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Nonoscillatory solutions to third-order neutral dynamic equations on time scales

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Abstract

In this paper, we establish the existence of nonoscillatory solutions to third-order nonlinear neutral dynamic equations on time scales of the form $(r_1(t)(r_2(t)(x(t)+p(t)x(g(t)))^{\Delta})^{\Delta})^{\Delta}+f(t,x(h(t)))=0$ by employing Kranoselskii's fixed point theorem. Three examples are included to illustrate the significance of the conclusions.

Keywords: third-order neutral dynamic equations; time scales; nonoscillatory solutions; Kranoselskii's fixed point theorem

1 Introduction

In this paper, we study third-order nonlinear neutral dynamic equations of the form

$$\big(r_1(t)\big(r_2(t)\big(x(t)+p(t)x\big(g(t)\big)\big)^{\Delta}\big)^{\Delta}\big)^{\Delta}+f\big(t,x\big(h(t)\big)\big)=0 \tag{1}$$

on a time scale $\mathbb{T}$ satisfying $\inf\mathbb{T}=t_0$ and $\sup\mathbb{T}=\infty$.

Throughout this paper we shall assume that:

(C1) $r_1,r_2\in C_{rd}(\mathbb{T},(0,\infty))$ such that

$$\int_{t_0}^{\infty}\frac{1}{r_1(t)}\Delta t=\infty,\qquad \int_{t_0}^{\infty}\frac{1}{r_2(t)}\Delta t=\infty.$$

(C2) $p\in C_{rd}(\mathbb{T},\mathbb{R})$ and there exists a constant p_0 with $|p_0|<1$ such that

$$\lim_{t\to\infty}p(t)=p_0.$$

(C3) $g,h\in C_{rd}(\mathbb{T},\mathbb{T})$, $g(t)\le t$, $\lim_{t\to\infty}g(t)=\lim_{t\to\infty}h(t)=\infty$, and

$$\lim_{t\to\infty}\frac{R_\lambda(g(t))}{R_\lambda(t)}=\eta_\lambda\in(0,1],\qquad \lambda=1,2,$$

where

$$R_1(t)=1+\int_{t_0}^{t}\frac{1}{r_2(s)}\Delta s,\qquad R_2(t)=1+\int_{t_0}^{t}\int_{t_0}^{s}\frac{1}{r_1(u)r_2(s)}\Delta u\Delta s.$$

If $p_0\in(-1,0]$, there exists a sequence $\{c_k\}_{k\ge0}$ such that $\lim_{k\to\infty}c_k=\infty$ and $g(c_{k+1})=c_k$.

(C4) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $f(t,x)$ is nondecreasing in x and $xf(t,x) > 0$ for $t \in \mathbb{T}$ and $x \neq 0$.

Hilger introduced the theory of time scales in his Ph.D. thesis [1] in 1988; see also [2]. More details of time scale calculus can be found in [3–6] and omitted here. In the last few years, there has been some research achievement as regards the existence of nonoscillatory solutions to neutral dynamic equations on time scales; see the papers [7–11] and the references therein.

Definition 1.1 By a solution of (1) we mean a continuous function $x(t)$ which is defined on $\mathbb{T}$ and satisfies (1) for $t \geq t_0$. A solution $x(t)$ of (1) is said to be eventually positive (or eventually negative) if there exists $c \in \mathbb{T}$ such that $x(t) > 0$ (or $x(t) < 0$) for all $t \geq c$ in $\mathbb{T}$. A solution x of (1) is said to be nonoscillatory if it is either eventually positive or eventually negative; otherwise, it is oscillatory.

In 1990s, some significative results for existence of nonoscillatory solutions to neutral functional differential equations were given in [7, 9]. In 2007, Zhu and Wang [11] discussed the existence of nonoscillatory solutions to first-order nonlinear neutral dynamic equations

$$\left[x(t) + p(t)x\big(g(t)\big)\right]^{\Delta} + f\big(t, x\big(h(t)\big)\big) = 0$$

on a time scale $\mathbb{T}$. In 2013, Gao and Wang [10] considered the second-order nonlinear neutral dynamic equations

$$\left[r(t)\big(x(t) + p(t)x\big(g(t)\big)\big)^{\Delta}\right]^{\Delta} + f\big(t, x\big(h(t)\big)\big) = 0 \tag{2}$$

under the condition $\int_{t_0}^{\infty} \frac{1}{r(s)} \Delta s < \infty$, and established the existence of nonoscillatory solutions to (2) on a time scale. In 2014, Deng and Wang [8] studied the same problem of (2) under the condition $\int_{t_0}^{\infty} \frac{1}{r(s)} \Delta s = \infty$.

In this paper, we shall establish the existence of nonoscillatory solutions to (1) by employing Kranoselskii's fixed point theorem, and we give three examples to show the versatility of the results.

For simplicity, throughout this paper, we denote $(a,b) \cap \mathbb{T} = (a,b)_{\mathbb{T}}$, where $a, b \in \mathbb{R}$, and $[a,b]_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ are denoted similarly.

2 Preliminary results

Let $C([T_0,\infty)_{\mathbb{T}}, \mathbb{R})$ denote all continuous functions mapping $[T_0,\infty)_{\mathbb{T}}$ into $\mathbb{R}$, and $R_0(t) \equiv 1$, $t \in [T_0,\infty)_{\mathbb{T}}$. For $\lambda = 0, 1, 2$, we define

$$BC_{\lambda}\,[T_0,\infty)_{\mathbb{T}} = \left\{x : x \in C\big([T_0,\infty)_{\mathbb{T}}, \mathbb{R}\big) \text{ and } \sup_{t \in [T_0,\infty)_{\mathbb{T}}} \left|\frac{x(t)}{R_{\lambda}^2(t)}\right| < \infty\right\}. \tag{3}$$

Endowing $BC_{\lambda}\,[T_0,\infty)_{\mathbb{T}}$ with the norm $\|x\|_{\lambda} = \sup_{t \in [T_0,\infty)_{\mathbb{T}}} |\frac{x(t)}{R_{\lambda}^2(t)}|$, $(BC_{\lambda}\,[T_0,\infty)_{\mathbb{T}}, \|\cdot\|_{\lambda})$ is a Banach space. Let $X \subseteq BC_{\lambda}\,[T_0,\infty)_{\mathbb{T}}$, we say that X is uniformly Cauchy if for any given $\epsilon > 0$, there exists a $T_1 \in [T_0,\infty)_{\mathbb{T}}$ such that, for any $x \in X$,

$$\left|\frac{x(t_1)}{R_{\lambda}^2(t_1)} - \frac{x(t_2)}{R_{\lambda}^2(t_2)}\right| < \epsilon \quad \text{for all } t_1, t_2 \in [T_1,\infty)_{\mathbb{T}}.$$

X is said to be equi-continuous on $[a,b]_{\mathbb{T}}$ if, for any given $\epsilon > 0$, there exists a $\delta > 0$ such that, for any $x \in X$ and $t_1, t_2 \in [a,b]_{\mathbb{T}}$ with $|t_1 - t_2| < \delta$,

$$\left| \frac{x(t_1)}{R_\lambda^2(t_1)} - \frac{x(t_2)}{R_\lambda^2(t_2)} \right| < \epsilon.$$

We have the following lemma, which is an analog of the Arzela-Ascoli theorem on time scales.

Lemma 2.1 ([11, Lemma 4]) *Suppose that $X \subseteq BC_\lambda\ [T_0,\infty)_{\mathbb{T}}$ is bounded and uniformly Cauchy. Further, suppose that X is equi-continuous on $[T_0,T_1]_{\mathbb{T}}$ for any $T_1 \in [T_0,\infty)_{\mathbb{T}}$. Then X is relatively compact.*

In this section, our approach to the existence of nonoscillatory solutions to (1) is based largely on the application of Kranoselskii's fixed point theorem (see [7]). For the sake of convenience, we state here this theorem as follows.

Lemma 2.2 (Kranoselskii's fixed point theorem) *Suppose that X is a Banach space and Ω is a bounded, convex, and closed subset of X. Suppose further that there exist two operators $U, S : \Omega \to X$ such that*
(i) *$Ux + Sy \in \Omega$ for all $x, y \in \Omega$;*
(ii) *U is a contraction mapping;*
(iii) *S is completely continuous.*
Then $U + S$ has a fixed point in Ω.

If $x(t)$ is an eventually negative solution of (1), then $y(t) = -x(t)$ will satisfy

$$\big(r_1(t)\big(r_2(t)\big(y(t) + p(t)y\big(g(t)\big)\big)^\Delta\big)^\Delta\big)^\Delta - f\big(t, -y\big(h(t)\big)\big) = 0.$$

We may note that $\bar{f}(t,u) := -f(t,-u)$ is nondecreasing in u and $u\bar{f}(t,u) > 0$ for $t \in \mathbb{T}$ and $u \neq 0$. Therefore, we will restrict our attention to eventually positive solutions of (1) in the following.

In the sequel, we use the notation

$$z(t) := x(t) + p(t)x\big(g(t)\big) \tag{4}$$

and have the following lemma.

Lemma 2.3 ([8, Lemma 2.3]) *Suppose that $x(t)$ is an eventually positive solution of* (1) *and $\lim_{t\to\infty} \frac{z(t)}{R_\lambda^i(t)} = a$ for $\lambda = 1, 2$ and $i = 0, 1$. Then we have*:
(i) *If a is finite, then*

$$\lim_{t\to\infty} \frac{x(t)}{R_\lambda^i(t)} = \frac{a}{1 + p_0 \eta_\lambda^i}.$$

(ii) *If a is infinite, then $\frac{x(t)}{R_\lambda^i(t)}$ is unbounded, or*

$$\limsup_{t\to\infty} \frac{x(t)}{R_\lambda^i(t)} = +\infty.$$

Let S^+ denote the set of all eventually positive solutions of (1) and

$$A(\alpha,\beta,\gamma)=\left\{x\in S^+:\lim_{t\to\infty}x(t)=\alpha,\lim_{t\to\infty}\frac{x(t)}{R_1(t)}=\beta,\lim_{t\to\infty}\frac{x(t)}{R_2(t)}=\gamma\right\}.$$

Now, we give the first theorem for a classification scheme of eventually positive solutions to (1).

