Ould Bouh *Advances in Difference Equations* 2014, 2014:319
http://www.advancesindifferenceequations.com/content/2014/1/319

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On a fourth order elliptic equation with supercritical exponent

Kamal Ould Bouh*

*Correspondence:
hbouh@taibahu.edu.sa;
kamal.bouh@gmail.com
Department of Mathematics, Taibah University, P.O. Box 30002, Almadinah Almunawwarah, Kingdom of Saudi Arabia

Abstract

This paper is concerned with the semi-linear elliptic problem involving nearly critical exponent (P_ε): $\Delta^2 u = |u|^{8/(n-4)+\varepsilon}u$ in Ω, $\Delta u = u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, and ε is a positive real parameter. We show that, for ε small, (P_ε) has no sign-changing solutions with low energy which blow up at exactly three points. Moreover, we prove that (P_ε) has no bubble-tower sign-changing solutions.

MSC: 35J20; 35J60

Keywords: nonlinear problem; critical exponent; sign-changing solutions; bubble-tower solution

1 Introduction and results

We consider the following semi-linear elliptic problem with supercritical nonlinearity:

$$(P_\varepsilon)\qquad \begin{cases} \Delta^2 u = |u|^{p-1+\varepsilon}u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, ε is a positive real parameter and $p + 1 = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embedding of $H^2(\Omega) \cap H_0^1(\Omega)$ into $L^{p+1}(\Omega)$.

When the biharmonic operator in (P_ε) is replaced by the Laplacian operator, there are many works devoted to the study of the counterpart of (P_ε); see for example [1–6], and the references therein.

When $\varepsilon < 0$, many works have been devoted to the study of the solutions of (P_ε) see for example [7–9]. In the critical case, this problem is not compact, that is, when $\varepsilon = 0$ it corresponds exactly to the limiting case of the Sobolev embedding $H^2(\Omega) \cap H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, and thus we lose the compact embedding. In fact, van Der Vorst showed in [10] that (P_0) has no positive solutions if Ω is a starshaped domain. Whereas Ebobisse and Ould Ahmedou proved in [11] that (P_0) has a positive solution provided that some homology group of Ω is non-trivial. This topological condition is sufficient, but not necessary, as examples of contractible domains Ω on which a positive solution exists show [12].

In the supercritical case, $\varepsilon > 0$, the problem (P_ε) becomes more delicate since we lose the Sobolev embedding which is an important point to overcome. The problem (P_ε) was studied in [7] where the authors show that there is no one-bubble solution to the problem and there is a one-bubble solution to the slightly subcritical case under some suitable conditions. However, we proved in [13] that (P_ε) has no sign-changing solutions which blow up exactly at two points. In this work we will show the non-existence of sign-changing solutions of (P_ε) having three concentration points.

We note that problem (P_ε) has a variational structure. The related functional is

$$\inf J(u), \quad \text{where } J(u) := \frac{\int_\Omega |\Delta u|^2}{(\int_\Omega |u|^{p+1+\varepsilon})^{2/(p+1+\varepsilon)}}, u \in H^2(\Omega) \cap H_0^1(\Omega), u \not\equiv 0.$$

J satisfies the Palais-Smale condition in the subcritical case, while this condition fails in the critical case. Such a failure is due to the functions

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{(n-4)/2}}{(1+\lambda^2|x-a|^2)^{(n-4)/2}}, \quad c_0 = \big(n(n-4)\big(n^2-4\big)\big)^{(n-4)/8}, \lambda > 0, a \in \mathbb{R}^n. \tag{1.1}$$

c_0 is chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the following problem:

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n. \tag{1.2}$$

When we study problem (1.2) in a bounded smooth domain Ω, we need to introduce the function $P\delta_{(a,\lambda)}$ which is the projection of $\delta_{(a,\lambda)}$ on $H_0^1(\Omega)$. It satisfies

$$\Delta^2 P\delta_{(a,\lambda)} = \Delta^2 \delta_{(a,\lambda)} \quad \text{in } \Omega, \qquad \Delta P\delta_{(a,\lambda)} = P\delta_{(a,\lambda)} = 0 \quad \text{on } \partial\Omega.$$

These functions are almost positive solutions of (1.2).

We denote by G the Green's function defined by, $\forall x \in \Omega$,

$$\Delta^2 G(x,\cdot) = c_n \delta_x \quad \text{in } \Omega, \qquad \Delta G(x,\cdot) = G(x,\cdot) = 0 \quad \text{on } \partial\Omega,$$

where δ_x is the Dirac mass at x and $c_n = (n-4)(n-2)w_n$, with w_n is the area of the unit sphere of $\mathbb{R}^n$. We denote by H the regular part of G, that is,

$$H(x_1,x_2) = |x_1 - x_2|^{4-n} - G(x_1,x_2) \quad \text{for } (x_1,x_2) \in \Omega^2.$$

For $x = (x_1,x_2) \in \Omega^2 \setminus \Gamma$, with $\Gamma = \{(y,y) : y \in \Omega\}$, we denote by $M(x)$ the matrix defined by

$$M(x) = (m_{ij})_{1 \le i,j \le 2}, \quad \text{where } m_{ii} = H(x_i,x_i), m_{12} = m_{21} = G(x_1,x_2), \tag{1.3}$$

and let $\rho(x)$ be its least eigenvalue.

The space $H^2(\Omega) \cap H_0^1(\Omega)$ is equipped with the norm $\|\cdot\|$ and its corresponding inner product $\langle\cdot,\cdot\rangle$ defined by

$$\|u\| = \left(\int_\Omega |\Delta u|^2\right)^{1/2} \quad \text{and} \quad \langle u,v\rangle = \int_\Omega \Delta u \Delta v, \quad u,v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{1.4}$$

Now, we are able to state our result.

Theorem 1.1 *Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 6$. If 0 is a regular value of $\rho(x)$, then there exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no sign-changing solutions u_ε which satisfy*

$$u_\varepsilon = P\delta_{(a_{\varepsilon,1},\lambda_{\varepsilon,1})} - P\delta_{(a_{\varepsilon,2},\lambda_{\varepsilon,2})} + P\delta_{(a_{\varepsilon,3},\lambda_{\varepsilon,3})} + v_\varepsilon, \tag{1.5}$$

with $|u_\varepsilon|_\infty^\varepsilon$ is bounded and

$$\begin{cases} a_{\varepsilon,i} \in \Omega, \quad \lambda_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to \infty \quad \text{for } i = 1,2,3, \\ \langle P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}, P\delta_{(a_{\varepsilon,j},\lambda_{\varepsilon,j})}\rangle \to 0 \quad \text{for } i \neq j \text{ and } \|v_\varepsilon\| \to 0 \text{ as } \varepsilon \to 0. \end{cases}$$

The second result deals with the phenomenon of bubble-tower solutions for the biharmonic problem (P_ε) with supercritical exponent. We will give a generalization of the result found in [13]. More precisely, we have the following.

