log(1+t)\bigr)(1+t)^{-x}=\sum_{n=0}^{\infty}CP_n^{(k)}(x;\lambda,\mu)\frac{t^n}{n!} \tag{4}$$

and

$$\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)(1+t)^{x}=\sum_{n=0}^{\infty}\hat{C}P_n^{(k)}(x;\lambda,\mu)\frac{t^n}{n!}. \tag{5}$$

For $\alpha\in\mathbb{Z}_{\geq 0}$, the Bernoulli polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!}\quad\text{(see [2, 5–11]).} \tag{6}$$

As is well known, the Frobenius-Euler polynomials of order α are given by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}H_n^{(\alpha)}(x\mid\lambda)\frac{t^n}{n!}\quad\text{(see [2–14]),} \tag{7}$$

where $\lambda\in\mathbb{C}$ with $\lambda\neq 1$ and $\alpha\in\mathbb{Z}_{\geq 0}$.

When $x=0$, $CP_n^{(k)}(0;\lambda,\mu)$ (or $\hat{C}P_n^{(k)}(0;\lambda,\mu)$) are called the poly-Cauchy of the first kind and Peters (or the poly-Cauchy of the second kind and Peters) mixed-type numbers.

The higher-order Cauchy polynomials of the first kind are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^{\alpha}(1+t)^{-x}=\sum_{n=0}^{\infty}\mathbb{C}_n^{(\alpha)}(x)\frac{t^n}{n!}\quad(\alpha\in\mathbb{Z}_{\geq 0}), \tag{8}$$

and the higher-order Cauchy polynomials of the second kind are given by

$$\left(\frac{t}{(1+t)\log(1+t)}\right)^{\alpha}(1+t)^{x}=\sum_{n=0}^{\infty}\hat{\mathbb{C}}_n^{(\alpha)}(x)\frac{t^n}{n!}\quad(\alpha\in\mathbb{Z}_{\geq 0}). \tag{9}$$

The Stirling number of the first kind is given by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l. \tag{10}$$

Thus, by (10), we get

$$\bigl(\log(1+t)\bigr)^m=m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!},\quad m\in\mathbb{Z}_{\geq 0}\text{ (see [1]).} \tag{11}$$

It is easy to show that

$$x^{(n)}=x(x+1)\cdots(x+n-1)=(-1)^n(-x)_n=\sum_{l=0}^{n}S_1(n,l)(-1)^{n-l}x^l. \tag{12}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the algebra of all formal power series in the variable t over $\mathbb{C}$ as follows:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{13}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L \mid p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L + M \mid p(x)\rangle = \langle L \mid p(x)\rangle + \langle M \mid p(x)\rangle$, $\langle cL \mid p(x)\rangle = c\langle L \mid p(x)\rangle$, where c is a complex constant in $\mathbb{C}$.

For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t) \mid x^n \rangle = a_n \quad (n \geq 0). \tag{14}$$

Then, by (13) and (14), we get

$$\langle t^k \mid x^n \rangle = n!\delta_{n,k} \quad (n, k \geq 0) \text{ (see [1, 15])}, \tag{15}$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$. Then, by (14), we see that $\langle f_L(t) \mid x^n\rangle = \langle L \mid x^n\rangle$. The map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra, and the umbral calculus is the study of umbral algebra. The order $O(f)$ of the power series $f(t)$ ($\neq 0$) is the smallest integer for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a delta series; if $O(f(t)) = 0$, then $f(t)$ is called an invertible series. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k \mid s_n(x)\rangle = n!\delta_{n,k}$ $(n, k \geq 0)$.

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t) \mid p(x)\rangle = \langle f(t) \mid g(t)p(x)\rangle = \langle g(t) \mid f(t)p(x)\rangle$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) \mid x^k \rangle \frac{t^k}{k!}, \qquad p(x) = \sum_{k=0}^{\infty} \langle t^k \mid p(x)\rangle \frac{x^k}{k!}. \tag{16}$$

By (16), we get

$$p^{(k)}(0) = \langle t^k \mid p(x)\rangle = \langle 1 \mid p^{(k)}(x)\rangle \quad (k \geq 0), \tag{17}$$

where $p^{(k)}(0) = \frac{d^k p(x)}{dx^k}|_{x=0}$.

Thus, by (17), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [1–3])}. \tag{18}$$

Let $s_n(x) \sim (g(t), f(t))$. Then the following equations are known in [1]:

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!} \quad \text{for all } x \in \mathbb{C}, \tag{19}$$

where $\bar{f}(t)$ is the compositional inverse for $f(t)$ with $f(\bar{f}(t)) = t$,

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle \frac{(\bar{f}(t))^j}{g(\bar{f}(t))} \,\middle|\, x^n \right\rangle x^j, \tag{20}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) P_{n-j}(y), \quad \text{where } P_n(x) = g(t) s_n(x), \tag{21}$$

and

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x), \quad f(t) s_n(x) = n s_{n-1}(x)\ (n \geq 0), \tag{22}$$

and

$$\frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \langle \bar{f}(t) \mid x^{n-l} \rangle s_l(x). \tag{23}$$

As is well known, the transfer formula for $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$ $(n \geq 1)$ is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x). \tag{24}$$

For $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, let

$$s_n(x) = \sum_{m=0}^{\infty} C_{n,m} r_n(x), \tag{25}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} \big(l(\bar{f}(t)) \big)^m \,\middle|\, x^n \right\rangle \quad \text{(see [1])}. \tag{26}$$

It is known that

$$\big\langle f(t) \mid x p(x) \big\rangle = \big\langle \partial_t f(t) \mid p(x) \big\rangle, \qquad e^{yt} p(x) = p(x+y), \tag{27}$$

where $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$ (see [1–3]).

In this paper, we consider Peters and poly-Cauchy mixed-type polynomials with umbral calculus viewpoint and investigate the properties of those polynomials which are derived from umbral calculus. Finally, we give some interesting identities of those polynomials associated with special polynomials.