Theorem 2.4 *If* $x(t)$ *is an eventually positive solution of* (1), *then* $x(t)$ *belongs to* $A(0,0,0)$, $A(b,0,0)$, $A(\infty,b,0)$, $A(\infty,\infty,b)$ *for some positive* b, *or* $A(\infty,\infty,0)$.

Proof Suppose that $x(t)$ is an eventually positive solution of (1). From (C2) and (C3), there exist $t_1\in[t_0,\infty)_{\mathbb{T}}$ and $|p_0|<p_1<1$ such that $x(t)>0$, $x(g(t))>0$, $x(h(t))>0$, and $|p(t)|\le p_1$ for $t\in[t_1,\infty)_{\mathbb{T}}$. By (1) and (C4), it follows that, for $t\in[t_1,\infty)_{\mathbb{T}}$,

$$\bigl(r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta}\bigr)^{\Delta}=-f\bigl(t,x\bigl(h(t)\bigr)\bigr)<0.$$

Hence, $r_1(t)(r_2(t)z^{\Delta}(t))^{\Delta}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. We claim that

$$r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta}>0,\quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{5}$$

Assume not; then there exists $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $r_1(t)(r_2(t)z^{\Delta}(t))^{\Delta}<0$ for $t\in[t_2,\infty)_{\mathbb{T}}$. So there exist a constant $c<0$ and $t_3\in[t_2,\infty)_{\mathbb{T}}$ such that $r_1(t)(r_2(t)z^{\Delta}(t))^{\Delta}\le c$ for $t\in[t_3,\infty)_{\mathbb{T}}$, which means that

$$\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta}\le\frac{c}{r_1(t)},\quad t\in[t_3,\infty)_{\mathbb{T}}. \tag{6}$$

Integrating (6) from t_3 to $t\in[\sigma(t_3),\infty)_{\mathbb{T}}$, we obtain

$$r_2(t)z^{\Delta}(t)\le r_2(t_3)z^{\Delta}(t_3)+c\int_{t_3}^{t}\frac{\Delta s}{r_1(s)}.$$

Letting $t\to\infty$, by (C1) we have $r_2(t)z^{\Delta}(t)\to-\infty$. Then there exists $t_4\in[t_3,\infty)_{\mathbb{T}}$ such that $r_2(t)z^{\Delta}(t)\le r_2(t_4)z^{\Delta}(t_4)<0$ for $t\in[t_4,\infty)_{\mathbb{T}}$, which implies that

$$z^{\Delta}(t)\le r_2(t_4)z^{\Delta}(t_4)\cdot\frac{1}{r_2(t)}. \tag{7}$$

Integrating (7) from t_4 to $t\in[\sigma(t_4),\infty)_{\mathbb{T}}$, we obtain

$$z(t)-z(t_4)\le r_2(t_4)z^{\Delta}(t_4)\int_{t_4}^{t}\frac{\Delta s}{r_2(s)}.$$

Letting $t\to\infty$, by (C1) we have $z(t)\to-\infty$. From (4), it follows that $p_0\in(-1,0]$, then there exists $t_5\in[t_4,\infty)_{\mathbb{T}}$ such that $z(t)<0$ or

$$x(t)<-p(t)x\bigl(g(t)\bigr)<p_1x\bigl(g(t)\bigr),\quad t\in[t_5,\infty)_{\mathbb{T}}.$$

By (C3), we can choose some positive integer k_0 such that $c_k \in [t_5, \infty)_{\mathbb{T}}$ for all $k \geq k_0$. Then for any $k \geq k_0 + 1$, we have

$$\begin{aligned} x(c_k) &< p_1 x\big(g(c_k)\big) = p_1 x(c_{k-1}) < p_1^2 x\big(g(c_{k-1})\big) = p_1^2 x(c_{k-2}) < \cdots \\ &< p_1^{k-k_0} x\big(g(c_{k_0+1})\big) = p_1^{k-k_0} x(c_{k_0}). \end{aligned}$$

The inequality above implies that $\lim_{k\to\infty} x(c_k) = 0$. It follows from (4) that $\lim_{k\to\infty} z(c_k) = 0$ and then contradicts $\lim_{t\to\infty} z(t) = -\infty$. So (5) holds, and

$$\lim_{t\to\infty} r_1(t)\big(r_2(t)z^\Delta(t)\big)^\Delta = L_2, \tag{8}$$

where $0 \leq L_2 < +\infty$.

From (5), we have $(r_2(t)z^\Delta(t))^\Delta > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, which means that $r_2(t)z^\Delta(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Hence, $r_2(t)z^\Delta(t)$ is either eventually positive or eventually negative. When $r_2(t)z^\Delta(t)$ is eventually negative, we have $\lim_{t\to\infty} r_2(t)z^\Delta(t) \leq 0$. Assume that there exists a constant $d < 0$ such that

$$\lim_{t\to\infty} r_2(t)z^\Delta(t) = d,$$

which means that

$$z^\Delta(t) \leq \frac{d}{r_2(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{9}$$

Integrating (9) from t_1 to $t \in [\sigma(t_1), \infty)_{\mathbb{T}}$, we obtain

$$z(t) \leq z(t_1) + d \int_{t_1}^t \frac{\Delta s}{r_2(s)}.$$

Letting $t \to \infty$, by (C1) we have $z(t) \to -\infty$. Similarly, it will cause the contradiction as before. Hence, $\lim_{t\to\infty} r_2(t)z^\Delta(t) = 0$. When $r_2(t)z^\Delta(t)$ is eventually positive, we have $\lim_{t\to\infty} r_2(t)z^\Delta(t) = b$ for some positive b or $\lim_{t\to\infty} r_2(t)z^\Delta(t) = +\infty$. Therefore,

$$\lim_{t\to\infty} r_2(t)z^\Delta(t) = L_1, \tag{10}$$

where $0 \leq L_1 \leq +\infty$.

When $r_2(t)z^\Delta(t)$ is eventually negative, which means that $z^\Delta(t)$ is eventually negative, then there exists $t_6 \in [t_1, \infty)_{\mathbb{T}}$ such that $z^\Delta(t) < 0$ for $t \in [t_6, \infty)_{\mathbb{T}}$. It follows that $z(t)$ is strictly decreasing on $[t_6, \infty)_{\mathbb{T}}$. Hence, $z(t)$ is either eventually positive or eventually negative. If $z(t)$ is eventually negative, we have $\lim_{t\to\infty} z(t) = -\infty$ or $\lim_{t\to\infty} z(t) < 0$. Similarly, it will cause the contradiction as before. Therefore, $z(t)$ is eventually positive, which means that $\lim_{t\to\infty} z(t) = b$ for some positive b or $\lim_{t\to\infty} z(t) = 0$.

When $r_2(t)z^\Delta(t)$ is eventually positive, it implies that $z^\Delta(t)$ is eventually positive. If $z(t)$ is eventually negative, we have $\lim_{t\to\infty} z(t) \leq 0$. Assume that $\lim_{t\to\infty} z(t) < 0$. It will cause a similar contradiction to the one before. So $\lim_{t\to\infty} z(t) = 0$. If $z(t)$ is eventually positive, we have $\lim_{t\to\infty} z(t) = b$ for some positive b or $\lim_{t\to\infty} z(t) = +\infty$.

Therefore,

$$\lim_{t\to\infty} z(t) = L_0,$$

where $0 \le L_0 \le +\infty$.

It follows from L'Hôpital's rule (see [5, Theorem 1.120]) and (8), (10) that

$$\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = \lim_{t\to\infty} \frac{z(t)}{R_1(t)} = L_1$$

and

$$\lim_{t\to\infty} r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta} = \lim_{t\to\infty} \frac{z(t)}{R_2(t)} = L_2.$$

When $L_0 = 0$ or $L_0 = b$ for some positive b, we have $L_1 = L_2 = 0$. When $L_0 = +\infty$, it implies that $z^{\Delta}(t)$ is eventually positive, which means that $r_2(t)z^{\Delta}(t)$ is eventually positive. It follows that $L_1 = b$ for some positive b or $L_1 = +\infty$. We have $L_2 = 0$ if $L_1 = b$ for some positive b, and $L_2 = 0$ or $L_2 = b$ for some positive b if $L_1 = +\infty$. Then by Lemma 2.3, we see that $x(t)$ must belong to $A(0,0,0)$, $A(b,0,0)$, $A(\infty,b,0)$, $A(\infty,\infty,b)$ for some positive b, or $A(\infty,\infty,0)$. The proof is complete. □

3 Main results

In this section, by employing Kranoselskii's fixed point theorem, we establish the existence criteria for each type of eventually positive solutions to (1).

Theorem 3.1 *Equation* (1) *has an eventually positive solution in* $A(\infty,\infty,b)$ *for some positive b if and only if there exists some constant* $K > 0$ *such that*

$$\int_{t_0}^{\infty} f\bigl(t, KR_2\bigl(h(t)\bigr)\bigr)\Delta t < \infty. \tag{11}$$

Proof Suppose that $x(t)$ is an eventually positive solution of (1) in $A(\infty,\infty,b)$, *i.e.*,

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{R_1(t)} = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{R_2(t)} = b. \tag{12}$$

Assume that $\lim_{t\to\infty} z(t) < \infty$ (or $\lim_{t\to\infty} \frac{z(t)}{R_1(t)} < \infty$). By Lemma 2.3 we have $\lim_{t\to\infty} x(t) < \infty$ (or $\lim_{t\to\infty} \frac{x(t)}{R_1(t)} < \infty$), which contradicts (12). Then we have

$$\lim_{t\to\infty} z(t) = \infty, \qquad \lim_{t\to\infty} \frac{z(t)}{R_1(t)} = \infty,$$
$$\lim_{t\to\infty} r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta} = \lim_{t\to\infty} \frac{z(t)}{R_2(t)} = (1 + p_0\eta_2)b$$

and there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) \ge \frac{b}{2}R_2(h(t))$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Integrating (1) from T_1 to $s \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we obtain

$$r_1(s)\bigl(r_2(s)z^{\Delta}(s)\bigr)^{\Delta} - r_1(T_1)\bigl(r_2(T_1)z^{\Delta}(T_1)\bigr)^{\Delta} = -\int_{T_1}^{s} f\bigl(u, x\bigl(h(u)\bigr)\bigr)\Delta u.$$

Letting $s \to \infty$, we have

$$\int_{T_1}^{\infty} f\bigl(u, x\bigl(h(u)\bigr)\bigr) \Delta u < \infty.$$

In view of (C4), it follows that

$$f\left(u, \frac{b}{2} R_2\bigl(h(u)\bigr)\right) \le f\bigl(u, x\bigl(h(u)\bigr)\bigr), \quad u \in [T_1, \infty)_{\mathbb{T}},$$

and

$$\int_{T_1}^{\infty} f\left(u, \frac{b}{2} R_2\bigl(h(u)\bigr)\right) \Delta u \le \int_{T_1}^{\infty} f\bigl(u, x\bigl(h(u)\bigr)\bigr) \Delta u < \infty,$$

which means that (11) holds. The necessary condition is proved.