Theorem 1.2 *Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$. There exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no solutions u_ε of the form*

$$u_\varepsilon = \sum_{i=1}^{k} \gamma_i P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})} + v_\varepsilon, \quad \text{with } \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \cdots \leq \lambda_{\varepsilon,k} \text{ and } |u_\varepsilon|_\infty^\varepsilon \text{ is bounded}, \tag{1.6}$$

where $k \geq 2$, $\gamma_i \in \{-1, 1\}$, $a_{\varepsilon,i} \in \Omega$, for each $i \leq j$, $\lambda_{\varepsilon,i}|a_{\varepsilon,i} - a_{\varepsilon,j}|$ is bounded and as $\varepsilon \to 0$, $\|v_\varepsilon\| \to 0$, $\lambda_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to +\infty$, $\langle P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}, P\delta_{(a_{\varepsilon,j},\lambda_{\varepsilon,j})}\rangle \to 0$ for $i \neq j$, and if $l \notin \{k-1, k\}$, $\lambda_{\varepsilon,l}|a_{\varepsilon,l} - a_{\varepsilon,l+1}| \to 0$, where $l = \min\{q : \gamma_q = \cdots = \gamma_k\}$.

The proof of our results will be by contradiction. Thus, throughout this paper we will assume that there exist solutions (u_ε) of (P_ε) which satisfy (1.5) or (1.6). In Section 2, we will obtain some information as regards such (u_ε) which allows us to develop Section 3 which deals with some useful estimates to the proof of our theorems. Finally, in Section 4, we combine these estimates to obtain a contradiction. Hence the proof of our results follows.

2 Preliminary results

In this section, we assume that there exist solutions (u_ε) of (P_ε) which satisfy

$$u_\varepsilon = \sum_{i=1}^{k} \gamma_i P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})} + v_\varepsilon, \tag{2.1}$$

with $|u_\varepsilon|_\infty^\varepsilon$ is bounded, $k \geq 2$, $a_{\varepsilon,i} \in \Omega$, and as $\varepsilon \to 0$, $\|v_\varepsilon\| \to 0$, $\lambda_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to +\infty$, $\langle P\delta_{(a_{\varepsilon,i},\lambda_{\varepsilon,i})}, P\delta_{(a_{\varepsilon,j},\lambda_{\varepsilon,j})}\rangle \to 0$ for $i \neq j$. Arguing as in [14] and [15], we see that for u_ε satisfying (2.1), there is a unique way to choose α_i, a_i, λ_i, and v such that

$$u_\varepsilon = \sum_{i=1}^{k} \gamma_i \alpha_i P\delta_{(a_i,\lambda_i)} + v, \tag{2.2}$$

$$\text{with } \begin{cases} \alpha_i \in \mathbb{R}, \qquad \alpha_i \to 1, \\ a_i \in \Omega, \qquad \lambda_i \in \mathbb{R}_+^*, \qquad \lambda_i d(a_i, \partial\Omega) \to +\infty, \\ v \to 0 \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega), \qquad v \in E, \end{cases} \tag{2.3}$$

where E denotes the subspace of $H_0^1(\Omega)$ defined by

$$E := \{w : \langle w, \varphi\rangle = 0, \forall\varphi \in \operatorname{Span}\{P\delta_i, \partial P\delta_i/\partial\lambda_i, \partial P\delta_i/\partial a_i^j, i \le k; j \le n\}\}. \tag{2.4}$$

Here, a_i^j denotes the jth component of a_i and in the sequel, in order to simplify the notations, we set $\delta_{(a_i,\lambda_i)} = \delta_i$ and $P\delta_{(a_i,\lambda_i)} = P\delta_i$. We always assume that u_ε (which satisfies (2.1)) is written as in (2.2) and (2.3) holds. From (2.1), it is easy to see that the following remark holds.

Lemma 2.1 [13] *Let u_ε satisfying the assumption of the theorems. λ_i occurring in* (2.2) *satisfies*

$$\lambda_i^\varepsilon \to 1 \quad \text{as } \varepsilon \to 0 \text{ for each } i \le k. \tag{2.5}$$

Remark 2.2 [2, 16] We recall the following estimate:

$$\delta_i^\varepsilon(x) - c_0^\varepsilon \lambda_i^{\varepsilon(n-4)/2} = O\big(\varepsilon \log\big(1 + \lambda_i^2|x - a_i|^2\big)\big) \quad \text{in } \Omega. \tag{2.6}$$

3 Some useful estimates

As usual in this type of problems, we first deal with the v-part of u_ε, in order to show that it is negligible with respect to the concentration phenomenon.

Lemma 3.1 *The function v defined in* (2.2), *satisfies the following estimate*:

$$\|v\| \le c\varepsilon + c \begin{cases} \sum_i \frac{1}{(\lambda_i d_i)^{n-4}} + \sum_{i\neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} & \text{if } n < 12, \\ \sum_i \frac{1}{(\lambda_i d_i)^{(n+4)/2-\varepsilon(n-4)}} + \sum_{i\neq j} \varepsilon_{ij}^{(n+4)/2(n-4)} (\log \varepsilon_{ij}^{-1})^{(n+4)/2n} & \text{if } n \ge 12, \end{cases}$$

where $d_i := d(a_i, \partial\Omega)$ for $i \le k$ and for $i \neq j$, ε_{ij} is defined by

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{(4-n)/2}. \tag{3.1}$$

Proof The proof is the same as that of Lemma 3.1 of [13], so we omit it. □

Now, we state the crucial points in the proof of our theorems.