2 Poly-Cauchy and Peters mixed-type polynomials

From (2), (3), and (19), we note that

$$CP_n^{(k)}(x;\lambda,\mu) \sim \left(\left(1+e^{-\lambda t}\right)^{\mu} \frac{1}{\mathrm{Lif}_k(-t)}, e^{-t}-1 \right) \tag{28}$$

and

$$\hat{C}P_n^{(k)}(x;\lambda,\mu) \sim \left(\left(1+e^{\lambda t}\right)^{\mu} \frac{1}{\mathrm{Lif}_k(-t)}, e^{t}-1 \right). \tag{29}$$

It is not difficult to show that

$$\begin{aligned}\left(1+e^{-\lambda t}\right)^{\mu} &= 2^{\mu}\left(1+\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-\lambda t)^j}{j!}\right)^{\mu} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j_1+\cdots+j_i=j} 2^{\mu-i}\binom{\mu}{i}\binom{j+i}{j_1+1,\ldots,j_i+1}\frac{(-\lambda t)^{j+i}}{(j+i)!}\end{aligned} \tag{30}$$

and

$$\begin{aligned}\left(1+(1+t)^{\lambda}\right)^{-\mu} &= 2^{-\mu}\left(1+\frac{1}{2}\sum_{j=0}^{\infty}\binom{\lambda}{j+1}t^{j+1}\right)^{-\mu} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j_1+\cdots+j_i=j} 2^{-(\mu+i)}\binom{-\mu}{i}\binom{\lambda}{j_1+1}\cdots\binom{\lambda}{j_i+1}t^{j+i}.\end{aligned} \tag{31}$$

From (14), we have

$$\begin{aligned}&CP_n^{(k)}(y;\lambda,\mu) \\ &\quad= \left\langle \sum_{l=0}^{\infty} CP_l^{(k)}(y;\lambda,\mu)\frac{t^l}{l!} \,\middle|\, x^n \right\rangle \\ &\quad= \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\mathrm{Lif}_k\left(\log(1+t)\right)(1+t)^{-y} \mid x^n \right\rangle \\ &\quad= \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu} \,\middle|\, \sum_{l=0}^{n}\binom{n}{l}C_l^{(k)}(y)x^{n-l} \right\rangle \\ &\quad= \sum_{l=0}^{n}\binom{n}{l}C_l^{(k)}(y)\left\langle \sum_{m=0}^{\infty}S_m(0;\lambda,\mu)\frac{t^m}{m!} \,\middle|\, x^{n-l} \right\rangle \\ &\quad= \sum_{l=0}^{n}\binom{n}{l}S_{n-l}(0;\lambda,\mu)C_l^{(k)}(y).\end{aligned} \tag{32}$$

Therefore, by (32), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$CP_n^{(k)}(x;\lambda,\mu) = \sum_{l=0}^{n}\binom{n}{l}S_{n-l}(0;\lambda,\mu)C_l^{(k)}(x).$$

Alternatively,

$$\begin{aligned}
CP_n^{(k)}(y;\lambda,\mu) &= \left\langle \sum_{l=0}^{\infty} CP_l^{(k)}(y;\lambda,\mu)\frac{t^l}{l!} \,\middle|\, x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(\log(1+t)\big) \mid \big(1+(1+t)^{\lambda}\big)^{-\mu}(1+t)^{-y}x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(\log(1+t)\big) \,\middle|\, \sum_{l=0}^{n}\binom{n}{l} S_l(-y;\lambda,\mu)x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l} S_l(-y;\lambda,\mu)\left\langle \mathrm{Lif}_k\big(\log(1+t)\big) \mid x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l} S_l(-y;\lambda,\mu) C_{n-l}^{(k)}(0).
\end{aligned} \tag{33}$$

Therefore, by (33), we obtain the following theorem.

Theorem 2 *For $n \geq 0$, let $C_{n-l}^{(k)}(0) = C_{n-l}^{(k)}$. Then we have*

$$CP_n^{(k)}(x;\lambda,\mu) = \sum_{l=0}^{n}\binom{n}{l} C_{n-l}^{(k)} S_l(-x;\lambda,\mu).$$

Remark By the same method, we get

$$\hat{C}P_n^{(k)}(x;\lambda,\mu) = \sum_{l=0}^{n}\binom{n}{l} S_{n-l}(0;\lambda,\mu)\hat{C}_l^{(k)}(x) \tag{34}$$

and

$$\hat{C}P_n^{(k)}(x;\lambda,\mu) = \sum_{l=0}^{n}\binom{n}{l} \hat{C}_{n-l}^{(k)} S_l(x;\lambda,\mu). \tag{35}$$

From (20) and (28), we have

$$\begin{aligned}
&CP_n^{(k)}(x;\lambda,\mu) \\
&\quad= \sum_{j=0}^{n}\frac{1}{j!}\left\langle \big(1+(1+t)^{\lambda}\big)^{-\mu}\,\mathrm{Lif}_k\big(\log(1+t)\big)\big(-\log(1+t)\big)^j \mid x^n\right\rangle x^j.
\end{aligned} \tag{36}$$

From (31), we note that

$$\begin{aligned}
&\left\langle \big(1+(1+t)^{\lambda}\big)^{-\mu}\,\mathrm{Lif}_k\big(\log(1+t)\big)\big(-\log(1+t)\big)^j \mid x^n\right\rangle \\
&\quad= \sum_{m=0}^{n-j}\frac{(-1)^j}{m!(m+1)^k}\sum_{l=0}^{n-j-m}\frac{(m+j)!}{(l+m+j)!}S_1(l+m+j,m+j) \\
&\qquad\times (n)_{l+m+j}\left\langle \big(1+(1+t)^{\lambda}\big)^{-\mu} \mid x^{n-l-m-j}\right\rangle \\
&\quad= \sum_{m=0}^{n-j}\frac{(-1)^j}{m!(m+1)^k}\sum_{l=0}^{n-j-m}\frac{(m+j)!}{(l+m+j)!}
\end{aligned}$$