Conversely, suppose that there exists some constant $K > 0$ such that (11) holds. There will be two cases to be considered: $0 \le p_0 < 1$ and $-1 < p_0 < 0$.

Case 1: $0 \le p_0 < 1$. Take p_1 such that $p_0 < p_1 < (1 + 4p_0)/5 < 1$, then $p_0 > (5p_1 - 1)/4$.

When $p_0 > 0$, since $\lim_{t\to\infty} p(t) = p_0$ and (11) hold, we can choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that $p(t) > 0$ for $t \in [T_0, \infty)_{\mathbb{T}}$, and

$$\frac{5p_1 - 1}{4} \le p(t) \le p_1 < 1, \qquad p(t)\frac{R_2(g(t))}{R_2(t)} \ge \frac{5p_1 - 1}{4}\eta_2, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{13}$$

$$\int_{T_0}^{\infty} f\bigl(t, KR_2\bigl(h(t)\bigr)\bigr) \Delta t \le \frac{(1 - p_1\eta_2)K}{8}. \tag{14}$$

When $p_0 = 0$, we can choose $0 < p_1 \le 1/13$ and the above T_0 such that

$$\bigl|p(t)\bigr| \le p_1, \quad t \in [T_0, \infty)_{\mathbb{T}}. \tag{15}$$

Furthermore, from (C3) there exists $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define the Banach space $BC_2\,[T_0, \infty)_{\mathbb{T}}$ as in (3) with $\lambda = 2$, and let

$$\Omega_1 = \left\{x(t) \in BC_2\,[T_0, \infty)_{\mathbb{T}} : \frac{K}{2} R_2(t) \le x(t) \le KR_2(t)\right\}. \tag{16}$$

It is easy to prove that Ω_1 is a bounded, convex, and closed subset of $BC_2\,[T_0, \infty)_{\mathbb{T}}$. By (C4), we have, for any $x \in \Omega_1$,

$$f\bigl(t, x\bigl(h(t)\bigr)\bigr) \le f\bigl(t, KR_2\bigl(h(t)\bigr)\bigr), \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

Now we define two operators U_1 and S_1: $\Omega_1 \to BC_2\,[T_0, \infty)_{\mathbb{T}}$ as follows

$$(U_1x)(t) = \begin{cases} \frac{3}{4}Kp_1\eta_2 R_2(t) - \frac{p(T_1)x(g(T_1))}{R_2(T_1)} R_2(t), & t \in [T_0, T_1)_{\mathbb{T}}, \\ \frac{3}{4}Kp_1\eta_2 R_2(t) - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

$$(S_1x)(t) = \begin{cases} \frac{3}{4}KR_2(t), & t \in [T_0, T_1)_{\mathbb{T}}, \\ \frac{3}{4}KR_2(t) + \int_{T_1}^{t}\int_{T_1}^{v}\int_{s}^{\infty} \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases} \tag{17}$$

Next, we will prove that U_1 and S_1 satisfy the conditions in Lemma 2.2.

(i) We prove that $U_1x + S_1y \in \Omega_1$ for any $x, y \in \Omega_1$. Note that, for any $x, y \in \Omega_1$, $\frac{K}{2}R_2(t) \le x(t) \le KR_2(t)$ and $\frac{K}{2}R_2(t) \le y(t) \le KR_2(t)$. For any $x, y \in \Omega_1$ and $t \in [T_1, \infty)_{\mathbb{T}}$, by (13), (14), and (16) we obtain

$$\begin{aligned}
&(U_1x)(t) + (S_1y)(t) \\
&\quad = \frac{3(1+p_1\eta_2)}{4}KR_2(t) - p(t)x\bigl(g(t)\bigr) + \int_{T_1}^{t}\int_{T_1}^{v}\int_{s}^{\infty}\frac{f(u, y(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v \\
&\quad \ge \frac{3(1+p_1\eta_2)}{4}KR_2(t) - p_1\eta_2KR_2(t) = \frac{3-p_1\eta_2}{4}KR_2(t) > \frac{K}{2}R_2(t).
\end{aligned}$$

On the other hand, for $t \in [T_1, \infty)_{\mathbb{T}}$ and $p(t) \ge 0$, we have

$$\begin{aligned}
&(U_1x)(t) + (S_1y)(t) \\
&\quad \le \frac{3(1+p_1\eta_2)}{4}KR_2(t) - \frac{K}{2}p(t)R_2\bigl(g(t)\bigr) + \frac{1-p_1\eta_2}{8}KR_2(t) \\
&\quad \le \frac{3(1+p_1\eta_2)}{4}KR_2(t) - \frac{K}{2}\frac{5p_1-1}{4}\eta_2R_2(t) + \frac{1-p_1\eta_2}{8}KR_2(t) \\
&\quad = \frac{7+\eta_2}{8}KR_2(t) \le KR_2(t).
\end{aligned}$$

For $t \in [T_1, \infty)_{\mathbb{T}}$, $p(t) < 0$, and $p_0 = 0$, we have $0 < p_1 \le 1/13$ and (15), and

$$\begin{aligned}
&(U_1x)(t) + (S_1y)(t) \\
&\quad \le \frac{3(1+p_1\eta_2)}{4}KR_2(t) - Kp(t)R_2\bigl(g(t)\bigr) + \frac{1-p_1\eta_2}{8}KR_2(t) \\
&\quad \le \frac{3(1+p_1\eta_2)}{4}KR_2(t) + p_1KR_2(t) + \frac{1-p_1\eta_2}{8}KR_2(t) \\
&\quad = \frac{7+8p_1+5p_1\eta_2}{8}KR_2(t) \le \frac{7+13p_1}{8}KR_2(t) \le KR_2(t).
\end{aligned}$$

Similarly, we can prove that $(U_1x)(t) + (S_1y)(t) \ge KR_2(t)/2$ for any $x, y \in \Omega_1$ and $t \in [T_0, T_1]_{\mathbb{T}}$. Then we prove that $(U_1x)(t) + (S_1y)(t) \le KR_2(t)$ for any $x, y \in \Omega_1$ and $t \in [T_0, T_1]_{\mathbb{T}}$. In fact, for $t \in [T_0, T_1]_{\mathbb{T}}$ and $p(t) \ge 0$, we have

$$\begin{aligned}
&(U_1x)(t) + (S_1y)(t) \\
&\quad = \frac{3(1+p_1\eta_2)}{4}KR_2(t) - \frac{p(T_1)x(g(T_1))}{R_2(T_1)}R_2(t) \\
&\quad \le \frac{3(1+p_1\eta_2)}{4}KR_2(t) - \frac{K}{2}\frac{5p_1-1}{4}\eta_2R_2(t) = \frac{6+p_1\eta_2+\eta_2}{8}KR_2(t) < KR_2(t).
\end{aligned}$$

For $t \in [T_0, T_1]_{\mathbb{T}}$, $p(t) < 0$, and $p_0 = 0$, we have $0 < p_1 \le 1/13$ and (15), and

$$\begin{aligned}
(U_1x)(t) + (S_1y)(t) &\le \frac{3(1+p_1\eta_2)}{4}KR_2(t) + p_1KR_2(t) \\
&= \frac{3+3p_1\eta_2+4p_1}{4}KR_2(t) < KR_2(t).
\end{aligned}$$

Therefore, we obtain $U_1x + S_1y \in \Omega_1$ for any $x, y \in \Omega_1$.

(ii) We prove that U_1 is a contraction mapping. In fact, noting that $g(t) \leq t$ and $R_2(t) \geq 1$ for $t \in [T_0, \infty)_{\mathbb{T}}$, for $x, y \in \Omega_1$ we have

$$\begin{aligned} \left| \frac{(U_1 x)(t)}{R_2^2(t)} - \frac{(U_1 y)(t)}{R_2^2(t)} \right| &= \left| p(T_1) \frac{R_2^2(g(T_1))}{R_2(t) R_2(T_1)} \frac{x(g(T_1)) - y(g(T_1))}{R_2^2(g(T_1))} \right| \\ &\leq p_1 \sup_{t \in [T_0, \infty)_{\mathbb{T}}} \left| \frac{x(t)}{R_2^2(t)} - \frac{y(t)}{R_2^2(t)} \right| \end{aligned}$$

for $t \in [T_0, T_1]_{\mathbb{T}}$, and

$$\begin{aligned} \left| \frac{(U_1 x)(t)}{R_2^2(t)} - \frac{(U_1 y)(t)}{R_2^2(t)} \right| &= \left| p(t) \frac{R_2^2(g(t))}{R_2^2(t)} \frac{x(g(t)) - y(g(t))}{R_2^2(g(t))} \right| \\ &\leq p_1 \sup_{t \in [T_0, \infty)_{\mathbb{T}}} \left| \frac{x(t)}{R_2^2(t)} - \frac{y(t)}{R_2^2(t)} \right| \end{aligned}$$

for $t \in [T_1, \infty)_{\mathbb{T}}$. It follows that

$$\|U_1 x - U_1 y\|_2 \leq p_1 \|x - y\|_2$$

for any $x, y \in \Omega_1$. Therefore, U_1 is a contraction mapping.

(iii) We prove that S_1 is a completely continuous mapping.

Firstly, for $t \in [T_0, \infty)_{\mathbb{T}}$, we have

$$(S_1 x)(t) \geq \frac{3}{4} K R_2(t) > \frac{K}{2} R_2(t)$$

and

$$(S_1 x)(t) \leq \frac{3}{4} K R_2(t) + \frac{1 - p_1 \eta_2}{8} K R_2(t) = \frac{7 - p_1 \eta_2}{8} K R_2(t) < K R_2(t).$$

That is, S_1 maps Ω_1 into Ω_1.

