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On the twisted Daehee polynomials with q-parameter

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Abstract
The nth twisted Daehee numbers with q-parameter are closely related to higher-order Bernoulli numbers and Bernoulli numbers of the second kind. In this paper, we give a p-adic integral representation of the twisted Daehee polynomials with q-parameter, and we derive some interesting properties related to the nth twisted Daehee polynomials with q-parameter.

Keywords: Bernoulli polynomials; Daehee polynomials with q-parameter; p-adic invariant integral

1 Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic invariant integral on $\mathbb{Z}_p$* is defined by Kim as follows:

$$I(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x) \quad \text{(see [1–3])}. \tag{1.1}$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I(f_1) = I(f) + f'(0), \quad \text{where } f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}. \tag{1.2}$$

As is well known, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{1.3}$$

and the *Stirling number of the second kind* is given by the generating function to be

$$\left(e^t-1\right)^m = m!\sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!} \tag{1.4}$$

(see [4–6]).

Unsigned Stirling numbers of the first kind are given by

$$x^{\underline{n}} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n}\left|S_1(n,l)\right|x^l. \tag{1.5}$$

Note that if we replace x to $-x$ in (1.3), then

$$\begin{aligned}(-x)_n = (-1)^n x^{\underline{n}} &= \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l \\ &= (-1)^n\sum_{l=0}^{n}\left|S_1(n,l)\right|x^l. \end{aligned}\tag{1.6}$$

Hence $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$.

For $r \in \mathbb{N}$, the *Bernoulli polynomials of order r* are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1, 4, 7–18])}. \tag{1.7}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the *Bernoulli numbers of order r*, and in the special case, $r = 1$, $B_n^{(1)}(x) = B_n(x)$ are called the *ordinary Bernoulli polynomials*.

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\ge 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \{\omega|\omega^{p^n} = 1\}$ is the cyclic group of order p^n.

We assume that q is an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. Then we define the q-analog of a falling factorial sequence as follows:

$$(x)_{n,q} = x(x-q)(x-2q)\cdots\left(x-(n-1)q\right) \quad (n\ge 1), \qquad (x)_{0,q} = 1.$$

Note that

$$\lim_{q\to 1}(x)_{n,q} = (x)_n = \sum_{l=0}^{n} S_1(n,l)x^l.$$

Recently, DS Kim and T Kim introduced the *Daehee polynomials* as follows:

$$D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n \, d\mu_0(y) \quad (n \geq 0) \text{ (see [2, 9, 19])}. \tag{1.8}$$

When $x = 0$, $D_n = D_n(0)$ are called the *nth Daehee numbers*. From (1.8), we can derive the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} \quad \text{(see [9])}. \tag{1.9}$$

In addition, DS Kim *et al.* consider the *Daehee polynomials with q-parameter*, which are defined by the generating function to be

$$\sum_{n=0}^{\infty} D_{n,q}\frac{t^n}{n!} = (1+qt)^{\frac{x}{q}} \frac{\log(1+qt)}{q((1+qt)^{\frac{1}{q}}-1)} \quad \text{(see [20, 21])}. \tag{1.10}$$

When $x = 0$, $D_{n,q} = D_{n,q}(0)$ are called the *Daehee numbers with q-parameter*.

From the viewpoint of a generalization of the Daehee polynomials with q-parameter, we consider the *twisted Daehee polynomials with q-parameter*, defined to be

$$\sum_{n=0}^{\infty} D_{n,\xi,q}\frac{t^n}{n!} = (1+q\xi t)^{\frac{x}{q}} \frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}}-1)}, \tag{1.11}$$

where $t, q \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$ and $\xi \in T_p$.

In this paper, we give a p-adic integral representation of the twisted Daehee polynomials with q-parameter, which is called the Witt-type formula for the twisted Daehee polynomials with q-parameter. We can derive some interesting properties related to the nth twisted Daehee polynomials with q-parameter.