$$
\begin{aligned}
&\times S_1(l+m+j,m+j)(n)_{l+m+j}\sum_{i=0}^{n-j-m-l}\sum_{r=0}^{\infty}\sum_{r_1+\cdots+r_i=r}2^{-(\mu+i)}\\
&\times\binom{-\mu}{i}\binom{\lambda}{r_1+1}\cdots\binom{\lambda}{r_i+1}\langle t^{r+i}\mid x^{n-l-m-j}\rangle\\
&=2^{-\mu}n!\sum_{m=0}^{n-j}\sum_{l=0}^{n-j-m}\sum_{i=0}^{n-j-m-l}\sum_{r_1+\cdots+r_i=n-j-m-l-i}\frac{2^{-i}(-1)^j(m+j)!}{m!(m+1)^k(l+m+j)!}\\
&\times\binom{-\mu}{i}\binom{\lambda}{r_1+1}\cdots\binom{\lambda}{r_i+1}S_1(l+m+j,m+j).
\end{aligned}
\tag{37}
$$

Therefore, by (36) and (37), we obtain the following theorem.

Theorem 3 *For $n\geq 0$, we have*

$$
\begin{aligned}
&CP_n^{(k)}(x;\lambda,\mu)\\
&=2^{-\mu}n!\sum_{j=0}^{n}\frac{(-1)^j}{j!}\Biggl\{\sum_{m=0}^{n-j}\sum_{l=0}^{n-j-m}\sum_{i=0}^{n-j-m-l}\sum_{r_1+\cdots+r_i=n-j-m-l-i}\frac{2^{-i}}{m!(m+1)^k}\\
&\times\frac{(m+j)!}{(l+m+j)!}\binom{-\mu}{i}\binom{\lambda}{r_1+1}\cdots\binom{\lambda}{r_i+1}S_1(l+m+j,m+j)\Biggr\}x^j.
\end{aligned}
$$

Remark By the same method as Theorem 3, we get

$$
\begin{aligned}
&\hat{C}P_n^{(k)}(x;\lambda,\mu)\\
&=2^{-\mu}n!\sum_{j=0}^{n}\frac{1}{j!}\Biggl\{\sum_{m=0}^{n-j}\sum_{l=0}^{n-j-m}\sum_{i=0}^{n-j-m-l}\sum_{r_1+\cdots+r_i=n-j-m-l-i}\frac{2^{-i}(-1)^m}{m!(m+1)^k}\\
&\times\frac{(m+j)!}{(l+m+j)!}\binom{-\mu}{i}\\
&\times\binom{\lambda}{r_1+1}\cdots\binom{\lambda}{r_i+1}S_1(l+m+j,m+j)\Biggr\}x^j.
\end{aligned}
\tag{38}
$$

From (28), we note that

$$
\left(1+e^{-\lambda t}\right)^{\mu}\frac{1}{\mathrm{Lif}_k(-t)}CP_n^{(k)}(x;\lambda,\mu)\sim\left(1,e^{-t}-1\right)
\tag{39}
$$

and

$$
x^n\sim(1,t).
\tag{40}
$$

By (24), (39), and (40), we get

$$
\begin{aligned}
&\left(1+e^{-\lambda t}\right)^{\mu}\frac{1}{\mathrm{Lif}_k(-t)}CP_n^{(k)}(x;\lambda,\mu)\\
&=x\left(\frac{t}{e^{-t}-1}\right)^n x^{n-1}
\end{aligned}
$$

$$= (-1)^n x\left(\frac{-t}{e^{-t}-1}\right)^n x^{n-1}$$
$$= (-1)^n \sum_{l=0}^{n-1} (-1)^l B_l^{(n)} \binom{n-1}{l} x^{n-l}. \tag{41}$$

Thus, by (41), we see that

$$\begin{aligned}
&CP_n^{(k)}(x;\lambda,\mu) \\
&= (-1)^n \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} B_l^{(n)} \left(1+e^{-\lambda t}\right)^{-\mu} \operatorname{Lif}_k(-t) x^{n-l} \\
&= (-1)^n \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} B_l^{(n)} \sum_{m=0}^{n-l} \frac{(-1)^m \binom{n-l}{m}}{(m+1)^k} \left(1+e^{-\lambda t}\right)^{-\mu} x^{n-l-m} \\
&= (-1)^n \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} B_l^{(n)} \sum_{m=0}^{n-l} \frac{(-1)^m \binom{n-l}{m}}{(m+1)^k} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_n=j} 2^{-\mu-i} \binom{-\mu}{i} \\
&\quad \times \binom{j+i}{j_1+1,\ldots,j_i+1} \frac{(-\lambda t)^{j+i}}{(j+i)!} x^{n-l-m} \\
&= (-1)^n \sum_{l=0}^{n} \sum_{m=0}^{n-l} \sum_{i=0}^{n-l-m} \sum_{j=0}^{n-l-m-i} \sum_{j_1+\cdots+j_i=n-l-m-i-r} (-1)^{n-r} \frac{2^{-\mu-i}\lambda^{n-l-m-r}}{(m+1)^k} \binom{n-1}{l} \\
&\quad \times \binom{n-l}{m} \binom{-\mu}{i} \binom{n-l-m-r}{j_1+1,\ldots,j_i+1} \binom{n-l-m}{r} B_l^{(n)} x^r.
\end{aligned} \tag{42}$$

Therefore, by (42), we obtain the following theorem.