Secondly, we prove the continuity of S_1. For $x \in \Omega_1$ and $t \in [T_0, \infty)_{\mathbb{T}}$, letting $x_n \in \Omega_1$ and $\|x_n - x\|_2 \to 0$ as $n \to \infty$, we have

$$\left| f\big(t, x_n\big(h(t)\big)\big) - f\big(t, x\big(h(t)\big)\big) \right| \to 0 \tag{18}$$

and

$$\left| f\big(t, x_n\big(h(t)\big)\big) - f\big(t, x\big(h(t)\big)\big) \right| \leq 2 f\big(t, K R_2\big(h(t)\big)\big)$$

as $n \to \infty$. For $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} &\left| \frac{(S_1 x_n)(t)}{R_2^2(t)} - \frac{(S_1 x)(t)}{R_2^2(t)} \right| \\ &\quad \leq \frac{1}{R_2^2(t)} \int_{T_1}^{t} \int_{T_1}^{v} \int_{s}^{\infty} \frac{|f(u, x_n(h(u))) - f(u, x(h(u)))|}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v \\ &\quad \leq \frac{1}{R_2(t)} \int_{T_1}^{\infty} \left| f\big(u, x_n\big(h(u)\big)\big) - f\big(u, x\big(h(u)\big)\big) \right| \Delta u. \end{aligned}$$

For $t \in [T_0, T_1]_{\mathbb{T}}$, we have $(S_1x_n)(t) - (S_1x)(t) = 0$. Then we obtain

$$\|S_1x_n - S_1x\|_2 \leq \sup_{t\in[t_0,\infty)_{\mathbb{T}}} \frac{1}{R_2(t)} \int_{T_1}^{\infty} \bigl|f\bigl(u, x_n\bigl(h(u)\bigr)\bigr) - f\bigl(u, x\bigl(h(u)\bigr)\bigr)\bigr| \Delta u.$$

Similar to Chen [7], by (18) and employing Lebesgue's dominated convergence theorem [5, Chapter 5], we conclude that

$$\|S_1x_n - S_1x\|_2 \to 0$$

as $n \to \infty$. That is, S_1 is continuous.

Thirdly, we prove that $S_1\Omega_1$ is relatively compact. According to Lemma 2.1, it suffices to show that $S_1\Omega_1$ is bounded, uniformly Cauchy and equi-continuous. It is obvious that $S_1\Omega_1$ is bounded. Since $\int_{t_0}^{\infty} f(t, KR_2(h(t)))\Delta t < \infty$ and $R_2(t) \to \infty$ as $t \to \infty$, for any given $\epsilon > 0$ there exists a sufficiently large $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $R_2(T_2) > 3K/\epsilon$ and $\frac{1}{R_2(T_2)} \int_{T_1}^{\infty} f(t, KR_2(h(t)))\Delta t < \epsilon/4$. Then, for any $x \in \Omega_1$ and $t_1, t_2 \in [T_2, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
&\left|\frac{(S_1x)(t_1)}{R_2^2(t_1)} - \frac{(S_1x)(t_2)}{R_2^2(t_2)}\right| \\
&\quad\leq \left|\frac{1}{R_2^2(t_1)} \int_{T_1}^{t_1} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u\Delta s\Delta v\right. \\
&\quad\quad \left.- \frac{1}{R_2^2(t_2)} \int_{T_1}^{t_2} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u\Delta s\Delta v\right| + \frac{3}{4}K\left|\frac{1}{R_2(t_1)} - \frac{1}{R_2(t_2)}\right| \\
&\quad\leq \frac{1}{R_2^2(t_1)} \int_{T_1}^{t_1} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u\Delta s\Delta v \\
&\quad\quad + \frac{1}{R_2^2(t_2)} \int_{T_1}^{t_2} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u\Delta s\Delta v + \frac{3}{4}K\left(\frac{1}{R_2(t_1)} + \frac{1}{R_2(t_2)}\right) \\
&\quad\leq \frac{1}{R_2(T_2)} \int_{T_1}^{\infty} f\bigl(u, x\bigl(h(u)\bigr)\bigr)\Delta u + \frac{1}{R_2(T_2)} \int_{T_1}^{\infty} f\bigl(u, x\bigl(h(u)\bigr)\bigr)\Delta u + \frac{3K}{2R_2(T_2)} \\
&\quad< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, $S_1\Omega_1$ is uniformly Cauchy.

Then, for $x \in \Omega_1$, if $t_1, t_2 \in \mathbb{T}$ with $T_1 \leq t_1 < t_2 < T_2 + 1$, we have

$$\begin{aligned}
&\left|\frac{(S_1x)(t_1)}{R_2^2(t_1)} - \frac{(S_1x)(t_2)}{R_2^2(t_2)}\right| \\
&\quad\leq \left|\frac{1}{R_2^2(t_1)} \int_{T_1}^{t_1} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u\Delta s\Delta v\right. \\
&\quad\quad \left.- \frac{1}{R_2^2(t_2)} \int_{T_1}^{t_2} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u\Delta s\Delta v\right| + \frac{3}{4}K\left|\frac{1}{R_2(t_1)} - \frac{1}{R_2(t_2)}\right| \\
&\quad\leq \frac{1}{R_2^2(T_1)} \int_{t_1}^{t_2} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, KR_2(h(u)))}{r_1(s)r_2(v)} \Delta u\Delta s\Delta v + \frac{3}{4}K\left|\frac{1}{R_2(t_1)} - \frac{1}{R_2(t_2)}\right|.
\end{aligned}$$

If $t_1, t_2 \in \mathbb{T}$ with $t_1 < T_1 \leq t_2 < T_2 + 1$, we have

$$\begin{aligned}
&\left|\frac{(S_1x)(t_1)}{R_2^2(t_1)} - \frac{(S_1x)(t_2)}{R_2^2(t_2)}\right| \\
&\quad\leq \frac{1}{R_2^2(t_2)} \int_{T_1}^{t_2} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v + \frac{3}{4}K\left|\frac{1}{R_2(t_1)} - \frac{1}{R_2(t_2)}\right| \\
&\quad\leq \frac{1}{R_2^2(T_1)} \int_{T_1}^{t_2} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, KR_2(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v + \frac{3}{4}K\left|\frac{1}{R_2(t_1)} - \frac{1}{R_2(t_2)}\right|.
\end{aligned}$$

If $t_1, t_2 \in [T_0, T_1]_{\mathbb{T}}$, we always have

$$\left|\frac{(S_1x)(t_1)}{R_2^2(t_1)} - \frac{(S_1x)(t_2)}{R_2^2(t_2)}\right| = \frac{3}{4}K\left|\frac{1}{R_2(t_1)} - \frac{1}{R_2(t_2)}\right|.$$

Therefore, there exists $0 < \delta < 1$ such that

$$\left|\frac{(S_1x)(t_1)}{R_2^2(t_1)} - \frac{(S_1x)(t_2)}{R_2^2(t_2)}\right| < \epsilon$$

whenever $t_1, t_2 \in [T_0, T_2 + 1]_{\mathbb{T}}$ and $|t_2 - t_1| < \delta$. That is, $S_1\Omega_1$ is equi-continuous.

It follows from Lemma 2.1 that $S_1\Omega_1$ is relatively compact, and then S_1 is completely continuous.

By Lemma 2.2, there exists $x \in \Omega_1$ such that $(U_1 + S_1)x = x$, which implies that $x(t)$ is a solution of (1). In particular, for $t \in [T_1, \infty)_{\mathbb{T}}$ we have

$$x(t) = \frac{3(1 + p_1\eta_2)K}{4} R_2(t) - p(t)x\bigl(g(t)\bigr) + \int_{T_1}^{t} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

Since

$$\int_{T_1}^{t} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \leq \int_{T_1}^{t} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, KR_2(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v$$

for $t \in [T_1, \infty)_{\mathbb{T}}$ and

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{R_2(t)} \int_{T_1}^{t} \int_{T_1}^{v} \int_{s}^{\infty} \frac{f(u, KR_2(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \\
&\quad= \lim_{t\to\infty} \int_{t}^{\infty} f\bigl(u, KR_2\bigl(h(u)\bigr)\bigr) \Delta u = 0,
\end{aligned}$$

we have

$$\lim_{t\to\infty} \frac{z(t)}{R_2(t)} = \frac{3(1 + p_1\eta_2)K}{4} \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{R_2(t)} = \frac{3(1 + p_1\eta_2)K}{4(1 + p_0\eta_2)} > 0.$$

It is obvious that

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{R_1(t)} = \infty.$$

The sufficiency holds when $0 \leq p_0 < 1$.

Case 2: $-1 < p_0 < 0$. Take p_1 so that $-p_0 < p_1 < (1-4p_0)/5 < 1$, then $p_0 < (1-5p_1)/4$. Since $\lim_{t\to\infty} p(t) = p_0$ and (11) hold, we can choose a sufficiently large $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$\frac{5p_1-1}{4} \le -p(t) \le p_1 < 1, \quad t \in [T_0,\infty)_{\mathbb{T}}. \tag{19}$$

From (C3) there exists $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Similarly, we introduce the Banach space $BC_2\ [T_0,\infty)_{\mathbb{T}}$ and its subset Ω_1 as in (16). Define the operator S_1 as in (17) and the operator U_1' on Ω_1 as follows:

$$\bigl(U_1'x\bigr)(t) = \begin{cases} -\frac{3}{4}Kp_1\eta_2R_2(t) - \frac{p(T_1)x(g(T_1))}{R_2(T_1)}R_2(t), & t \in [T_0,T_1)_{\mathbb{T}}, \\ -\frac{3}{4}Kp_1\eta_2R_2(t) - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

Next, we prove that $U_1'x + S_1y \in \Omega_1$ for any $x,y \in \Omega_1$. In fact, for any $x,y \in \Omega_1$ and $t \in [T_1,\infty)_{\mathbb{T}}$, by (14) and (19) we obtain

$$\begin{aligned} &\bigl(U_1'x\bigr)(t) + (S_1y)(t) \\ &\quad= \frac{3(1-p_1\eta_2)}{4}KR_2(t) - p(t)x\bigl(g(t)\bigr) + \int_{T_1}^t \int_{T_1}^v \int_s^\infty \frac{f(u,y(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v \\ &\quad\ge \frac{3(1-p_1\eta_2)}{4}KR_2(t) + \frac{K}{2}\frac{5p_1-1}{4}\eta_2R_2(t) \\ &\quad= \frac{6-p_1\eta_2-\eta_2}{8}KR_2(t) > \frac{K}{2}R_2(t) \end{aligned}$$

and

$$\begin{aligned} \bigl(U_1'x\bigr)(t) + (S_1y)(t) &\le \frac{3(1-p_1\eta_2)}{4}KR_2(t) + p_1\eta_2KR_2(t) + \frac{1-p_1\eta_2}{8}KR_2(t) \\ &= \frac{7+p_1\eta_2}{8}KR_2(t) < KR_2(t). \end{aligned}$$

That is, $U_1'x + S_1y \in \Omega_1$ for any $x,y \in \Omega_1$.