Proposition 3.2 *Assume that $n \ge 5$ and let α_i, a_i and λ_i be the variables defined in* (2.2) *with $k = 3$ and $\gamma_1 = -\gamma_2 = \gamma_3$. We have*

$$\begin{aligned} &\left|\alpha_i c_1 \frac{n-4}{2}\frac{H(a_i,a_i)}{\lambda_i^{n-4}} + \sum_{j\neq i}(-1)^{i+j}\alpha_j c_1\left(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{n-4}{2}\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}}\right) + \alpha_i\frac{n-4}{2}c_2\varepsilon\right| \\ &\quad \le c\varepsilon^2 + c\begin{cases} \sum_k \frac{1}{(\lambda_k d_k)^{n-2}} + \sum_{j\neq i}(\varepsilon_{ij}^{\frac{n}{n-4}}\log\varepsilon_{ij}^{-1} + \varepsilon_{ij}^2(\log\varepsilon_{12}^{-1})^{\frac{2(n-4)}{n}}) & \text{if } n \ge 6, \\ \sum_k \frac{1}{(\lambda_k d_k)^2} + \sum_{j\neq i}\varepsilon_{ij}^2(\log\varepsilon_{12}^{-1})^{2/5} & \text{if } n = 5, \end{cases} \end{aligned} \tag{3.2}$$

where $i, j \in \{1, 2, 3\}$ with $i \neq j$ and c_1, c_2 are positive constants.

Proof Let

$$c_1 = c_0^{\frac{2n}{n-4}} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+4)/2}}$$

and

$$c_2 = \frac{n-4}{2} c_0^{\frac{2n}{n-4}} \int_{\mathbb{R}^n} \log\bigl(1+|x|^2\bigr) \frac{|x|^2-1}{(1+|x|^2)^{n+1}}\, dx.$$

It suffices to prove the proposition for $i = 1$. Multiplying (P_ε) by $\lambda_1 \partial P\delta_1/\partial\lambda_1$ and integrating on Ω, we obtain

$$\begin{aligned} &\alpha_1 \int_\Omega \delta_1^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} - \alpha_2 \int_\Omega \delta_2^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} + \alpha_3 \int_\Omega \delta_3^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\quad = \int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1}. \end{aligned} \tag{3.3}$$

Using [17], we derive

$$\int_\Omega \delta_1^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} = \frac{n-4}{2} c_1 \frac{H(a_1,a_1)}{\lambda_1^{n-4}} + O\biggl(\frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^{n-1}}\biggr), \tag{3.4}$$

$$\int_\Omega \delta_j^p \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} = c_1 \biggl(\lambda_1 \frac{\partial\varepsilon_{1j}}{\partial\lambda_1} + \frac{n-4}{2} \frac{H(a_1,a_j)}{(\lambda_1\lambda_j)^{(n-4)/2}}\biggr) + R_j, \tag{3.5}$$

where $j = 2, 3$ and R_j satisfies

$$R_j = O\biggl(\sum_{k=1,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n-1}} + \varepsilon_{1j}^{\frac{n}{n-4}} \log \varepsilon_{1j}^{-1}\biggr). \tag{3.6}$$

For the other term of (3.3), we have

$$\begin{aligned} &\int_\Omega |u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\quad = \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{p-1+\varepsilon} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3) \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\qquad + (p+\varepsilon) \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\qquad + O\biggl(\|v\|^2 + \sum_{i\neq j} \varepsilon_{ij}^{\frac{n}{n-4}} \log \varepsilon_{ij}^{-1}\biggr). \end{aligned} \tag{3.7}$$

Concerning the last integral, it can be written as

$$\begin{aligned} &\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} \\ &\quad = \int_\Omega (\alpha_1 P\delta_1)^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial\lambda_1} + O\biggl(\int_{\Omega\setminus A_j} P\delta_j^{p-1} P\delta_1 |v| + \int_{A_j} P\delta_1^{p-1} P\delta_2 |v|\biggr), \end{aligned} \tag{3.8}$$

where $A_j = \{x : 2\alpha_j P\delta_j \leq \alpha_1 P\delta_1\}$ for $j = 2, 3$.

Observe that, for $n \geq 12$, we have $p - 1 = 8/(n-4) \leq 1$, thus

$$\int_{\Omega\setminus A_j} P\delta_j^{p-1} P\delta_1 |v| + \int_{A_j} P\delta_1^{p-1} P\delta_j |v| \leq c \int_\Omega |v| (\delta_1 \delta_j)^{\frac{n+4}{2(n-4)}}$$
$$\leq c\|v\| \varepsilon_{1j}^{(n+4)/2(n-4)} \left(\log \varepsilon_{1j}^{-1}\right)^{(n+4)/2n}. \tag{3.9}$$

But for $n < 12$, we have

$$\int_{\Omega\setminus A_j} P\delta_j^{p-1} P\delta_1 |v| + \int_A P\delta_1^{p-1} P\delta_j |v| \leq c\varepsilon_{1j} \left(\log \varepsilon_{1j}^{-1}\right)^{(n-4)/n} \|v\|. \tag{3.10}$$

For the other integral in (3.8), using [16, 17], and Remark 2.2, we get

$$\int_\Omega P\delta_1^{p-1+\varepsilon} v \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1}$$
$$= O\left(\|v\| \left[\varepsilon + \left(\frac{1}{(\lambda_1 d_1)^{\inf(n-4,(n+4)/2)}} \text{ (if } n \neq 12) + \frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^4} \text{ (if } n = 12) \right) \right] \right). \tag{3.11}$$

It remains to estimate the second integral of (3.7). We have

$$\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3|^{p-1+\varepsilon} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + \alpha_3 P\delta_3) \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1}$$
$$= \int_\Omega (\alpha_1 P\delta_1)^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} - \int_\Omega (\alpha_2 P\delta_2)^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} + \int_\Omega (\alpha_3 P\delta_3)^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1}$$
$$- (p+\varepsilon) \left(\int_\Omega \alpha_2 P\delta_2 (\alpha_1 P\delta_1)^{p-1+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} - \int_\Omega \alpha_3 P\delta_3 (\alpha_1 P\delta_1)^{p-1+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \right)$$
$$+ O\left(\sum \varepsilon_{1j}^{\frac{n}{n-4}} \log \varepsilon_{1j}^{-1} \right). \tag{3.12}$$