Theorem 4 *For $n \geq 0$, we have*

$$\begin{aligned}
&CP_n^{(k)}(x;\lambda,\mu) \\
&= \frac{\lambda^n}{2^\mu} \sum_{r=0}^{n} \left(-\lambda^{-1}\right)^r \Biggl\{ \sum_{l=0}^{n-r} \sum_{m=0}^{n-r-l} \sum_{i=0}^{n-r-l-m} \sum_{j_1+\cdots+j_i=n-r-l-m-i} \frac{2^{-i}\lambda^{-l-m}}{(m+1)^k} \binom{n-1}{l} \\
&\quad \times \binom{n-l}{m} \binom{-\mu}{i} \binom{n-r-l-m}{j_1+1,\ldots,j_i+1} \binom{n-l-m}{r} B_l^{(n)} \Biggr\} x^r.
\end{aligned}$$

Remark By the same method as Theorem 4, we get

$$\begin{aligned}
&\hat{C}P_n^{(k)}(x;\lambda,\mu) \\
&= \frac{\lambda^n}{2^\mu} \sum_{r=0}^{n} \lambda^{-r} \Biggl\{ \sum_{l=0}^{n-r} \sum_{m=0}^{n-l-r} \sum_{i=0}^{n-l-m-r} \sum_{j_1+\cdots+j_i=n-r-l-m-i} \frac{(-1)^m 2^{-i}\lambda^{-l-m}}{(m+1)^k} \binom{n-1}{l} \\
&\quad \times \binom{n-l}{m} \binom{-\mu}{i} \binom{n-r-l-m}{j_1+1,\ldots,j_i+1} \binom{n-l-m}{r} B_l^{(m)} \Biggr\} x^r.
\end{aligned} \tag{43}$$

From (12), we note that

$$x^{(n)} = x(x+1)\cdots(x+n-1) \sim \bigl(1, 1-e^{-t}\bigr). \tag{44}$$

Thus, by (44), we see that

$$(-1)^n x^{(n)} = (-x)_n = \sum_{m=0}^{n} S_1(n,m)(-x)^m \sim \bigl(1, e^{-t}-1\bigr) \tag{45}$$

and

$$\bigl(1+e^{-\lambda t}\bigr)^{\mu} \frac{1}{\operatorname{Lif}_k(-t)} CP_n^{(k)}(x;\lambda,\mu) \sim \bigl(1, e^{-t}-1\bigr). \tag{46}$$

From (45) and (46), we have

$$\begin{aligned}
&\bigl(1+e^{-\lambda t}\bigr)^{\mu} \frac{1}{\operatorname{Lif}_k(-t)} CP_n^{(k)}(x;\lambda,\mu) \\
&\quad = (-1)^n x^{(n)} \\
&\quad = \sum_{l=0}^{n} S_1(n,l)(-x)^l.
\end{aligned} \tag{47}$$

Thus, by (47), we get

$$\begin{aligned}
&CP_n^{(k)}(x;\lambda,\mu) \\
&\quad = \sum_{l=0}^{n} S_1(n,l)(-1)^l \bigl(1+e^{-\lambda t}\bigr)^{-\mu} \operatorname{Lif}_k(-t) x^l \\
&\quad = \sum_{l=0}^{n} S_1(n,l)(-1)^l \sum_{m=0}^{l} \frac{(-1)^m \binom{l}{m}}{(m+1)^k} \bigl(1+e^{-\lambda t}\bigr)^{-\mu} x^{l-m} \\
&\quad = \sum_{l=0}^{n} S_1(n,l)(-1)^l \sum_{m=0}^{l} \frac{(-1)^m \binom{l}{m}}{(m+1)^k} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_i=j} 2^{-\mu-i} \\
&\qquad \times \binom{-\mu}{i} \binom{j+i}{j_1+1,\ldots,j_i+1} \frac{(-\lambda)^{j+i}}{(j+i)!} t^{j+i} x^{l-m} \\
&\quad = \sum_{l=0}^{n}\sum_{m=0}^{l}\sum_{i=0}^{l-m}\sum_{r=0}^{l-m-i} \sum_{j_1+\cdots+j_i=l-m-i-r} (-1)^r \frac{2^{-\mu-i}\lambda^{l-m-r}}{(m+1)^k} \\
&\qquad \times \binom{l}{m}\binom{-\mu}{i}\binom{l-m-r}{j_1+1,\ldots,j_i+1}\binom{l-m}{r} S_1(n,l) x^r \\
&\quad = 2^{-\mu} \sum_{r=0}^{n} \bigl(-\lambda^{-1}\bigr)^r \Biggl\{ \sum_{l=r}^{n}\sum_{m=0}^{l-r}\sum_{i=0}^{l-r-m} \sum_{j_1+\cdots+j_i=l-r-m-i} \frac{2^{-i}\lambda^{l-m}}{(m+1)^k} \\
&\qquad \times \binom{l}{m}\binom{-\mu}{i}\binom{l-m-r}{j_1+1,\ldots,j_i+1}\binom{l-m}{r} S_1(n,l) \Biggr\} x^r.
\end{aligned} \tag{48}$$

Therefore, by (48), we obtain the following theorem.

Theorem 5 *For $n \geq 0$, we have*

$$\begin{aligned} & CP_n^{(k)}(x;\lambda,\mu) \\ &= 2^{-\mu}\sum_{r=0}^{n}\left(-\lambda^{-1}\right)^r\left\{\sum_{l=r}^{n}\sum_{m=0}^{l-r}\sum_{i=0}^{l-r-m}\sum_{j_1+\cdots+j_i=l-r-m-i}\frac{2^{-i}\lambda^{l-m}}{(m+1)^k}\right. \\ &\quad\left.\times\binom{l}{m}\binom{-\mu}{i}\binom{l-m-r}{j_1+1,\ldots,j_i+1}\binom{l-m}{r}S_1(n,l)\right\}x^r. \end{aligned}$$

It is easy to see that

$$\left(1+e^{\lambda t}\right)^{\mu}\frac{1}{\operatorname{Lif}_k(-t)}\hat{C}P_n^{(k)}(x;\lambda,\mu)\sim\left(1,e^t-1\right) \tag{49}$$

and

$$(x)_n = x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l\sim\left(1,e^t-1\right). \tag{50}$$

By the same method as Theorem 5, we get

$$\begin{aligned} & \hat{C}P_n^{(k)}(x;\lambda,\mu) \\ &= 2^{-\mu}\sum_{r=0}^{n}\lambda^{-r}\left\{\sum_{l=r}^{n}\sum_{m=0}^{l-r}\sum_{i=0}^{l-r-m}\sum_{j_1+\cdots+j_i=l-r-m-i}\frac{(-1)^m 2^{-i}\lambda^{l-m}}{(m+1)^k}\right. \\ &\quad\left.\times\binom{l}{m}\binom{-\mu}{i}\binom{l-m-r}{j_1+1,\ldots,j_i+1}\binom{l-m}{r}S_1(n,l)\right\}x^r. \end{aligned} \tag{51}$$