The remainder of the proof is similar to the case $0 \le p_0 < 1$ and we omit it here. By Lemma 2.2, there exists $x \in \Omega_1$ such that $(U_1' + S_1)x = x$, which implies that $x(t)$ is a solution of (1). In particular, for $t \in [T_1,\infty)_{\mathbb{T}}$ we have

$$x(t) = \frac{3(1-p_1\eta_2)K}{4}R_2(t) - p(t)x\bigl(g(t)\bigr) + \int_{T_1}^t \int_{T_1}^v \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v.$$

Letting $t \to \infty$, we have

$$\lim_{t\to\infty}\frac{z(t)}{R_2(t)} = \frac{3(1-p_1\eta_2)K}{4} \quad \text{and} \quad \lim_{t\to\infty}\frac{x(t)}{R_2(t)} = \frac{3(1-p_1\eta_2)K}{4(1+p_0\eta_2)} > 0.$$

It is obvious that

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty}\frac{x(t)}{R_1(t)} = \infty.$$

The sufficiency holds when $-1 < p_0 < 0$.

The proof is complete. □

Theorem 3.2 *Equation* (1) *has an eventually positive solution in* $A(\infty, b, 0)$ *for some positive b if and only if there exists some constant* $K > 0$ *such that*

$$\int_{t_0}^{\infty}\int_{s}^{\infty}\frac{f(u, KR_1(h(u)))}{r_1(s)}\Delta u\Delta s < \infty. \tag{20}$$

Proof Suppose that $x(t)$ is an eventually positive solution of (1) in $A(\infty, b, 0)$, *i.e.*,

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty}\frac{x(t)}{R_1(t)} = b, \qquad \lim_{t\to\infty}\frac{x(t)}{R_2(t)} = 0.$$

Similarly, we have

$$\begin{aligned}
&\lim_{t\to\infty} z(t) = \infty, \\
&\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = \lim_{t\to\infty}\frac{z(t)}{R_1(t)} = (1 + p_0\eta_1)b, \\
&\lim_{t\to\infty} r_1(t)\big(r_2(t)z^{\Delta}(t)\big)^{\Delta} = \lim_{t\to\infty}\frac{z(t)}{R_2(t)} = 0
\end{aligned}$$

and there exists $T_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) \ge bR_1(t)/2$, $x(g(t)) \ge bR_1(g(t))/2$, $x(h(t)) \ge bR_1(h(t))/2$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Integrating (1) from $s \in [T_1, \infty)_{\mathbb{T}}$ to $v \in [\sigma(s), \infty)_{\mathbb{T}}$, we obtain

$$r_1(v)\big(r_2(v)z^{\Delta}(v)\big)^{\Delta} - r_1(s)\big(r_2(s)z^{\Delta}(s)\big)^{\Delta} = -\int_{s}^{v} f\big(u, x\big(h(u)\big)\big)\Delta u.$$

Letting $v \to \infty$, we have

$$r_1(s)\big(r_2(s)z^{\Delta}(s)\big)^{\Delta} = \int_{s}^{\infty} f\big(u, x\big(h(u)\big)\big)\Delta u,$$

or

$$\big(r_2(s)z^{\Delta}(s)\big)^{\Delta} = \frac{\int_{s}^{\infty} f(u, x(h(u)))\Delta u}{r_1(s)}. \tag{21}$$

Integrating (21) from T_1 to $t \in [\sigma(T_1), \infty)_{\mathbb{T}}$, we have

$$r_2(t)z^{\Delta}(t) - r_2(T_1)z^{\Delta}(T_1) = \int_{T_1}^{t}\int_{s}^{\infty}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s.$$

Letting $t \to \infty$, we have

$$\int_{T_1}^{\infty}\int_{s}^{\infty}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s < \infty.$$

In view of (C4), it follows that

$$f\left(u, \frac{b}{2}R_1\big(h(u)\big)\right) \le f\big(u, x\big(h(u)\big)\big), \quad u \in [T_1, \infty)_{\mathbb{T}},$$

and

$$\int_{T_1}^{\infty}\int_{s}^{\infty}\frac{f(u,bR_1(h(u))/2)}{r_1(s)}\Delta u\Delta s\leq\int_{T_1}^{\infty}\int_{s}^{\infty}\frac{f(u,x(h(u)))}{r_1(s)}\Delta u\Delta s<\infty,$$

which means that (20) holds. The necessary condition is proved.

Conversely, suppose that there exists some constant $K>0$ such that (20) holds. There will be two cases to be considered: $0\leq p_0<1$ and $-1<p_0<0$.

Case 1: $0\leq p_0<1$. Take p_1 such that $p_0<p_1<(1+4p_0)/5<1$, then $p_0>(5p_1-1)/4$.

When $p_0>0$, since $\lim_{t\to\infty}p(t)=p_0$ and (20) hold, we can choose a sufficiently large $T_0\in[t_0,\infty)_{\mathbb{T}}$ such that $p(t)>0$ for $t\in[T_0,\infty)_{\mathbb{T}}$, and

$$\begin{aligned}&\frac{5p_1-1}{4}\leq p(t)\leq p_1<1,\qquad p(t)\frac{R_1(g(t))}{R_1(t)}\geq\frac{5p_1-1}{4}\eta_1,\quad t\in[T_0,\infty)_{\mathbb{T}},\\&\int_{T_0}^{\infty}\int_{s}^{\infty}\frac{f(u,KR_1(h(u)))}{r_1(s)}\Delta u\Delta s\leq\frac{(1-p_1\eta_1)K}{8}.\end{aligned}$$

When $p_0=0$, we can choose $0<p_1\leq 1/13$ and the above T_0 such that

$$\bigl|p(t)\bigr|\leq p_1,\quad t\in[T_0,\infty)_{\mathbb{T}}.$$

Furthermore, from (C3) there exists $T_1\in(T_0,\infty)_{\mathbb{T}}$ such that $g(t)\geq T_0$ and $h(t)\geq T_0$ for $t\in[T_1,\infty)_{\mathbb{T}}$.

Define the Banach space $BC_1\,[T_0,\infty)_{\mathbb{T}}$ as in (3) with $\lambda=1$, and let

$$\Omega_2=\left\{x(t)\in BC_1\,[T_0,\infty)_{\mathbb{T}}:\frac{K}{2}R_1(t)\leq x(t)\leq KR_1(t)\right\}. \tag{22}$$

It is easy to prove that Ω_1 is a bounded, convex, and closed subset of $BC_1\,[T_0,\infty)_{\mathbb{T}}$. By (C4), we have, for any $x\in\Omega_2$,

$$f\bigl(t,x\bigl(h(t)\bigr)\bigr)\leq f\bigl(t,KR_1\bigl(h(t)\bigr)\bigr),\quad t\in[T_1,\infty)_{\mathbb{T}}.$$

Now we define two operators U_2 and $S_2:\Omega_2\to BC_1\,[T_0,\infty)_{\mathbb{T}}$ as follows:

$$\begin{aligned}&(U_2x)(t)=\begin{cases}\frac{3}{4}Kp_1\eta_1R_1(t)-\frac{p(T_1)x(g(T_1))}{R_1(T_1)}R_1(t), & t\in[T_0,T_1)_{\mathbb{T}},\\ \frac{3}{4}Kp_1\eta_1R_1(t)-p(t)x(g(t)), & t\in[T_1,\infty)_{\mathbb{T}},\end{cases}\\&(S_2x)(t)=\begin{cases}\frac{3}{4}KR_1(t), & t\in[T_0,T_1)_{\mathbb{T}},\\ \frac{3}{4}KR_1(t)+\int_t^{\infty}\int_v^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v, & t\in[T_1,\infty)_{\mathbb{T}}.\end{cases}\end{aligned} \tag{23}$$

Next, we can prove that U_2 and S_2 satisfy the conditions in Lemma 2.2. The proof is similar to the case $0\leq p_0<1$ of Theorem 3.1 and omitted here.

By Lemma 2.2, there exists $x\in\Omega_2$ such that $(U_2+S_2)x=x$, which implies that $x(t)$ is a solution of (1). In particular, for $t\in[T_1,\infty)_{\mathbb{T}}$ we have

$$x(t)=\frac{3(1+p_1\eta_1)K}{4}R_1(t)-p(t)x\bigl(g(t)\bigr)+\int_t^{\infty}\int_v^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v.$$

Since

$$\int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v \le \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,KR_1(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v$$

for $t \in [T_1,\infty)_{\mathbb{T}}$ and

$$\begin{aligned}&\lim_{t\to\infty} \frac{1}{R_1(t)} \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,KR_1(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\\ &\quad = -\lim_{t\to\infty} \int_t^\infty \int_s^\infty \frac{f(u,KR_1(h(u)))}{r_1(s)}\Delta u\Delta s = 0,\end{aligned}$$

we have

$$\lim_{t\to\infty} \frac{z(t)}{R_1(t)} = \frac{3(1+p_1\eta_1)K}{4} \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{R_1(t)} = \frac{3(1+p_1\eta_1)K}{4(1+p_0\eta_1)} > 0,$$

which implies that

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{R_2(t)} = 0.$$

The sufficiency holds when $0 \le p_0 < 1$.