Now, using Remark 2.2 and [17], we have

$$\int_\Omega P\delta_1^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = \frac{n-4}{2} \left(c_2 \varepsilon + 2c_1 \frac{H(a_1,a_1)}{\lambda_1^{n-4}} \right)$$
$$+ O\left(\varepsilon^2 + \frac{\log(\lambda_1 d_1)}{(\lambda_1 d_1)^{n-1}} + \frac{1}{(\lambda_1 d_1)^2} \text{ (if } n = 5) \right), \tag{3.13}$$

$$\int_\Omega P\delta_j^{p+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = c_1 \left(\lambda_1 \frac{\partial \varepsilon_{1j}}{\partial \lambda_1} + \frac{n-4}{2} \frac{H(a_1,a_j)}{(\lambda_1 \lambda_j)^{(n-4)/2}} \right) + T_j, \tag{3.14}$$

$$p \int_\Omega P\delta_j P\delta_1^{p-1+\varepsilon} \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} = c_1 \left(\lambda_1 \frac{\partial \varepsilon_{1j}}{\partial \lambda_1} + \frac{n-4}{2} \frac{H(a_1,a_j)}{(\lambda_1 \lambda_j)^{(n-4)/2}} \right) + T_j, \tag{3.15}$$

where for $i = 2, 3$,

$$T_i = O\left(\varepsilon \varepsilon_{1j} \left(\log \varepsilon_{1j}^{-1}\right)^{\frac{n-4}{n}} \right) + \left(\varepsilon_{1j}^{\frac{n}{n-4}} \left(\log \varepsilon_{1j}^{-1}\right) + \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^n} \text{ (if } n \geq 8) \right)$$
$$+ \left(\frac{\varepsilon_{1j} (\log \varepsilon_{1j}^{-1})^{\frac{n-4}{n}}}{(\lambda_i d_i)^{n-4}} \text{ (if } n < 8) \right).$$

Therefore, combining (3.3)-(3.15), and Lemma 3.1, the proof of Proposition 3.2 follows. □

Proposition 3.3 *Let $n \geq 6$. We have the following estimate*:

$$\alpha_i \frac{1}{\lambda_i^{n-3}} \frac{\partial H(a_i,a_i)}{\partial a_i} - \frac{2}{\lambda_i} \sum_{j\neq i} (-1)^{i+j} \alpha_j \left(\frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{1}{(\lambda_i \lambda_j)^{(n-4)/2}} \frac{\partial H}{\partial a_i}(a_i,a_j) \right)$$
$$= O\left(\sum_k \frac{1}{(\lambda_k d_k)^{n-2}} + \sum_{j\neq i} \varepsilon_{ij}^{\frac{n}{n-4}} \log \varepsilon_{ij}^{-1} + \varepsilon_{ij}^2 \left(\log \varepsilon_{1j}^{-1}\right)^{\frac{2(n-4)}{n}} + \varepsilon^2 + \frac{\varepsilon}{(\lambda_i d_i)^{n-3}} \right),$$

where $i,j \in \{1,2,3\}$ and $j \neq i$.

Proof The proof is similar to the proof of Proposition 3.2. But there exist some integrals which have different estimates. We will focus in those integrals. In fact, (3.3), (3.7)-(3.12) are also true if we change $\lambda_1 \partial P\delta_1/\partial \lambda_1$ by $(1/\lambda_1)\partial P\delta_1/\partial a_1$. It remains to deal with the other equations. Following [17], we get

$$\int_\Omega \delta_1^p \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} = -\frac{1}{2} \frac{c_1}{\lambda_1^{n-3}} \frac{\partial H(a_1,a_1)}{\partial a_1} + O\left(\frac{1}{(\lambda_1 d_1)^{n-1}} \right), \tag{3.16}$$

$$\int_\Omega \delta_j^p \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} = \frac{c_1}{\lambda_1} \left(\frac{\partial \varepsilon_{1j}}{\partial a_1} - \frac{1}{(\lambda_1 \lambda_j)^{(n-4)/2}} \frac{\partial H}{\partial a_1}(a_1,a_j) \right)$$
$$+ O\left(\sum_{k=1,j} \frac{1}{(\lambda_k d_k)^{n-1}} + \lambda_j |a_1 - a_j| \varepsilon_{1j}^{(n-1)/(n-4)} \right), \tag{3.17}$$

$$\int_\Omega P\delta_1^{p+\varepsilon} \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} = -c_0^\varepsilon \lambda_1^{\varepsilon(n-4)/2} \frac{c_1}{\lambda_1^{n-3}} \frac{\partial H(a_1,a_1)}{\partial a_1} + O\left(\frac{1}{(\lambda_1 d_1)^{n-2}} + \frac{\varepsilon}{(\lambda_1 d_1)^{n-3}} \right), \tag{3.18}$$

$$\int_\Omega P\delta_j^{p+\varepsilon} \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial \lambda_1} = c_0^\varepsilon \lambda_j^{\varepsilon(n-4)/2} \left(P\delta_j, \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} \right) + O\left(\varepsilon \varepsilon_{1j} \left(\log \varepsilon_{1j}^{-1}\right)^{(n-4)/n} \right) + T_j, \tag{3.19}$$

$$\int_\Omega P\delta_j \frac{1}{\lambda_1} \frac{\partial (P\delta_1^{p+\varepsilon})}{\partial a_1} = c_0^\varepsilon \lambda_1^{\varepsilon(n-4)/2} \left(P\delta_j, \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1} \right) + O\left(\varepsilon \varepsilon_{1j} \left(\log \varepsilon_{1j}^{-1}\right)^{(n-4)/n} \right) + T_j. \tag{3.20}$$

The proof of Proposition 3.3 is thereby completed. □

4 Proof of the theorems

Proof of Theorem 1.1

Arguing by contradiction, let us assume that problem (P_ε) has solutions (u_ε) as stated in Theorem 1.1. Recall that u_ε is written as

$$u_\varepsilon = \alpha_{\varepsilon,1} P\delta_{(a_{\varepsilon,1},\lambda_{\varepsilon,1})} - \alpha_{\varepsilon,2} P\delta_{(a_{\varepsilon,2},\lambda_{\varepsilon,2})} + \alpha_{\varepsilon,3} P\delta_{(a_{\varepsilon,3},\lambda_{\varepsilon,3})} + v_\varepsilon,$$

with v_ε orthogonal to each $P\delta_{(a_i,\lambda_i)}$ and their derivatives with respect to λ_i and $(a_i)_k$, where $(a_i)_k$ denotes the kth component of a_i (see (2.2) and (2.3)). For simplicity, we will write $\alpha_i := \alpha_{\varepsilon,i}$, $\lambda_i := \lambda_{\varepsilon,i}$, and $a_i := a_{\varepsilon,i}$. From Proposition 3.2, for each $i = 1,2,3$, with $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = -1$. We have

$$(E_i) \quad c_1 \frac{n-4}{2} \frac{H(a_i,a_i)}{\lambda_i^{n-4}} + \gamma_i c_1 \sum_{j\neq i} \gamma_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2} \frac{H(a_i,a_j)}{(\lambda_i \lambda_j)^{(n-4)/2}} \right) + \frac{n-4}{2} c_2 \varepsilon$$
$$= o\left(\varepsilon + \sum_{j=1}^{3} \frac{1}{(\lambda_j d_j)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj} \right).$$