2 Witt-type formula for the *n*th twisted Daehee polynomials with *q*-parameter

First, we consider the following integral representation associated with falling factorial sequences:

$$\xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q} \, d\mu_0(y), \quad \text{where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \text{ and } \xi \in T_p. \tag{2.1}$$

By (2.1),

$$\begin{aligned}\sum_{n=0}^{\infty} \xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q} \, d\mu_0(y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \left(\frac{x+y}{q}\right)_n d\mu_0(y)\frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x+y}{q}} \, d\mu_0(y), \end{aligned} \tag{2.2}$$

where $t, q \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$. For $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$, put $f(x) = (1 + q\xi t)^{\frac{x+y}{q}}$. By (1.1), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x+y}{q}}\, d\mu_0(y) &= (1+q\xi t)^{\frac{x}{q}} \frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}} - 1)} \\ &= \sum_{n=0}^{\infty} D_{n,\xi,q}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$D_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q}\, d\mu_0(y).$$

In (2.3), by replacing t by $\frac{1}{\xi q}(e^{\xi t} - 1)$, we have

$$\sum_{n=0}^{\infty} D_{n,\xi,q}(x) \frac{1}{\xi^n q^n} \frac{(e^{\xi t} - 1)^n}{n!} = e^{\frac{\xi t x}{q}} \frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}} - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{\xi^n}{q^n} \frac{t^n}{n!} \tag{2.4}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \frac{1}{n!} \left(e^{\xi t} - 1\right)^n &= \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m S_2(m,n) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{D_{n,\xi,q}(x)}{\xi^n q^n} \xi^m S_2(m,n) \frac{t^m}{m!}. \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we obtain the following corollary.

Corollary 2.2 *For $n \geq 0$, we have*

$$B_n(x) = \sum_{m=0}^{n} D_{m,\xi,q}(x) \xi^{-m} q^{n-m} S_2(n,m).$$

By Theorem 2.1,

$$\begin{aligned} D_{n,\xi,q}(x) &= \xi^n \int_{\mathbb{Z}_p} (x+y)_{n,q}\, d\mu_0(y) \\ &= \xi^n q^n \sum_{l=0}^{n} \frac{1}{q^l} S_1(n,l) \int_{\mathbb{Z}_p} (x+y)^l\, d\mu_0(y). \end{aligned} \tag{2.6}$$

By (1.2), we can derive easily that

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{(x+y)t}\, d\mu_0(y) &= \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \\ &= \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^l\, d\mu_0(y) \frac{t^l}{l!}, \end{aligned} \tag{2.7}$$

and so

$$B_n(x) = \int_{\mathbb{Z}_p} (x+y)^n \, d\mu_0(y). \tag{2.8}$$

By (1.6), (2.7), and (2.8), we obtain the following corollary.

Corollary 2.3 *For $n \geq 0$, we have*

$$D_{n,\xi,q}(x) = \xi^n \sum_{l=0}^{n} q^{n-l} S_1(n,l) B_l(x) = \xi^n \sum_{l=0}^{n} \left|S_1(n,l)\right| (-q)^{n-l} B_l(x).$$

From now on, we consider *twisted Daehee polynomials of order $k \in \mathbb{N}$ with q-parameter*. Twisted Daehee polynomials of order $k \in \mathbb{N}$ with q-parameter are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$D^{(k)}_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_{n,q} \, d\mu_0(x_1) \cdots d\mu_0(x_k), \tag{2.9}$$

where n is a nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $D^{(k)}_{n,\xi,q} = D^{(k)}_{n,\xi,q}(0)$ are called the *Daehee numbers of order k with q-parameter*.

From (2.9), we can derive the generating function of $D^{(k)}_{n,\xi,q}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} D^{(k)}_{n,\xi,q}(x) \frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\frac{x_1+\cdots+x_k+x}{q}}{n} d\mu_0(x_1) \cdots d\mu_0(x_k) t^n \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x_1+\cdots+x_k+x}{q}} \, d\mu_0(x_1) \cdots d\mu_0(x_k) \\
&\quad = (1+q\xi t)^{\frac{x}{q}} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{x_1+\cdots+x_k}{q}} \, d\mu_0(x_1) \cdots d\mu_0(x_k) \\
&\quad = (1+q\xi t)^{\frac{x}{q}} \left(\frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}} - 1)} \right)^k.
\end{aligned} \tag{2.10}$$

Note that, by (2.9),

$$D^{(k)}_{n,\xi,q}(x) = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^m \, d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{2.11}$$

Since

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_k+x)t} \, d\mu_0(x_1) \cdots d\mu_0(x_k) = \left(\frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B^{(k)}_n(x) \frac{t^n}{n!},$$

we can derive easily

$$B^{(k)}_n(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^n \, d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{2.12}$$