Case 2: $-1 < p_0 < 0$. We introduce the Banach space $BC_1\ [T_0,\infty)_{\mathbb{T}}$ and its subset Ω_2 as in (22). Define the operator S_2 as in (23) and the operator U_2' on Ω_2 as follows:

$$\big(U_2'x\big)(t) = \begin{cases} -\frac{3}{4}Kp_1\eta_1R_1(t) - \frac{p(T_1)x(g(T_1))}{R_1(T_1)}R_1(t), & t \in [T_0,T_1)_{\mathbb{T}}, \\ -\frac{3}{4}Kp_1\eta_1R_1(t) - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

The following proof is similar to the case $-1 < p_0 < 0$ in Theorem 3.1 and we omit it here. By Lemma 2.2, there exists $x \in \Omega_2$ such that $(U_2' + S_2)x = x$, which implies that $x(t)$ is a solution of (1). In particular, for $t \in [T_1,\infty)_{\mathbb{T}}$ we have

$$x(t) = \frac{3(1-p_1\eta_1)K}{4}R_1(t) - p(t)x\big(g(t)\big) + \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v.$$

Similarly, we have

$$\lim_{t\to\infty} \frac{z(t)}{R_1(t)} = \frac{3(1-p_1\eta_1)K}{4} \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{R_1(t)} = \frac{3(1-p_1\eta_1)K}{4(1+p_0\eta_1)} > 0,$$

which implies that

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{R_2(t)} = 0.$$

The sufficiency holds when $-1 < p_0 < 0$.

The proof is complete. □

Theorem 3.3 *Equation* (1) *has an eventually positive solution in* $A(b,0,0)$ *for some positive* b *if and only if there exists some constant* $K>0$ *such that*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{f(u,K)}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v<\infty. \tag{24}$$

Proof Suppose that $x(t)$ is an eventually positive solution of (1) in $A(b,0,0)$, *i.e.*,

$$\lim_{t\to\infty}x(t)=b,\qquad \lim_{t\to\infty}\frac{x(t)}{R_1(t)}=0,\qquad \lim_{t\to\infty}\frac{x(t)}{R_2(t)}=0.$$

Then

$$\lim_{t\to\infty}z(t)=(1+p_0)b,$$

$$\lim_{t\to\infty}r_2(t)z^{\Delta}(t)=\lim_{t\to\infty}\frac{z(t)}{R_1(t)}=0,$$

$$\lim_{t\to\infty}r_1(t)\bigl(r_2(t)z^{\Delta}(t)\bigr)^{\Delta}=\lim_{t\to\infty}\frac{z(t)}{R_2(t)}=0,$$

and there exists $T_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)\ge b/2$, $x(g(t))\ge b/2$, $x(h(t))\ge b/2$ for $t\in[T_1,\infty)_{\mathbb{T}}$. Integrating (1) from $s\in[T_1,\infty)_{\mathbb{T}}$ to $v\in[\sigma(s),\infty)_{\mathbb{T}}$, we obtain

$$r_1(v)\bigl(r_2(v)z^{\Delta}(v)\bigr)^{\Delta}-r_1(s)\bigl(r_2(s)z^{\Delta}(s)\bigr)^{\Delta}=-\int_s^v f\bigl(u,x\bigl(h(u)\bigr)\bigr)\Delta u.$$

Letting $v\to\infty$, we have

$$r_1(s)\bigl(r_2(s)z^{\Delta}(s)\bigr)^{\Delta}=\int_s^{\infty}f\bigl(u,x\bigl(h(u)\bigr)\bigr)\Delta u,$$

or

$$\bigl(r_2(s)z^{\Delta}(s)\bigr)^{\Delta}=\frac{\int_s^{\infty}f(u,x(h(u)))\Delta u}{r_1(s)}. \tag{25}$$

Integrating (25) from $v\in[T_1,\infty)_{\mathbb{T}}$ to $w\in[\sigma(v),\infty)_{\mathbb{T}}$, we have

$$r_2(w)z^{\Delta}(w)-r_2(v)z^{\Delta}(v)=\int_v^w\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)}\Delta u\Delta s.$$

Letting $w\to\infty$, we have

$$r_2(v)z^{\Delta}(v)=-\int_v^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)}\Delta u\Delta s,$$

or

$$z^{\Delta}(v)=-\int_v^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s. \tag{26}$$

Integrating (26) from T_1 to $t\in[\sigma(T_1),\infty)_{\mathbb{T}}$, we have

$$z(t)-z(T_1)=-\int_{T_1}^t\int_v^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v.$$

Letting $t \to \infty$, we have

$$\int_{T_1}^{\infty} \int_{v}^{\infty} \int_{s}^{\infty} \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v < \infty.$$

In view of (C4), it follows that

$$f\left(u, \frac{b}{2}\right) \le f\big(u, x\big(h(u)\big)\big), \quad u \in [T_1, \infty)_{\mathbb{T}},$$

and

$$\int_{T_1}^{\infty} \int_{v}^{\infty} \int_{s}^{\infty} \frac{f(u,b/2)}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le \int_{T_1}^{\infty} \int_{v}^{\infty} \int_{s}^{\infty} \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v < \infty,$$

which means that (24) holds. The necessary condition is proved.

Conversely, suppose that there exists some constant $K > 0$ such that (24) holds. There will be two cases to be considered: $0 \le p_0 < 1$ and $-1 < p_0 < 0$.

Case 1: $0 \le p_0 < 1$. Take p_1 such that $p_0 < p_1 < (1 + 4p_0)/5 < 1$, then $p_0 > (5p_1 - 1)/4$.

When $p_0 > 0$, since $\lim_{t\to\infty} p(t) = p_0$ and (24) hold, we can choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that $p(t) > 0$ for $t \in [T_0, \infty)_{\mathbb{T}}$, and

$$\frac{5p_1 - 1}{4} \le p(t) \le p_1 < 1, \quad t \in [T_0, \infty)_{\mathbb{T}},$$
$$\int_{T_0}^{\infty} \int_{v}^{\infty} \int_{s}^{\infty} \frac{f(u,K)}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le \frac{(1-p_1)K}{8}.$$

When $p_0 = 0$, we can choose $0 < p_1 \le 1/13$ and the above T_0 such that

$$\big|p(t)\big| \le p_1, \quad t \in [T_0, \infty)_{\mathbb{T}}.$$

Furthermore, from (C3) there exists $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define the Banach space $BC_0\ [T_0, \infty)_{\mathbb{T}}$ as in (3) with $\lambda = 0$, and let

$$\Omega_3 = \left\{x(t) \in BC_0\ [T_0, \infty)_{\mathbb{T}} : \frac{K}{2} \le x(t) \le K\right\}. \tag{27}$$

It is easy to prove that Ω_3 is a bounded, convex, and closed subset of $BC_0\ [T_0, \infty)_{\mathbb{T}}$. By (C4), we have, for any $x \in \Omega_3$,

$$f\big(t, x\big(h(t)\big)\big) \le f(t, K), \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

Now we define two operators U_3 and $S_3 : \Omega_3 \to BC_0\ [T_0, \infty)_{\mathbb{T}}$ as follows:

$$(U_3x)(t) = \begin{cases} \frac{3}{4}Kp_1 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \\ (U_3x)(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \end{cases}$$
$$(S_3x)(t) = \begin{cases} \frac{3}{4}K + \int_t^{\infty} \int_v^{\infty} \int_s^{\infty} \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v, & t \in [T_1, \infty)_{\mathbb{T}}, \\ (S_3x)(T_1), & t \in [T_0, T_1)_{\mathbb{T}}. \end{cases} \tag{28}$$

Next, we can prove that U_3 and S_3 satisfy the conditions in Lemma 2.2. The proof is similar to the case $0 \leq p_0 < 1$ of Theorem 3.1 and omitted here.

By Lemma 2.2, there exists $x \in \Omega_3$ such that $(U_3 + S_3)x = x$, which implies that $x(t)$ is a solution of (1). In particular, for $t \in [T_1, \infty)_{\mathbb{T}}$ we have

$$x(t) = \frac{3(1+p_1)K}{4} - p(t)x\bigl(g(t)\bigr) + \int_t^{\infty}\int_v^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v.$$

Letting $t \to \infty$, we have

$$\lim_{t\to\infty} z(t) = \frac{3(1+p_1)K}{4} \quad \text{and} \quad \lim_{t\to\infty} x(t) = \frac{3(1+p_1)K}{4(1+p_0)} > 0,$$

which implies that

$$\lim_{t\to\infty}\frac{x(t)}{R_1(t)} = \lim_{t\to\infty}\frac{x(t)}{R_2(t)} = 0.$$

The sufficiency holds when $0 \leq p_0 < 1$.

Case 2: $-1 < p_0 < 0$. We introduce the Banach space $BC_0\ [T_0, \infty)_{\mathbb{T}}$ and its subset Ω_3 as in (27). Define the operator S_3 as in (28) and the operator U_3' on Ω_3 as follows:

$$\bigl(U_3'x\bigr)(t) = \begin{cases} -\frac{3}{4}Kp_1 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \\ (U_3'x)(T_1), & t \in [T_0, T_1)_{\mathbb{T}}. \end{cases}$$

The following proof is similar to the case $-1 < p_0 < 0$ in Theorem 3.1 and we omit it here. By Lemma 2.2, there exists $x \in \Omega_3$ such that $(U_3' + S_3)x = x$, which implies that $x(t)$ is a solution of (1). In particular, for $t \in [T_1, \infty)_{\mathbb{T}}$ we have

$$x(t) = \frac{3(1-p_1)K}{4} - p(t)x\bigl(g(t)\bigr) + \int_t^{\infty}\int_v^{\infty}\int_s^{\infty}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v.$$

Similarly, we have

$$\lim_{t\to\infty} z(t) = \frac{3(1-p_1)K}{4} \quad \text{and} \quad \lim_{t\to\infty} x(t) = \frac{3(1-p_1)K}{4(1+p_0)} > 0,$$

which implies that

$$\lim_{t\to\infty}\frac{x(t)}{R_1(t)} = \lim_{t\to\infty}\frac{x(t)}{R_2(t)} = 0.$$

The sufficiency holds when $-1 < p_0 < 0$.