Furthermore, an easy computation shows that

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{n-4}{2}\varepsilon_{ij}\left(1 - 2\frac{\lambda_j}{\lambda_i}\varepsilon_{ij}^{2/n-4}\right) \quad \text{for } i,j = 1,2,3, j \neq i, \tag{4.1}$$

$$-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - 2\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geq \frac{n-4}{2}\varepsilon_{ij} \quad \text{for } \lambda_i \leq \lambda_j. \tag{4.2}$$

On the other hand, following the proof of Proposition 3.3, we have, for each $i = 1,2,3$,

$$(F_i) \quad \frac{1}{\lambda_i^{n-3}} \frac{\partial H(a_i,a_i)}{\partial a_i} - \sum_{j \neq i} 2\frac{(-1)^{j+i}}{\lambda_i}\left(\frac{\partial \varepsilon_{ji}}{\partial a_i} - \frac{\partial H(a_j,a_i)}{\partial a_i}\frac{1}{(\lambda_j\lambda_i)^{(n-4)/2}}\right)$$
$$= o\left(\sum_j \frac{1}{(\lambda_j d_j)^{n-3}} + \sum_{r \neq j} \varepsilon_{rj}^{\frac{n-3}{n-4}} + \varepsilon^{\frac{n-3}{n-4}}\right). \tag{4.3}$$

We distinguish many cases depending on the set

$$\mathcal{F} := \{(i,j) : i \neq j \text{ and } \min(\lambda_i, \lambda_j)|a_i - a_j| \text{ is bounded}\}$$

and we will prove that all these cases cannot occur.

We remark that if $(i,j) \in \mathcal{F}$ we derive $\lambda_i/\lambda_j \to 0$ or ∞ and $d_i/d_j = 1 + o(1)$ as $\varepsilon \to 0$.

Furthermore, the behavior of ε_{ij} depends on the set $\mathcal{F}$. In fact we have, assuming that $\lambda_i \leq \lambda_j$,

$$c\left(\frac{\lambda_i}{\lambda_j}\right)^{(n-4)/2} \leq \varepsilon_{ij} \leq \left(\frac{\lambda_i}{\lambda_j}\right)^{(n-4)/2} \quad \text{if } (i,j) \in \mathcal{F}, \tag{4.4}$$

$$\varepsilon_{ij} = \frac{1}{(\lambda_i\lambda_j|a_i - a_j|^2)^{(n-4)/2}} + o(\varepsilon_{ij}) \quad \text{if } (i,j) \notin \mathcal{F}. \tag{4.5}$$

First we start by proving the following crucial lemmas.

Remark 4.1 Ordering the λ_i's: $\lambda_{i_1} \leq \lambda_{i_2} \leq \lambda_{i_3}$, adding $(E_{i_1}) + 2(E_{i_2}) + 4(E_{i_3})$, and using (4.2), it is easy to derive a contradiction if we have $\varepsilon_{13} = o(\sum(\lambda_i d_i)^{4-n} + \sum \varepsilon_{rj} + \varepsilon)$.

Lemma 4.2 *Let $n \geq 4$. Then there exists a positive constant $\underline{c}_0 > 0$ such that*

(i) $\underline{c}_0^{-1} \leq \dfrac{d_1}{d_3} \leq \underline{c}_0$;

(ii) $\underline{c}_0^{-1} \leq \dfrac{\lambda_1}{\lambda_3} \leq \underline{c}_0$;

(iii) $\underline{c}_0^{-1} \leq \dfrac{|a_1 - a_3|}{d_i} \leq \underline{c}_0^{-1}$ *for $i = 1,3$.*

Proof The proof will be by contradiction.

Proof of (i). Assume that $d_1/d_3 \to 0$. In this case, we have

$$|a_1 - a_3| \geq cd_3 \quad \text{and} \quad \varepsilon_{13} = \frac{1}{(\lambda_1\lambda_3|a_1 - a_3|^2)^{(n-4)/2}} + o(\varepsilon_{13}), \tag{4.6}$$

which implies that $\varepsilon_{13} = o((\lambda_1 d_1)^{4-n} + (\lambda_3 d_3)^{4-n})$. Using Remark 4.1, we derive a contradiction. In the same way, we prove that $d_3/d_1 \nrightarrow 0$. Hence the proof of Claim (i) is completed.

Proof of (ii). Assume that $\lambda_1/\lambda_3 \to 0$. By Claim (i), we have $(\lambda_3 d_3)^{-1} = o((\lambda_1 d_1)^{-1})$. Four cases may occur.

Case 1. $\lambda_2/\lambda_3 \nrightarrow 0$ or $\{(1,2),(2,3)\} \cap \mathcal{F} = \phi$. Using (4.5), (E_2) implies that

$$\frac{H(a_2,a_2)}{\lambda_2^{n-4}} + \varepsilon_{12} + \varepsilon_{23} + \varepsilon = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \varepsilon_{13}\right).$$

By Claim (i) and (E_3), we obtain $\varepsilon_{13} = o((\lambda_1 d_1)^{4-n})$. By Remark 4.1, this case cannot occur.

Case 2. $\lambda_2/\lambda_3 \to 0$, $\{(1,2),(2,3)\} \cap \mathcal{F} \neq \phi$, and $\lambda_2/\lambda_1 \to +\infty$. In this case, it is easy to obtain $\varepsilon_{13} = o(\varepsilon_{12} + \varepsilon_{23})$. Using Remark 4.1, we derive a contradiction.

Case 3. $\lambda_2/\lambda_3 \to 0$, $(2,3) \in \mathcal{F}$, $(1,2) \notin \mathcal{F}$, and $\lambda_2/\lambda_1 \nrightarrow +\infty$. In this case, we see that $\lambda_2|a_2 - a_3|$ is bounded and $\lambda_2|a_1 - a_2| \to +\infty$. Hence, we derive that $\lambda_2|a_1 - a_3| \to +\infty$, which implies that $\lambda_k|a_1 - a_3| \to +\infty$ for $k = 1,3$. Thus

$$\varepsilon_{13} = \frac{1 + o(1)}{(\lambda_1\lambda_3|a_1 - a_3|^2)^{(n-4)/2}} = \left(\frac{\lambda_2}{\lambda_3}\right)^{(n-4)/2} \frac{1 + o(1)}{(\lambda_1\lambda_2|a_1 - a_3|^2)^{(n-4)/2}} = o(\varepsilon_{23}).$$

Then by Remark 4.1, we get a contradiction.