From (20) and (28), we have

$$\begin{aligned} & CP_n^{(k)}(x;\lambda,\mu) \\ &= \sum_{j=0}^{n}\frac{1}{j!}\left\langle\left(1+(1+t)^{\lambda}\right)^{-\mu}\operatorname{Lif}_k\left(\log(1+t)\right)\left(-\log(1+t)\right)^j \mid x^n\right\rangle x^j. \end{aligned} \tag{52}$$

Now, we observe that

$$\begin{aligned} & \left\langle\left(1+(1+t)^{\lambda}\right)^{-\mu}\operatorname{Lif}_k\left(\log(1+t)\right)\left(-\log(1+t)\right)^j \mid x^n\right\rangle \\ &= (-1)^j\left\langle\log(1+t)^j \,\middle|\, \sum_{m=0}^{\infty}CP_m^{(k)}(0;\lambda,\mu)\frac{t^m}{m!}x^n\right\rangle \\ &= (-1)^j\sum_{m=0}^{n}\binom{n}{m}CP_m^{(k)}(0;\lambda,\mu)\left\langle\left(\log(1+t)\right)^j \mid x^{n-m}\right\rangle \\ &= (-1)^j\sum_{m=0}^{n}\binom{n}{m}CP_m^{(k)}(0;\lambda,\mu)j!S_1(n-m,j). \end{aligned} \tag{53}$$

Therefore, by (52) and (53), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$CP_n^{(k)}(x;\lambda,\mu) = \sum_{j=0}^{n}(-1)^j\left\{\sum_{m=0}^{n}\binom{n}{m}S_1(n-m,j)CP_m^{(k)}(0;\lambda,\mu)\right\}x^j.$$

Remark By the same method as Theorem 6, we get

$$\hat{C}P_n^{(k)}(x;\lambda,\mu) = \sum_{j=0}^{n}\left\{\sum_{m=0}^{n}\binom{n}{m}S_1(n-m,j)\hat{C}P_n^{(k)}(0;\lambda,\mu)\right\}x^j. \tag{54}$$

From (21), we have

$$CP_n^{(k)}(x+y;\lambda,\mu) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}CP_{n-j}^{(k)}(x;\lambda,\mu)y^{(j)} \tag{55}$$

and

$$\hat{C}P_n^{(k)}(x+y;\lambda,\mu) = \sum_{j=0}^{n}\binom{n}{j}\hat{C}P_{n-j}^{(k)}(x;\lambda,\mu)(y)_j. \tag{56}$$

By (22) and (28), we get

$$\left(e^{-t}-1\right)CP_n^{(k)}(x;\lambda,\mu) = nCP_{n-1}^{(k)}(x;\lambda,\mu) \tag{57}$$

and

$$\left(e^{-t}-1\right)CP_n^{(k)}(x;\lambda,\mu) = CP_n^{(k)}(x-1;\lambda,\mu) - CP_n^{(k)}(x;\lambda,\mu). \tag{58}$$

Therefore, by (57) and (58), we obtain the following theorem.

Theorem 7 *For $n \geq 0$, we have*

$$CP_n^{(k)}(x-1;\lambda,\mu) - CP_n^{(k)}(x;\lambda,\mu) = nCP_{n-1}^{(k)}(x;\lambda,\mu).$$

Remark By the same method as Theorem 7, we get

$$\hat{C}P_n^{(k)}(x+1;\lambda,\mu) - \hat{C}P_n^{(k)}(x;\lambda,\mu) = n\hat{C}P_{n-1}^{(k)}(x;\lambda,\mu). \tag{59}$$

From (22), (28), and (29), we have

$$\begin{aligned}CP_{n+1}^{(k)}(x;1,\mu) &= -xCP_n^{(k)}(x+1;1,\mu) + \mu\sum_{m=0}^{n}\left(-\frac{1}{2}\right)^{m+1}(n)_m CP_{n-m}^{(k)}(x;1,\mu)\\&\quad + 2^{-\mu}\sum_{r=0}^{n}(-1)^r\left\{\sum_{m=r}^{n}\sum_{l=r}^{m}\sum_{i=0}^{l-r}\sum_{j_1+\cdots+j_i=l-i-r}\frac{2^{-i}}{(m-l+2)^k}\binom{m}{l}\right.\\&\quad\left.\times\binom{-\mu}{i}\binom{l-r}{j_1+1,\ldots,j_i+1}\binom{l}{r}S_1(n,m)\right\}(x+1)^r\end{aligned} \tag{60}$$

and

$$\begin{aligned}
&\hat{C}P_{n+1}^{(k)}(x;1,\mu)\\
&= x\hat{C}P_{n}^{(k)}(x-1;1,\mu) + \mu\sum_{m=0}^{n}\left(-\frac{1}{2}\right)^{m+1}(n)_m\hat{C}P_{n-m}^{(k)}(x;1,\mu)\\
&\quad - 2^{-\mu}\sum_{r=0}^{n}\left\{\sum_{m=r}^{n}\sum_{l=r}^{m}\sum_{i=0}^{l-r}\sum_{j_1+\cdots+j_i=l-i-r}\frac{(-1)^{m-l}2^{-i}}{(m-l+2)^k}\binom{m}{l}\binom{-\mu}{i}\right.\\
&\quad \left.\times\binom{l-r}{j_1+1,\ldots,j_i+1}\binom{l}{r}S_1(n,m)\right\}(x-1)^r. \qquad (61)
\end{aligned}$$