Thus, by (2.11) and (2.12), we have

$$\begin{aligned} D_{n,\xi,q}^{(k)}(x) &= \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} B_m^{(k)}(x) \\ &= \xi^n \sum_{m=0}^{n} q^{n-m} S_1(n,m) B_m^{(k)}(x) \\ &= \xi^n \sum_{m=0}^{n} \left|S_1(n,m)\right| (-q)^{n-m} B_m^{(k)}(x). \end{aligned} \tag{2.13}$$

In (2.10), by replacing t by $\frac{1}{q\xi}(e^{\xi t}-1)$, we get

$$\sum_{n=0}^{\infty} D_{n,\xi,q}^{(k)}(x) \frac{(e^{\xi t}-1)^n}{\xi^n q^n n!} = e^{\frac{\xi t x}{q}} \left(\frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}}-1} \right)^k = \sum_{n=0}^{\infty} \frac{\xi^n B_n^{(k)}(x)}{q^n} \frac{t^n}{n!} \tag{2.14}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} \frac{1}{n!} \left(e^{\xi t}-1\right)^n &= \sum_{n=0}^{\infty} \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} \sum_{l=n}^{\infty} S_2(l,n) \xi^l \frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty} \left(\xi^m \sum_{n=0}^{m} \frac{D_{n,\xi,q}^{(k)}(x)}{\xi^n q^n} S_2(m,n) \right) \frac{t^m}{m!}. \end{aligned} \tag{2.15}$$

By (2.13), (2.14), and (2.15), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$D_{n,\xi,q}^{(k)}(x) = \xi^n \sum_{m=0}^{n} q^{n-m} S_1(n,m) B_m^{(k)}(x) = \xi^n \sum_{m=0}^{n} \left|S_1(n,m)\right| (-q)^{n-m} B_m^{(k)}(x)$$

and

$$B_n^{(k)}(x) = \sum_{m=0}^{n} D_{m,\xi,q}^{(k)}(x) \xi^{-m} q^{n-m} S_2(n,m).$$

Now, we consider the *twisted Daehee polynomials of the second kind with q-parameter* as follows:

$$\hat{D}_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} (-y+x)_{n,q} \, d\mu_0(y) \quad (n \geq 0). \tag{2.16}$$

In the special case $x = 0$, $\hat{D}_{n,\xi,q}(0) = \hat{D}_{n,\xi,q}$ are called the *twisted Daehee numbers of the second kind with q-parameter*.

By (2.16), we have

$$\hat{D}_{n,\xi,q}(x) = \xi^n q^n \int_{\mathbb{Z}_p} \left(\frac{-y+x}{q} \right)_n d\mu_0(y), \tag{2.17}$$

and so we can derive the generating function of $\hat{D}_{n,\xi,q}(x)$ by (1.1) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\hat{D}_{n,\xi,q}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}q^n\xi^n\int_{\mathbb{Z}_p}\left(\frac{-y+x}{q}\right)_n d\mu_0(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}q^n\xi^n\int_{\mathbb{Z}_p}\binom{\frac{-y+x}{q}}{n}d\mu_0(y)t^n\\ &= \int_{\mathbb{Z}_p}(1+q\xi t)^{\frac{-y+x}{q}}\,d\mu_0(y)\\ &= (1+q\xi t)^{\frac{x}{q}}\frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}}-1)}(1+q\xi t)^{\frac{1}{q}}. \end{aligned}\tag{2.18}$$

From (1.3), (1.6), and (2.17), we get

$$\begin{aligned}\hat{D}_{n,\xi,q}(x) &= q^n\xi^n\int_{\mathbb{Z}_p}\left(\frac{-y+x}{q}\right)_n d\mu_0(y)\\ &= q^n\xi^n\int_{\mathbb{Z}_p}\sum_{l=0}^{n}\frac{S_1(n,l)}{q^l}(-y+x)^l\,d\mu_0(y)\\ &= \xi^n\sum_{l=0}^{n}S_1(n,l)(-1)^l\int_{\mathbb{Z}_p}(y-x)^l\,d\mu_0(y)q^{n-l}\\ &= \xi^n\sum_{l=0}^{n}S_1(n,l)(-1)^lB_l(-x)q^{n-l}\\ &= (-\xi)^n\sum_{l=0}^{n}\left|S_1(n,l)\right|B_l(-x)q^{n-l}. \end{aligned}\tag{2.19}$$