The proof is complete. □

Theorem 3.4 *Equation* (1) *has an eventually positive solution in* $A(\infty, \infty, 0)$, *then*

$$\int_{t_0}^{\infty} f\biggl(u, \frac{3}{4}R_1\bigl(h(u)\bigr)\biggr)\Delta u < \infty, \qquad \int_{t_0}^{\infty}\int_s^{\infty}\frac{f(u,R_2(h(u)))}{r_1(s)}\Delta u\Delta s = \infty. \tag{29}$$

Conversely, if there exists a nonnegative constant M such that $|p(t)R_2(t)| \leq M$ and

$$\int_{t_0}^{\infty} f\big(u, R_2\big(h(u)\big)\big)\Delta u < \infty, \qquad \int_{t_0}^{\infty}\int_{s}^{\infty} \frac{f(u,(M+3/4)R_1(h(u)))}{r_1(s)}\Delta u\Delta s = \infty, \tag{30}$$

then (1) *has an eventually positive solution in $A(\infty,\infty,0)$.*

Proof Suppose that $x(t)$ is an eventually positive solution of (1) in $A(\infty,\infty,0)$, *i.e.*,

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty}\frac{x(t)}{R_1(t)} = \infty, \qquad \lim_{t\to\infty}\frac{x(t)}{R_2(t)} = 0.$$

Similarly, we have

$$\begin{aligned}
&\lim_{t\to\infty} z(t) = \infty,\\
&\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = \lim_{t\to\infty}\frac{z(t)}{R_1(t)} = \infty,\\
&\lim_{t\to\infty} r_1(t)\big(r_2(t)z^{\Delta}(t)\big)^{\Delta} = \lim_{t\to\infty}\frac{z(t)}{R_2(t)} = 0,
\end{aligned}$$

and there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that $3R_1(t)/4 \leq x(t) \leq R_2(t)$ for $t \in [T_0,\infty)_{\mathbb{T}}$. From (C3) there exists $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Integrating (1) from T_1 to $s \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we obtain

$$r_1(s)\big(r_2(s)z^{\Delta}(s)\big)^{\Delta} - r_1(T_1)\big(r_2(T_1)z^{\Delta}(T_1)\big)^{\Delta} = -\int_{T_1}^{s} f\big(u, x\big(h(u)\big)\big)\Delta u.$$

Letting $s \to \infty$, we have

$$r_1(T_1)\big(r_2(T_1)z^{\Delta}(T_1)\big)^{\Delta} = \int_{T_1}^{\infty} f\big(u, x\big(h(u)\big)\big)\Delta u, \tag{31}$$

which implies that

$$\int_{T_1}^{\infty} f\bigg(u, \frac{3}{4}R_1\big(h(u)\big)\bigg)\Delta u < \infty$$

by the monotonicity of f and $3R_1(h(t))/4 \leq x(h(t))$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Substituting s for T_1 in (31), we have

$$\big(r_2(s)z^{\Delta}(s)\big)^{\Delta} = \frac{\int_s^{\infty} f(u,x(h(u)))\Delta u}{r_1(s)}. \tag{32}$$

Integrating (32) from T_1 to $t \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we have

$$r_2(t)z^{\Delta}(t) - r_2(T_1)z^{\Delta}(T_1) = \int_{T_1}^{t}\int_{s}^{\infty}\frac{f(u,x(h(u)))}{r_1(s)}\Delta u\Delta s.$$

Letting $t \to \infty$, we have

$$\int_{T_1}^{\infty}\int_{s}^{\infty}\frac{f(u,x(h(u)))}{r_1(s)}\Delta u\Delta s = \infty.$$

By the monotonicity of f and $x(h(t)) \leq R_2(h(t))$ for $t \in [T_1, \infty)_{\mathbb{T}}$, it follows that

$$f\big(u, x\big(h(u)\big)\big) \leq f\big(u, R_2\big(h(u)\big)\big), \quad u \in [T_1, \infty)_{\mathbb{T}},$$

and

$$\int_{T_1}^{\infty} \int_s^{\infty} \frac{f(u, R_2(h(u)))}{r_1(s)} \Delta u \Delta s \geq \int_{T_1}^{\infty} \int_s^{\infty} \frac{f(u, x(h(u)))}{r_1(s)} \Delta u \Delta s = \infty,$$

which means that (29) holds. The necessary condition is proved.

Conversely, if there exists a positive constant M such that $|p(t)R_2(t)| \leq M$ and (30) hold, then $\lim_{t\to\infty} p(t) = 0$ and we can choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\big|p(t)\big| \leq p_1 < 1, \qquad \big|p(t)R_2(t)\big| \leq M, \qquad \left(2M + \frac{3}{2}\right) R_1(t) \leq \frac{1}{4} R_2(t), \quad t \in [T_0, \infty)_{\mathbb{T}},$$
$$\int_{T_0}^{\infty} f\big(u, R_2\big(h(u)\big)\big) \Delta u \leq \frac{1 - p_1}{8}.$$

From (C3) there exists $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define the Banach space $BC_2\ [T_0, \infty)_{\mathbb{T}}$ as in (3) with $\lambda = 2$, and let

$$\Omega_4 = \left\{ x(t) \in BC_2\ [T_0, \infty)_{\mathbb{T}} : \left(M + \frac{3}{4}\right) R_1(t) \leq x(t) \leq R_2(t) \right\}.$$

It is easy to prove that Ω_4 is a bounded, convex, and closed subset of $BC_2\ [T_0, \infty)_{\mathbb{T}}$. According to (C3) and (C4), we have, for any $x \in \Omega_4$,

$$x\big(h(t)\big) \geq \left(M + \frac{3}{4}\right) R_1\big(h(t)\big), f\big(t, x\big(h(t)\big)\big) \leq f\big(t, R_2\big(h(t)\big)\big), \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

Now we define two operators U_4 and $S_4 : \Omega_4 \to BC_2\ [T_0, \infty)_{\mathbb{T}}$ as follows:

$$(U_4 x)(t) = \begin{cases} (M + \frac{3}{4}) R_1(t) - \frac{p(T_1)x(g(T_1))}{R_2(T_1)} R_2(t), & t \in [T_0, T_1)_{\mathbb{T}}, \\ (M + \frac{3}{4}) R_1(t) - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

$$(S_4 x)(t) = \begin{cases} (M + \frac{3}{4}) R_1(t), & t \in [T_0, T_1)_{\mathbb{T}}, \\ (M + \frac{3}{4}) R_1(t) + \int_{T_1}^t \int_{T_1}^v \int_s^{\infty} \frac{f(u, x(h(u)))}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Next, we can prove that U_4 and S_4 satisfy the conditions in Lemma 2.2. The proof is similar to Theorem 3.1 and omitted here. By Lemma 2.2, there exists $x \in \Omega_4$ such that $(U_4 + S_4)x = x$, which implies that $x(t)$ is a solution of (1). In particular, for $t \in [T_1, \infty)_{\mathbb{T}}$ we have

$$x(t) = \left(2M + \frac{3}{2}\right) R_1(t) - p(t) x\big(g(t)\big) + \int_{T_1}^t \int_{T_1}^v \int_s^{\infty} \frac{f(u, x(h(u)))}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v.$$

Since $x(h(t)) \geq (M + 3/4) R_1(h(t))$ and

$$\int_{T_1}^t \int_{T_1}^v \int_s^{\infty} \frac{f(u, x(h(u)))}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v \leq \int_{T_1}^t \int_{T_1}^v \int_s^{\infty} \frac{f(u, R_2(h(u)))}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v$$

for $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$\lim_{t\to\infty} \frac{R_1(t)}{R_2(t)} = \lim_{t\to\infty} \frac{1 + \int_{t_0}^{t} \frac{1}{r_2(s)} \Delta s}{1 + \int_{t_0}^{t}\int_{t_0}^{s} \frac{1}{r_1(u) r_2(s)} \Delta u \Delta s} = \lim_{t\to\infty} \frac{1}{\int_{t_0}^{t} \frac{1}{r_1(u)} \Delta u} = 0,$$

$$\lim_{t\to\infty} \frac{1}{R_2(t)} \int_{T_1}^{t}\int_{T_1}^{v}\int_{s}^{\infty} \frac{f(u, R_2(h(u)))}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v$$
$$= \lim_{t\to\infty} \int_{t}^{\infty} f\big(u, R_2\big(h(u)\big)\big) \Delta u = 0,$$

$$\lim_{t\to\infty} \frac{1}{R_1(t)} \int_{T_1}^{t}\int_{T_1}^{v}\int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v$$
$$= \lim_{t\to\infty} \int_{T_1}^{t}\int_{s}^{\infty} \frac{f(u, x(h(u)))}{r_1(s)} \Delta u \Delta s$$
$$\geq \lim_{t\to\infty} \int_{T_1}^{t}\int_{s}^{\infty} \frac{f(u, (M + 3/4) R_1(h(u)))}{r_1(s)} \Delta u \Delta s = \infty.$$

It follows that

$$\lim_{t\to\infty} z(t) = \infty, \qquad \lim_{t\to\infty} \frac{z(t)}{R_1(t)} = \infty, \qquad \lim_{t\to\infty} \frac{z(t)}{R_2(t)} = 0.$$

Since $|p(t)x(g(t))| \leq |p(t)R_2(t)| \leq M$, by Lemma 2.3 we have

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{R_1(t)} = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{R_2(t)} = 0.$$

The proof is complete. □

When $p(t) \geq 0$ eventually, we have the following theorem.

Theorem 3.5 *If there exist a constant $K > 0$ and $T_0 \in [t_0, \infty)_{\mathbb{T}}$ with $T_0 > 0$ such that, for $t \in [T_0, \infty)_{\mathbb{T}}$,*

$$0 \leq p(t) \leq K g(t) e^{-t}, \tag{33}$$

$$\int_{t}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty} \frac{f(u, e^{-h(u)})}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v \geq (K + 1) e^{-t} \tag{34}$$

and

$$\int_{t}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty} \frac{f(u, 1/h(u))}{r_1(s) r_2(v)} \Delta u \Delta s \Delta v \leq \frac{1}{t}, \tag{35}$$

then (1) *has an eventually positive solution in* $A(0, 0, 0)$.