By (14) and (27), we get

$$\begin{aligned}
&CP_n^{(k)}(y;\lambda,\mu)\\
&= \left\langle \sum_{l=0}^{\infty} CP_l^{(k)}(y;\lambda,\mu)\frac{t^l}{l!} \,\middle|\, x^n \right\rangle\\
&= \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\operatorname{Lif}_k\left(\log(1+t)\right)(1+t)^{-y} \mid x\cdot x^{n-1}\right\rangle\\
&= \left\langle \partial_t\left(\left(1+(1+t)^{\lambda}\right)^{-\mu}\operatorname{Lif}_k\left(\log(1+t)\right)(1+t)^{-y}\right) \mid x^{n-1}\right\rangle\\
&= \left\langle \left(\partial_t\left(1+(1+t)^{\lambda}\right)^{-\mu}\right)\operatorname{Lif}_k\left(\log(1+t)\right)(1+t)^{-y} \mid x^{n-1}\right\rangle\\
&\quad + \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\left(\partial_t\operatorname{Lif}_k\left(\log(1+t)\right)\right)(1+t)^{-y} \mid x^{n-1}\right\rangle\\
&\quad + \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\operatorname{Lif}_k\left(\log(1+t)\right)\left(\partial_t(1+t)^{-y}\right) \mid x^{n-1}\right\rangle\\
&= -\mu\lambda\left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu-1}\operatorname{Lif}_k\left(\log(1+t)\right)(1+t)^{-(y-\lambda+1)} \mid x^{n-1}\right\rangle\\
&\quad - y\left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\left(\operatorname{Lif}_k\left(\log(1+t)\right)\right)(1+t)^{-y-1} \mid x^{n-1}\right\rangle\\
&\quad + \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\left(\partial_t\operatorname{Lif}_k\left(\log(1+t)\right)\right)(1+t)^{-y} \mid x^{n-1}\right\rangle\\
&= -\mu\lambda CP_{n-1}^{(k)}(y-\lambda+1;\lambda,\mu+1) - yCP_{n-1}^{(k)}(y+1;\lambda,\mu)\\
&\quad + \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\frac{\operatorname{Lif}_{k-1}(\log(1+t))-\operatorname{Lif}_k(\log(1+t))}{(1+t)\log(1+t)}(1+t)^{-y} \mid x^{n-1}\right\rangle. \qquad (62)
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
&\left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\frac{\operatorname{Lif}_{k-1}(\log(1+t))-\operatorname{Lif}_k(\log(1+t))}{(1+t)\log(1+t)}(1+t)^{-y} \,\middle|\, x^{n-1}\right\rangle\\
&= \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\frac{\operatorname{Lif}_{k-1}(\log(1+t))-\operatorname{Lif}_k(\log(1+t))}{t}(1+t)^{-y} \,\middle|\right.\\
&\quad \left.\frac{t}{(1+t)\log(1+t)}x^{n-1}\right\rangle\\
&= \sum_{l=0}^{n-1}\binom{n-1}{l}\hat{\mathbb{C}}_{n-1-l}^{(1)}(0)\\
&\quad \times\left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu}\frac{\operatorname{Lif}_{k-1}(\log(1+t))-\operatorname{Lif}_k(\log(1+t))}{t}(1+t)^{-y} \,\middle|\, x^l\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{n-1} \binom{n-1}{l} \hat{\mathbb{C}}_{n-1-l}^{(1)}(0) \\
&\quad \times \left\langle \left(1+(1+t)^{\lambda}\right)^{-\mu} \frac{\mathrm{Lif}_{k-1}(\log(1+t)) - \mathrm{Lif}_{k}(\log(1+t))}{t} (1+t)^{-y} \,\middle|\, t\frac{x^{l+1}}{l+1} \right\rangle \\
&= \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l+1} \hat{\mathbb{C}}_{n-1-l}^{(1)}(0) \left\{ CP_{l+1}^{(k-1)}(y;\lambda,\mu) - CP_{l+1}^{(k)}(y;\lambda,\mu) \right\}.
\end{aligned} \tag{63}
$$

Therefore, by (62) and (63), we obtain the following theorem.

Theorem 8 *For $n \geq 0$, we have*

$$
\begin{aligned}
&CP_{n}^{(k)}(x;\lambda,\mu) \\
&\quad = -\mu\lambda CP_{n-1}^{(k)}(x-\lambda+1;\lambda,\mu+1) - xCP_{n-1}^{(k)}(x+1;\lambda,\mu) \\
&\qquad + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l+1} \hat{C}_{n-1-l} \left\{ CP_{l+1}^{(k-1)}(x;\lambda,\mu) - CP_{l+1}^{(k)}(x;\lambda,\mu) \right\},
\end{aligned}
$$

where $\hat{C}_{n-1-l} = \hat{\mathbb{C}}_{n-1-l}^{(1)}(0)$.

Remark By the same method as Theorem 8, we get

$$
\begin{aligned}
&\hat{C}P_{n}^{(k)}(x;\lambda,\mu) \\
&\quad = -\mu\lambda \hat{C}P_{n-1}^{(k)}(x+\lambda-1;\lambda,\mu+1) + x\hat{C}P_{n-1}^{(k)}(x-1;\lambda,\mu) \\
&\qquad + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l+1} \hat{C}_{n-1-l} \left(\hat{C}P_{l+1}^{(k-1)}(x;\lambda,\mu) - \hat{C}P_{l+1}^{(k)}(x;\lambda,\mu) \right).
\end{aligned} \tag{64}
$$

By (23), we get

$$
\begin{aligned}
&\frac{d}{dx} CP_{n}^{(k)}(x;\lambda,\mu) \\
&\quad = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle -\log(1+t) \mid x^{n-l} \right\rangle CP_{l}^{(k)}(x;\lambda,\mu) \\
&\quad = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \sum_{m=1}^{\infty} \frac{(-1)^m}{m} t^m \,\middle|\, x^{n-l} \right\rangle CP_{l}^{(k)}(x;\lambda,\mu) \\
&\quad = \sum_{l=0}^{n-1} \binom{n}{l} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left\langle t^m \mid x^{n-l} \right\rangle CP_{l}^{(k)}(x;\lambda,\mu) \\
&\quad = \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l} CP_{l}^{(k)}(x;\lambda,\mu)(n-l-1)! \\
&\quad = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l}}{(n-l)l!} CP_{l}^{(k)}(x;\lambda,\mu).
\end{aligned} \tag{65}
$$