Case 4. $\lambda_2/\lambda_3 \to 0$, $(1,2) \in \mathcal{F}$, and $\lambda_2/\lambda_1 \nrightarrow +\infty$. In this case, it is easy to get $\varepsilon_{23} = o(\varepsilon_{12})$.

Using the formula $[(E_1) + (E_2) - (E_3)]$, we deduce that $\varepsilon = o(\varepsilon_{12} + \varepsilon_{13})$, which implies that $\varepsilon_{13} = o(\varepsilon_{12})$. Hence by Remark 4.1, we derive a contradiction and Claim (ii) is thereby completed.

Proof of (iii). Without loss of generality, we can assume that $d_1 \leq d_3$. First, as in the proof of Claim (i), we get $|a_1 - a_3| \leq c_0 d_1$. Now assume that $|a_1 - a_3|/d_1 \to 0$, which implies

$$\frac{H(a_i,a_i)}{\lambda_i^{n-4}} = o(\varepsilon_{13}) \quad \text{for } i = 1,3.$$

Two cases may occur.

Case 1. $\lambda_1 \leq \lambda_2$ or $\{(1,2),(2,3)\} \cap \mathcal{F} = \phi$. Using (E_2), we obtain

$$\frac{H(a_2,a_2)}{\lambda_2^{n-4}} = o(\varepsilon_{13}), \qquad \varepsilon_{i2} = o(\varepsilon_{13}) \quad \text{for } i = 1,3 \quad \text{and} \quad \varepsilon = o(\varepsilon_{13}),$$

and we derive a contradiction from (E_1).

Case 2. $\lambda_2 \leq \lambda_1$ and $\{(1,2),(2,3)\} \cap \mathcal{F} \neq \phi$. Let $k \in \{1,3\}$ such that $(2,k) \in \mathcal{F}$. Using Claim (ii) and the fact that $\lambda_2 \leq \lambda_1$, we derive that $\varepsilon_{2k} \geq c(\lambda_2/\lambda_k)^{(n-4)/2}$, which implies that $d_2 \sim d_k$, $\lambda_2/\lambda_k \to 0$, and $\lambda_2|a_2 - a_k|$ is bounded. Using (4.3) for $i = k$, we get

$$-\lambda_2|a_2 - a_k|\varepsilon_{2k}^{\frac{n}{n-4}} + \frac{\lambda_1\lambda_3}{\lambda_k}|a_1 - a_3|\varepsilon_{13}^{\frac{n}{n-4}} = o\left(\frac{1}{(\lambda_2 d_2)^{n-3}} + \sum_{r\neq j}\varepsilon_{rj}^{\frac{n-3}{n-4}} + \varepsilon^{\frac{n-3}{n-4}}\right). \tag{4.7}$$

Since $\lambda_2|a_2 - a_k|$ is bounded and $\varepsilon_{13} \simeq (\lambda_1\lambda_3|a_2 - a_k|^2)^{(4-n)/2}$, we derive that

$$\varepsilon_{13}^{\frac{n-3}{n-4}} = o\left(\frac{1}{(\lambda_2 d_2)^{n-3}} + \varepsilon_{12}^{\frac{n-3}{n-4}} + \varepsilon_{23}^{\frac{n-3}{n-4}} + \varepsilon^{\frac{n-3}{n-4}}\right),$$

which implies that

$$\varepsilon_{13} = o\left(\frac{1}{(\lambda_2 d_2)^{n-4}} + \varepsilon_{12} + \varepsilon_{23} + \varepsilon\right). \tag{4.8}$$

By Remark 4.1, we get a contradiction. □

Lemma 4.3 *There exists a positive constant* $\underline{c}'_0$ *such that*

(i) $\underline{c}'_0 \lambda_1 \le \lambda_2$;

(ii) $d_i \ge \underline{c}'_0$ *for* $i = 1, 3$.

Proof Without loss of generality, we can assume that $d_1 \le d_3$.

Proof of (i). Assume that $\lambda_2/\lambda_1 \to 0$. First we claim that $d_1/d_2 \nrightarrow 0$. In fact, arguing by contradiction we assume that $d_1/d_2 \to 0$, we get $d_1 \to 0$, $|a_1 - a_2| \ge cd_2$, and $|a_2 - a_3| \ge cd_2$. Hence, $\{(1,2),(2,3)\} \cap \mathcal{F} = \phi$. From (E_2), we obtain

$$\frac{H(a_2,a_2)}{\lambda_2^{n-4}} + \varepsilon_{12} + \varepsilon_{23} + \varepsilon = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \frac{1}{(\lambda_3 d_3)^{n-4}} + \varepsilon_{13}\right). \tag{4.9}$$

Let ν_i be the outward normal vector at a_i. Since d_1, d_3, and $|a_1 - a_3|$ are of the same order, we have (see [18] and [19])

$$\frac{1}{\lambda_1^{n-3}} \frac{\partial H(a_1,a_1)}{\partial \nu_1} \sim \frac{c}{(\lambda_1 d_1)^{n-3}} \quad \text{and} \quad \frac{\partial G(a_1,a_3)}{\partial \nu_1} \le 0. \tag{4.10}$$

Using (F_1), we get $1/(\lambda_1 d_1)^{n-3} = o(\varepsilon_{13}^{(n-3)/(n-4)})$, which implies that $1/(\lambda_1 d_1)^{n-4} = o(\varepsilon_{13})$. From (E_1), we derive a contradiction. Hence our claim is proved.

Thus there exists a positive constant c so that $d_1 \ge cd_2$. Now, since we have assumed that $\lambda_2/\lambda_1 \to 0$, Lemma 4.2 implies that $\varepsilon_{13} = o((\lambda_2 d_2)^{4-n})$. Finally, using Remark 4.1, we get a contradiction and the proof of Claim (i) follows.