Proof From (C3) there exists $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Define the Banach space $BC_0\ [T_0, \infty)_{\mathbb{T}}$ as in (3) with $\lambda = 0$, and let

$$\Omega_5 = \big\{x(t) \in BC_0\ [T_0, \infty)_{\mathbb{T}} : x(t) \in \big[e^{-t}, 1/t\big] \text{ for } t \in [T_1, \infty)_{\mathbb{T}} \text{ and}$$
$$x(t) \in \big[e^{-T_1}, 1/t\big] \text{ for } t \in [T_0, T_1]_{\mathbb{T}}\big\}.$$

It is easy to prove that Ω_5 is a bounded, convex, and closed subset of $BC_0\ [T_0,\infty)_{\mathbb{T}}$. Define an operator S_5 on Ω_5 as follows:

$$(S_5x)(t) = \begin{cases} -p(t)x(g(t)) + \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v, & t \in [T_1,\infty)_{\mathbb{T}}, \\ (S_5x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}. \end{cases}$$

We prove that $S_5x \in \Omega_5$ for any $x \in \Omega_5$. In fact, from (33)-(35), for $t \in [T_1,\infty)_{\mathbb{T}}$ we have

$$\begin{aligned}(S_5x)(t) &= -p(t)x\bigl(g(t)\bigr) + \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \\ &\le \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,1/h(u))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le \frac{1}{t}\end{aligned}$$

and

$$\begin{aligned}(S_5x)(t) &\ge -\frac{p(t)}{g(t)} + \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,e^{-h(u)})}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \\ &\ge -Ke^{-t} + (K+1)e^{-t} = e^{-t}.\end{aligned}$$

It follows that $e^{-T_1} \le (S_5x)(t) \le 1/t$ for $t \in [T_0,T_1]_{\mathbb{T}}$. Hence, $S_5x \in \Omega_5$ for any $x \in \Omega_5$. Similarly, we can prove that the operators $U_5 = 0$ and S_5 satisfy all the conditions in Lemma 2.2. The rest of the proof is similar to that of Theorem 3.1 and omitted here. By Lemma 2.2, there exists $x \in \Omega_5$ such that $S_5x = x$, which implies that $x(t)$ is a solution of (1). In particular, for $t \in [T_1,\infty)_{\mathbb{T}}$ we have

$$x(t) = -p(t)x\bigl(g(t)\bigr) + \int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

In view of (C4), for any $x \in \Omega_5$ we have

$$f\bigl(t,x\bigl(h(t)\bigr)\bigr) \le f\bigl(t,1/h(t)\bigr), \quad t \in [T_1,\infty)_{\mathbb{T}}.$$

Letting $t \to \infty$, we obtain

$$\lim_{t\to\infty} z(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} x(t) = 0,$$

which implies that

$$\lim_{t\to\infty} \frac{x(t)}{R_1(t)} = \lim_{t\to\infty} \frac{x(t)}{R_2(t)} = 0.$$

The proof is complete. □

While $p(t)$ is eventually negative, we have another result. The proof is similar to that of Theorem 3.5 and hence we omit it here.

Theorem 3.6 *If there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ with $T_0 > 0$ such that, for $t \in [T_0,\infty)_{\mathbb{T}}$,*

$$p(t)e^{-g(t)} \le -e^{-t}$$

and

$$\int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,1/h(u))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \le \frac{1}{t} + \frac{p(t)}{g(t)},$$

then (1) *has an eventually positive solution in* $A(0,0,0)$.

4 Examples

In this section, the application of our results will be shown in three examples. The first example is given to demonstrate Theorems 3.1-3.4.

Example 4.1 Let $c \ge 1$ and $\mathbb{T} = \bigcup_{n=1}^\infty [(2n-1)c, 2nc]$. Consider the equation

$$\left(t\left(t\left(x(t) - \frac{t+1}{2t}x(t-2c)\right)^\Delta\right)^\Delta\right)^\Delta + \frac{(t+\sigma(t))x(t)}{t^2(\sigma(t))^2(1+t^2)} = 0, \tag{36}$$

where $r_1(t) = r_2(t) = t$, $p(t) = -(t+1)/2t$, $p_0 = -1/2$, $g(t) = t-2c$, $h(t) = t$, $f(t,x) = \frac{(t+\sigma(t))x(t)}{t^2(\sigma(t))^2(1+t^2)}$, $t_0 = c$.

It is obvious that the coefficients of (36) satisfy (C1)-(C4), and by (C3) we have

$$R_1(t) = 1 + \int_c^t \frac{1}{s}\Delta s \le 1 + \frac{1}{c}(t-c) = \frac{t}{c} < 1 + t^2,$$
$$R_2(t) = 1 + \int_c^t \int_c^s \frac{1}{u\cdot s}\Delta u \Delta s \le 1 + \frac{1}{c^2}\int_c^t s\Delta s \le 1 + \frac{t^2-c^2}{2c^2} < 1 + t^2.$$

Therefore,

$$\int_c^\infty f\big(t, R_2\big(h(t)\big)\big)\Delta t = \int_c^\infty \frac{(t+\sigma(t))R_2(t)}{t^2(\sigma(t))^2(1+t^2)}\Delta t < \int_c^\infty \frac{t+\sigma(t)}{t^2(\sigma(t))^2}\Delta t = \frac{1}{c^2} < \infty,$$
$$\int_c^\infty \int_s^\infty \frac{f(u, R_1(h(u)))}{r_1(s)}\Delta u \Delta s = \int_c^\infty \int_s^\infty \frac{(u+\sigma(u))R_1(u)}{u^2(\sigma(u))^2(1+u^2)s}\Delta u \Delta s$$
$$< \int_c^\infty \int_s^\infty \frac{u+\sigma(u)}{u^2(\sigma(u))^2 s}\Delta u \Delta s = \int_c^\infty \frac{1}{s^3}\Delta s < \infty,$$
$$\int_c^\infty \int_v^\infty \int_s^\infty \frac{f(u,1)}{r_1(s)r_2(v)}\Delta u \Delta s \Delta v$$
$$< \int_c^\infty \int_v^\infty \int_s^\infty \frac{u+\sigma(u)}{u^2(\sigma(u))^2 s\cdot v}\Delta u \Delta s \Delta v$$
$$= \int_c^\infty \int_v^\infty \frac{1}{vs^3}\Delta s \Delta v = \int_c^\infty \int_c^{\sigma(s)} \frac{1}{vs^3}\Delta v \Delta s < \frac{1}{c}\int_c^\infty \frac{1}{s^2}\Delta s < \infty,$$
$$\int_c^\infty \int_s^\infty \frac{f(u, R_2(h(u)))}{r_1(s)}\Delta u \Delta s < \int_c^\infty \int_s^\infty \frac{u+\sigma(u)}{u^2(\sigma(u))^2 s}\Delta u \Delta s = \int_c^\infty \frac{1}{s^3}\Delta s < \infty.$$

By Theorems 3.1-3.4, we see that (36) has eventually positive solutions $x_1(t) \in A(\infty, \infty, b)$, $x_2(t) \in A(\infty, b, 0)$, $x_3(t) \in A(b, 0, 0)$, but it has no eventually positive solution in $A(\infty, \infty, 0)$.

Then we give the second example to demonstrate Theorem 3.4.

Example 4.2 For any given time scale $\mathbb{T}$, let $t_0 \geq 1$. Consider the equation

$$\left(\left(\left(\left(1+\frac{1}{t^2}\right)x(t)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}+\frac{1}{t^2}x\left(\sqrt[3]{t}\right)=0, \tag{37}$$

where $r_1(t) = r_2(t) = 1$, $p(t) = 1/t^2$, $p_0 = 0$, $g(t) = t$, $h(t) = \sqrt[3]{t}$, $f(t,x) = x/t^2$.

It is obvious that the coefficients of (37) satisfy (C1)-(C4), and by (C3) we have

$$\begin{aligned}
R_1(t) &= 1+\int_{t_0}^{t}\Delta s = 1+t-t_0 \leq t \leq t^2,\\
R_2(t) &= 1+\int_{t_0}^{t}\int_{t_0}^{s}\Delta u\Delta s = 1+\int_{t_0}^{t}(s-t_0)\Delta s\\
&< 1+\frac{1}{2}\int_{t_0}^{t}\bigl(s+\sigma(s)\bigr)\Delta s = 1+\frac{1}{2}\bigl(t^2-t_0^2\bigr) \leq t^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\bigl|p(t)R_2(t)\bigr| \leq 1,\\
&\int_{t_0}^{\infty} f\bigl(u,R_2\bigl(h(u)\bigr)\bigr)\Delta u \leq \int_{t_0}^{\infty}\frac{u^{2/3}}{u^2}\Delta u = \int_{t_0}^{\infty}\frac{1}{u^{4/3}}\Delta u < \infty,\\
&\int_{t_0}^{\infty}\int_{s}^{\infty}\frac{f(u,(M+3/4)R_1(h(u)))}{r_1(s)}\Delta u\Delta s\\
&\quad > \int_{t_0}^{\infty}\int_{s}^{\infty} f(u,M+3/4)\Delta u\Delta s = \left(M+\frac{3}{4}\right)\int_{t_0}^{\infty}\int_{s}^{\infty}\frac{1}{u^2}\Delta u\Delta s\\
&\quad \geq \left(M+\frac{3}{4}\right)\int_{t_0}^{\infty}\frac{1}{s}\Delta s = \infty.
\end{aligned}$$

It follows that (37) has an eventually positive solution $x(t) \in A(\infty,\infty,0)$ in terms of Theorem 3.4.

The third example illustrates Theorem 3.5.

Example 4.3 Let $\mathbb{T} = [1,\infty)$. Consider the equation

$$\left(e^{-\frac{t}{6}}\left(e^{-\frac{t}{3}}\left(x(t)+(t-1)e^{-t}x(t-1)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}+e^{-t}x\left(\frac{t}{3}\right)=0, \tag{38}$$

where $r_1(t) = e^{-t/6}$, $r_2(t) = e^{-t/3}$, $p(t) = (t-1)e^{-t}$, $p_0 = 0$, $g(t) = t-1$, $h(t) = t/3$, $f(t,x) = e^{-t}x$.

It is obvious that the coefficients of (38) satisfy (C1)-(C4), and we have

$$\begin{aligned}
&\int_{t}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{f(u,e^{-h(u)})}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\\
&\quad = \int_{t}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{e^{-4u/3}}{e^{-s/6}\cdot e^{-v/3}}\,du\,ds\,dv\\
&\quad = \frac{27}{35}e^{-\frac{5}{6}t},
\end{aligned}$$

$$\begin{aligned}
&\int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u,1/h(u))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v \\
&\quad = \int_t^\infty \int_v^\infty \int_s^\infty \frac{3/u \cdot e^{-u}}{e^{-s/6} \cdot e^{-v/3}} \, du \, ds \, dv \\
&\quad \le \int_t^\infty \int_v^\infty \int_s^\infty \frac{3e^{-u}}{e^{-s/6} \cdot e^{-v/3}} \, du \, ds \, dv \\
&\quad = \frac{36}{5} e^{-\frac{1}{2}t}.
\end{aligned}$$

Take $K = 1$, and there exists a sufficiently large $T_0 \in [1, \infty)$ such that, for $t \in [T_0, \infty)$, the conditions (33)-(35) hold. By Theorem 3.5, we see that (38) has an eventually positive solution $x(t) \in A(0, 0, 0)$.

Competing interests
The author declares that he has no competing interests.

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