By the same method as (65), we get

$$\begin{aligned}&\frac{d}{dx}\hat{C}P_n^{(k)}(x;\lambda,\mu)\\&\quad=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{(n-l)l!}\hat{C}P_l^{(k)}(x;\lambda,\mu).\end{aligned}\tag{66}$$

Now, we compute the following equation in two different ways:

$$\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\bigl(\log(1+t)\bigr)^m\mid x^n\bigr\rangle.$$

On the one hand,

$$\begin{aligned}&\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\bigl(\log(1+t)\bigr)^m\mid x^n\bigr\rangle\\&\quad=\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\mid\bigl(\log(1+t)\bigr)^m x^n\bigr\rangle\\&\quad=\sum_{l=0}^{n-m}m!\binom{n}{l+m}S_1(l+m,m)\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\mid x^{n-l-m}\bigr\rangle\\&\quad=\sum_{l=0}^{n-m}m!\binom{n}{l}S_1(n-l,m)\hat{C}P_l^{(k)}(0;\lambda,\mu).\end{aligned}\tag{67}$$

On the other hand,

$$\begin{aligned}&\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\bigl(\log(1+t)\bigr)^m\mid x^n\bigr\rangle\\&\quad=\bigl\langle\partial_t\bigl(\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\bigl(\log(1+t)\bigr)^m\bigr)\mid x^{n-1}\bigr\rangle\\&\quad=\bigl\langle\bigl(\partial_t\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\bigr)\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\bigl(\log(1+t)\bigr)^m\mid x^{n-1}\bigr\rangle\\&\qquad+\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\bigl(\partial_t\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\bigr)\bigl(\log(1+t)\bigr)^m\mid x^{n-1}\bigr\rangle\\&\qquad+\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)\bigl(\partial_t\bigl(\log(1+t)\bigr)^m\bigr)\mid x^{n-1}\bigr\rangle.\end{aligned}\tag{68}$$

Therefore, by (67) and (68), we obtain the following theorem.

Theorem 9 *For $n\in\mathbb{N}$ with $n\geq 2$, let $n-1\geq m\geq 1$. Then we have*

$$\begin{aligned}&m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\hat{C}P_l^{(k)}(0;\lambda,\mu)\\&\quad=-\mu\lambda m\sum_{l=0}^{n-1-m}\binom{n-1}{l}S_1(n-1-l,m)\hat{C}P_l^{(k)}(\lambda-1;\lambda,\mu+1)\\&\qquad+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)\hat{C}P_l^{(k-1)}(-1;\lambda,\mu)\\&\qquad+(m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)\hat{C}P_l^{(k)}(-1;\lambda,\mu).\end{aligned}$$

Remark By the same method as Theorem 9, we get

$$\begin{aligned}
&m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)CP_l^{(k)}(0;\lambda,\mu)\\
&\quad=-\mu\lambda m\sum_{l=0}^{n-1-m}\binom{n-1}{l}S_1(n-1-l,m)CP_l^{(k)}(1-\lambda;\lambda,\mu+1)\\
&\qquad+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)CP_l^{(k-1)}(1;\lambda,\mu)\\
&\qquad+(m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)CP_l^{(k)}(-1;\lambda,\mu),
\end{aligned}$$

where $n-1\geq m\geq 1$.

Let us consider the following two Sheffer sequences:

$$CP_n^{(k)}(x;\lambda,\mu)\sim\left(\left(1+e^{-\lambda t}\right)^{\mu}\frac{1}{\mathrm{Lif}_k(-t)},e^{-t}-1\right) \tag{69}$$

and

$$B_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right)\quad(s\in\mathbb{Z}_{\geq 0}). \tag{70}$$

Let

$$CP_n^{(k)}(x;\lambda,\mu)=\sum_{m=0}^{n}C_{n,m}B_m^{(s)}(x). \tag{71}$$

Then, by (26), we get

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\frac{(\frac{e^{-\log(1+t)}-1}{-\log(1+t)})^s}{(1+e^{\lambda\log(1+t)})^{\mu}}\mathrm{Lif}_k\bigl(\log(1+t)\bigr)\bigl(-\log(1+t)\bigr)^m\,\Big|\,x^n\right\rangle\\
&=\frac{(-1)^m}{m!}\Bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\mathrm{Lif}_k\bigl(\log(1+t)\bigr)\\
&\qquad\times(1+t)^{-s}\left(\frac{t}{\log(1+t)}\right)^s\,\Big|\,\bigl(\log(1+t)\bigr)^m x^n\Bigr\rangle\\
&=\frac{(-1)^m}{m!}\sum_{l=0}^{n-m}m!\binom{n}{l+m}S_1(l+m,m)\sum_{i=0}^{n-l-m}\binom{n-l-m}{i}\mathbb{C}_i^{(s)}\\
&\qquad\times\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\mathrm{Lif}_k\bigl(\log(1+t)\bigr)(1+t)^{-s}\mid x^{n-l-m-i}\bigr\rangle\\
&=(-1)^m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\sum_{i=0}^{l}\binom{l}{i}\mathbb{C}_i^{(s)}CP_{l-i}^{(k)}(s;\lambda,\mu).
\end{aligned} \tag{72}$$

Therefore, by (71) and (72), we obtain the following theorem.