Proof of (ii). Assume that $d_1 \to 0$. Note that Claim (i) and (E_2) imply that (4.9) holds. Now, following the proof of (i), we obtain a contradiction. □

We turn now to the proof of Theorem 1.1. By the previous lemmas, we know that λ_1 and λ_3 are of the same order, $|a_1 - a_3| \ge c$ and $\lambda_2 \ge c\lambda_i$, for $i = 1, 3$ where c is a positive constant. Hence, (E_2) implies that (4.9) holds. Furthermore, for $i = 1, 3$ (E_i) implies that

$$\frac{H(a_i,a_i)}{\lambda_i^{n-4}} - \frac{G(a_1,a_3)}{(\lambda_1\lambda_3)^{n-4}} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \frac{1}{(\lambda_3 d_3)^{n-4}} + \varepsilon_{13}\right). \tag{4.11}$$

We denote by $r(x)$ the eigenvector associated to $\rho(x)$ whose norm is 1. We point out that we can choose $r(x)$ so that all their components are positive (see [18] and [19]).

Let $\Lambda_i = \lambda_i^{(4-n)/2}$, $\Lambda = (\Lambda_1, \Lambda_3)$, and $x = (a_1, a_3)$. From (4.11), we have

$$M(x) \cdot \frac{{}^t\Lambda}{\|\Lambda\|} = o(1). \tag{4.12}$$

The scalar product of (4.12) by $r(x)$ gives

$$\rho(x)r(x)\cdot\frac{{}^t\Lambda}{\|\Lambda\|}=o(1). \tag{4.13}$$

Since the components of $r(x)$ are positive and λ_1, λ_3 are of the same order, there exists a positive constant c, such that $r(x)\cdot\frac{{}^t\Lambda}{\|\Lambda\|}\geq c>0$. Hence, we get

$$\rho(x)=o(1). \tag{4.14}$$

We deduce from (4.3) and (4.11) that

$$\frac{\partial M}{\partial x_i}(x)\cdot\frac{{}^t\Lambda}{\|\Lambda\|}=o(1). \tag{4.15}$$

Observe that Λ may be written in the form

$$\Lambda=\beta r(x)+\overline{r}(x),\quad \text{with } r(x)\cdot\overline{r}(x)=0, \|\overline{r}\|=o(\beta) \text{ and } \beta\sim\|\Lambda\|. \tag{4.16}$$

Using (4.15), we get

$$\frac{\partial M}{\partial x_i}(x)\cdot{}^t r(x)+\frac{\partial M}{\partial x_i}(x)\cdot\frac{\overline{r}(x)}{\|\Lambda\|}=o(1). \tag{4.17}$$

Since $d_i\geq c_0$ for $i=1,3$ and $|a_1-a_3|\geq c_0$, the matrix $\frac{\partial M}{\partial x_i}(x)$ is bounded.

Furthermore, we have $\|\overline{r}\|=o(\|\Lambda\|)$, which implies that

$$\frac{\partial M}{\partial x_i}(x)\cdot{}^t r(x)=o(1). \tag{4.18}$$

Let us consider the equality

$$M(x)\cdot{}^t r(x)=\rho(x)\cdot{}^t r(x)$$

and derivative it with respect to x_i; we obtain

$$\frac{\partial M}{\partial x_i}(x)\cdot{}^t r(x)+M(x)\frac{\partial^t r}{\partial x_i}(x)=\frac{\partial\rho}{\partial x_i}(x)\cdot{}^t r(x)+\rho(x)\frac{\partial^t r}{\partial x_i}(x).$$

The scalar product with $r(x)$ gives

$$r(x)\cdot\frac{\partial M}{\partial x_i}(x)\cdot{}^t r(x)=\frac{\partial\rho}{\partial x_i}(x). \tag{4.19}$$

Using (4.18), we obtain

$$\frac{\partial\rho}{\partial x_i}(x)=o(1). \tag{4.20}$$

Hence, we derive a contradiction from (4.14), (4.20), and the fact that 0 is a regular value of ρ. Thus the proof of our theorem follows.

Proof of Theorem 1.2

Arguing by contradiction, let us assume that problem (P_ε) has solutions (u_ε) as stated in Theorem 1.2. From Section 2, these solutions have to satisfy (2.2) and (2.3).

As in the proof of Proposition 3.2, we have, for each $i = 1, \dots, k$,

$$(E_i) \quad c_1 \frac{n-4}{2} \frac{H(a_i,a_i)}{\lambda_i^{n-4}} + \gamma_i c_1 \sum_{j\neq i} \gamma_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2} \frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}} \right) + \frac{n-4}{2} c_2 \varepsilon = o\left(\varepsilon + \sum_{j=1}^{k} \frac{1}{(\lambda_j d_j)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj} \right).$$

Observe that, if $j < i$, we have $\lambda_j |a_i - a_j|$ is bounded (by the assumption) which implies that

$$\begin{aligned} &|a_i - a_j| = o(d_j), \qquad d_i/d_j = 1 + o(1), \quad \forall i,j \quad \text{and} \\ &\varepsilon_{ij} \geq c(\lambda_j/\lambda_i)^{(n-4)/2}, \quad \forall j < i, \end{aligned} \tag{4.21}$$

where c is a positive constant. Using (4.21), easy computations show that

$$\begin{aligned} &\varepsilon_{(i-1)j} + \varepsilon_{i(j+1)} = o(\varepsilon_{ij}), \quad \forall i < j, \\ &\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-4)/2}} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} \right) \quad \text{if } (i,j) \neq (1,1). \end{aligned} \tag{4.22}$$

Thus, using (4.22), (E_i) can be written as

$$(E'_1) \quad c_1 \frac{n-4}{2} \frac{H(a_1,a_1)}{\lambda_1^{n-4}} + c_1\gamma_1\gamma_2\lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + \frac{n-4}{2} c_2 \varepsilon = o\left(\varepsilon + \frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj} \right),$$

$$(E'_k) \quad c_1\gamma_{k-1}\gamma_k\lambda_k \frac{\partial \varepsilon_{(k-1)k}}{\partial \lambda_k} + \frac{n-4}{2} c_2 \varepsilon = o\left(\varepsilon + \frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj} \right),$$

and for $1 < i < k$,

$$(E'_i) \quad c_1\gamma_{i-1}\gamma_i\lambda_i \frac{\partial \varepsilon_{(i-1)i}}{\partial \lambda_i} + c_1\gamma_i\gamma_{i+1}\lambda_i \frac{\partial \varepsilon_{i(i+1)}}{\partial \lambda_i} + \frac{n-4}{2} c_2 \varepsilon = o\left(\varepsilon + \frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj} \right).$$

The proof will depend on the value of l which is defined in the theorem.