Theorem 10 *For $n \geq 0$, we have*

$$CP_n^{(k)}(x;\lambda,\mu) = \sum_{m=0}^{n}(-1)^m \left\{ \sum_{l=0}^{n-m}\sum_{i=0}^{l} \binom{n}{l}\binom{l}{i} S_1(n-l,m)\mathbb{C}_i^{(s)} CP_{l-i}^{(k)}(s;\lambda,\mu) \right\} B_n^{(s)}(x).$$

Remark By the same method as Theorem 10, we have

$$\begin{aligned}&\hat{C}P_n^{(k)}(x;\lambda,\mu)\\ &\quad = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m}\sum_{i=0}^{l} \binom{n}{l}\binom{l}{i} S_1(n-l,m)\hat{\mathbb{C}}_i^{(s)} \hat{C}P_{l-i}^{(k)}(s;\lambda,\mu) \right\} B_m^{(s)}(x).\end{aligned} \tag{73}$$

For $CP_n^{(k)}(x;\lambda,\mu) \sim ((1+e^{-\lambda t})^{\mu}\frac{1}{\mathrm{Lif}_k(-t)}, e^{-t}-1)$, $H_n^{(s)}(x \mid \lambda) \sim ((\frac{e^t-\lambda}{1-\lambda})^s, t)$, $s \in \mathbb{Z}_{\geq 0}$, $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, let us assume that

$$CP_n^{(k)}(x;\lambda,\mu) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x;\lambda). \tag{74}$$

From (26), we have

$$\begin{aligned}C_{n,m} &= \frac{(-1)^m}{m!}\Big\langle \big(1+(1+t)^{\lambda}\big)^{-\mu} \mathrm{Lif}_k\big(\log(1+t)\big)\\ &\quad \times (1+t)^{-s}\left(1+\frac{\lambda}{\lambda-1}t\right)^s \Bigm| \big(\log(1+t)\big)^m x^n \Big\rangle\\ &= \frac{(-1)^m}{m!}\sum_{l=0}^{n-m} m!\binom{n}{l+m} S_1(l+m,m) \sum_{i=0}^{\min\{s,n-l-m\}} \binom{s}{i}\left(\frac{\lambda}{\lambda-1}\right)^i\\ &\quad \times \big\langle \big(1+(1+t)^{\lambda}\big)^{-\mu}\mathrm{Lif}_k\big(\log(1+t)\big)(1+t)^{-s} \mid t^i x^{n-l-m}\big\rangle\\ &= (-1)^m \sum_{l=0}^{n-m}\sum_{i=0}^{\min\{s,n-l-m\}} \binom{n}{l+m}\binom{s}{i}\\ &\quad \times (n-l-m)_i \left(\frac{\lambda}{\lambda-1}\right)^i S_1(l+m,m) CP_{n-l-m-i}^{(k)}(s;\lambda,\mu)\\ &= (-1)^m \sum_{l=0}^{n-m}\sum_{i=0}^{\min\{s,l\}} \binom{n}{l}\binom{s}{i}(l)_i \left(\frac{\lambda}{\lambda-1}\right)^i S_1(n-l,m) CP_{l-i}^{(k)}(s;\lambda,\mu).\end{aligned} \tag{75}$$

Therefore, by (75) and (76), we obtain the following theorem.

Theorem 11 *For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, $n \geq 0$, we have*

$$\begin{aligned}&CP_n^{(k)}(x;\lambda,\mu)\\ &\quad = \sum_{m=0}^{n}(-1)^m \left\{ \sum_{l=0}^{n-m}\sum_{i=0}^{\min\{s,l\}} \binom{n}{l}\binom{s}{i}(l)_i \right.\\ &\qquad \left. \cdot \left(\frac{\lambda}{\lambda-1}\right)^i S_1(n-l,m) CP_{l-i}^{(k)}(s;\lambda,\mu) \right\} H_m^{(s)}(x;\lambda).\end{aligned}$$

Remark By the same method as Theorem 11, we get

$$\begin{aligned}
&\hat{C}P_n^{(k)}(x;\lambda,\mu)\\
&\quad=\sum_{m=0}^{n}\left\{\sum_{l=0}^{n-m}\sum_{i=0}^{\min\{s,l\}}\binom{n}{l}\binom{s}{i}(l)_i\right.\\
&\qquad\left.\cdot\left(\frac{1}{1-\lambda}\right)^i S_1(n-l,m)\hat{C}P_{l-i}^{(k)}(0;\lambda,\mu)\right\}H_m^{(s)}(x;\lambda).
\end{aligned}$$

For $CP_n^{(k)}(x;\lambda,\mu)\sim((1+e^{-\lambda t})^{\mu}\frac{1}{\operatorname{Lif}_k(-t)},e^{-t}-1)$ and $x^{(n)}\sim(1,1-e^{-t})$, let us assume that

$$CP_n^{(k)}(x;\lambda,\mu)=\sum_{m=0}^{\infty}C_{n,m}x^{(m)}. \tag{76}$$

By (26), we get

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\frac{1}{(1+e^{\lambda\log(1+t)})^{\mu}}\operatorname{Lif}_k\bigl(\log(1+t)\bigr)\bigl(1-e^{\log(1+t)}\bigr)^m\;\middle|\;x^n\right\rangle\\
&=\frac{1}{m!}\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(\log(1+t)\bigr)(-t)^m\mid x^n\bigr\rangle\\
&=\frac{(-1)^m}{m!}\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(\log(1+t)\bigr)\mid t^m x^n\bigr\rangle\\
&=(-1)^m\binom{n}{m}\bigl\langle\bigl(1+(1+t)^{\lambda}\bigr)^{-\mu}\operatorname{Lif}_k\bigl(\log(1+t)\bigr)\mid x^{n-m}\bigr\rangle\\
&=(-1)^m\binom{n}{m}CP_{n-m}^{(k)}(0;\lambda,\mu).
\end{aligned}\tag{77}$$

Therefore, by (77) and (78), we obtain the following theorem.

Theorem 12 *For $n\geq 0$, we have*

$$CP_n^{(k)}(x;\lambda,\mu)=\sum_{m=0}^{n}(-1)^m\binom{n}{m}CP_{n-m}^{(k)}(0;\lambda,\mu)x^{(m)}.$$

Remark By the same method as Theorem 12, we get

$$\hat{C}P_n^{(k)}(x;\lambda,\mu)=\sum_{m=0}^{n}\binom{n}{m}\hat{C}P_{n-m}^{(k)}(0;\lambda,\mu)(x)_m. \tag{78}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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