Case 1. $l = k$. From the definition of l we get $\gamma_{k-1}\gamma_k = -1$. Now using (4.1) and (E'_k), we derive that

$$\varepsilon = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r\neq j} \varepsilon_{ij} \right) \quad \text{and} \quad \varepsilon_{(k-1)k} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj} \right). \tag{4.23}$$

Now, using (4.23) and (E'_{k-1}), we derive the estimate of $\varepsilon_{(k-2)(k-1)}$ and by induction we get

$$\varepsilon_{(i-1)i} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj} \right) \quad \text{for each } i = 2, \dots, k. \tag{4.24}$$

Finally, using (4.22), (4.23), (4.24), and (E_1') we obtain

$$\frac{H(a_1,a_1)}{\lambda_1^{n-4}} = o\left(\frac{1}{(\lambda_1 d_1)^{n-4}}\right),$$

which gives a contradiction.

Case 2. $l = k-1$. Using (4.1), an easy computation implies that

$$\lambda_{k-1}\frac{\partial\varepsilon_{(k-1)k}}{\partial\lambda_{k-1}} - \lambda_k\frac{\partial\varepsilon_{(k-1)k}}{\partial\lambda_k} \geq c\varepsilon_{(k-1)k}. \tag{4.25}$$

Then from (E_{k-1}'), (E_k'), (4.1), (4.25), and the fact that $\gamma_{k-1}\gamma_k = 1$ and $\gamma_{k-2}\gamma_{k-1} = -1$ (since $l = k-1$), we obtain

$$c\varepsilon_{(k-1)k} + \varepsilon_{(k-2)(k-1)} = o\left(\varepsilon + \frac{1}{(\lambda_1 d_1)^{n-4}} + \sum_{r\neq j}\varepsilon_{rj}\right). \tag{4.26}$$

Now using (E_k') and (4.26) we get (4.23) and as before, (4.24) is satisfied. Hence we also derive a contradiction from (E_1').

Case 3. $l \notin \{k, k-1\}$. Recall that in this case we have assumed that $\lambda_l|a_l - a_{l+1}| \to 0$. This implies that

$$\lambda_l\frac{\partial\varepsilon_{l(l+1)}}{\partial\lambda_l} = \big((n-4)/2\big)\varepsilon_{l(l+1)}\big(1+o(1)\big). \tag{4.27}$$

Hence, using (E_l'), the definition of l and (4.1) we obtain the first part of (4.23). The second part follows from (E_k') and the first one. Finally, as before we derive a contradiction from (E_1').

Hence, our theorem is proved.

Competing interests

The author declares that they have no competing interests.

Acknowledgements

The author gratefully acknowledges the Deanship of Scientific Research at Taibah University on material and moral support, in particular by financing this research project.

Received: 26 August 2014 Accepted: 1 December 2014 Published: 30 Dec 2014

References

1. Bartsch, T, Weth, T: A note on additional properties of sign-changing solutions to superlinear elliptic equations. Topol. Methods Nonlinear Anal. **22**, 1-14 (2003)
2. Ben Ayed, M, El Mehdi, K, Grossi, M, Rey, O: A nonexistence result of single peaked solutions to a supercritical nonlinear problem. Commun. Contemp. Math. **2**, 179-195 (2003)
3. Ben Ayed, M, Ould Bouh, K: Nonexistence results of sign-changing solution to a supercritical nonlinear problem. Commun. Pure Appl. Anal. **7**(5), 1057-1075 (2008)
4. Castro, A, Cossio, J, Newberger, JM: A sign-changing solution for a supercritical Dirichlet problem. Rocky Mt. J. Math. **27**, 1041-1053 (1997)
5. Del Pino, M, Felmer, P, Musso, M: Two bubbles solutions in the supercritical Bahri-Coron's problem. Calc. Var. Partial Differ. Equ. **16**, 113-145 (2003)
6. Ould Bouh, K: Nonexistence result of sign-changing solutions for a supercritical problem of the scalar curvature type. Adv. Nonlinear Stud. **12**, 149-171 (2012)
7. Ben Ayed, M, El Mehdi, K: On a biharmonic equation involving nearly critical exponent. Nonlinear Differ. Equ. Appl. **13**(4), 485-509 (2006)
8. Ben Ayed, M, Ghoudi, R: Profile and existence of sign-changing solutions to an elliptic subcritical equation. Commun. Contemp. Math. **10**(6), 1183-1216 (2008)
9. Chou, KS, Geng, D: Asymptotics of positive solutions for a biharmonic equation involving critical exponent. Differ. Integral Equ. **13**, 921-940 (2000)

10. van der Vorst, RCAM: Fourth order elliptic equations with critical growth. C. R. Acad. Sci. Paris **320**, 295-299 (1995)
11. Ebobisse, F, Ould Ahmedou, M: On a nonlinear fourth order elliptic equation involving the critical Sobolev exponent. Nonlinear Anal. TMA **52**, 1535-1552 (2003)
12. Gazzola, F, Grunau, HC, Squassina, M: Existence and nonexistence results for critical growth biharmonic elliptic equations. Calc. Var. Partial Differ. Equ. **18**, 117-143 (2003)
13. Ould Bouh, K: Sign-changing solutions of a fourth-order elliptic equation with supercritical exponent. Electron. J. Differ. Equ. **2014**, 77 (2014)
14. Bahri, A, Coron, JM: On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of topology of the domain. Commun. Pure Appl. Math. **41**, 255-294 (1988)
15. Rey, O: The role of Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent. J. Funct. Anal. **89**, 1-52 (1990)
16. Rey, O: The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension 3. Adv. Differ. Equ. **4**, 581-616 (1999)
17. Bahri, A: Critical Point at Infinity in Some Variational Problems. Pitman Res. Notes Math. Ser., vol. 182. Longman, Harlow (1989)
18. Bahri, A, Li, YY, Rey, O: On a variational problem with lack of compactness: the topological effect of the critical points at infinity. Calc. Var. Partial Differ. Equ. **3**, 67-94 (1995)
19. Ben Ayed, M, Chtioui, H, Hammami, M: A Morse lemma at infinity for Yamabe type problems on domains. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **20**, 543-577 (2003)

10.1186/1687-1847-2014-319
Cite this article as: Ould Bouh: On a fourth order elliptic equation with supercritical exponent. *Advances in Difference Equations* 2014, 2014:319