By (1.10), it is easy to show that $B_n(-x) = (-1)^nB_n(x+1)$. Thus, from (2.19), we have the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$\hat{D}_{n,\xi,q}(x) = \xi^n\sum_{l=0}^{n}S_1(n,l)(-1)^lB_l(-x)q^{n-l} = \xi^n\sum_{l=0}^{n}\left|S_1(n,l)\right|B_l(x+1)(-q)^{n-l}.$$

By replacing t by $\frac{1}{q\xi}(e^{\xi t}-1)$ in (2.18), we have

$$\sum_{n=0}^{\infty}\hat{D}_{n,\xi,q}(x)\frac{1}{q^n\xi^n}\frac{(e^{\xi t}-1)^n}{n!} = e^{\frac{\xi t}{q}(x+1)}\frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}}-1} = \sum_{n=0}^{\infty}\frac{\xi^nB_n(x+1)}{q^n}\frac{t^n}{n!}\tag{2.20}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\hat{D}_{n,\xi,q}(x)\frac{1}{q^n\xi^n}\frac{(e^{\xi t}-1)^n}{n!} &= \sum_{n=0}^{\infty}\frac{\hat{D}_{n,\xi,q}(x)}{q^n\xi^n}\sum_{m=n}^{\infty}S_2(m,n)\frac{(\xi t)^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\hat{D}_{m,\xi,q}(x)S_2(n,m)q^{-m}\xi^{n-m}\right)\frac{t^n}{n!}. \end{aligned}\tag{2.21}$$

By (2.20) and (2.21), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$B_n(x+1) = \sum_{m=0}^{n} q^{n-m}\xi^{-m}\hat{D}_{m,\xi,q}(x)S_2(n,m).$$

Now, we consider *higher-order twisted Daehee polynomials of the second kind with q-parameter*. Higher-order twisted Daehee polynomials of the second kind with q-parameter are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$\hat{D}^{(k)}_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_{n,q}\, d\mu_0(x_1)\cdots d\mu_0(x_k), \tag{2.22}$$

where n is a nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $\hat{D}^{(k)}_{n,\xi,q} = \hat{D}^{(k)}_{n,\xi,q}(0)$ are called the *higher-order twisted Daehee numbers of the second kind with q-parameter*.

From (2.22), we can derive the generating function of $\hat{D}^{(k)}_{n,\xi,q}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \hat{D}^{(k)}_{n,\xi,q}(x)\frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\frac{-x_1-\cdots-x_k+x}{q}}{n} d\mu_0(x_1)\cdots d\mu_0(x_k) t^n \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+q\xi t)^{\frac{-x_1-\cdots-x_k+x}{q}} d\mu_0(x_1)\cdots d\mu_0(x_k) \\
&\quad = (1+q\xi t)^{\frac{x+k}{q}} \left(\frac{\log(1+q\xi t)}{q((1+q\xi t)^{\frac{1}{q}} - 1)} \right)^k.
\end{aligned} \tag{2.23}$$

By (2.22),

$$\begin{aligned}
&\hat{D}^{(k)}_{n,\xi,q}(x) \\
&\quad = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)^m\, d\mu_0(x_1)\cdots d\mu_0(x_k) \\
&\quad = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{(-q)^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k - x)^m\, d\mu_0(x_1)\cdots d\mu_0(x_k) \\
&\quad = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{(-q)^m} B^{(k)}_m(-x) \\
&\quad = \xi^n \sum_{m=0}^{n} q^{n-m} \left|S_1(n,m)\right| B^{(k)}_m(-x).
\end{aligned} \tag{2.24}$$

From (1.10), we know that $B^{(k)}_n(-x) = (-1)^n B^{(k)}_n(k+x)$. Hence, by (2.24), we obtain the following theorem.

Theorem 2.7 *For $n \geq 0$, we have*

$$\hat{D}^{(k)}_{n,\xi,q}(x) = \xi^n \sum_{m=0}^{n} (-1)^m q^{n-m} S_1(n,m) B^{(k)}_m(-x) = \xi^n \sum_{m=0}^{n} (-1)^m q^{n-m} \left|S_1(n,m)\right| B^{(k)}_m(x+k).$$

In (2.23), by replacing t by $\frac{1}{q\xi}(e^{\xi t}-1)$, we get

$$\sum_{n=0}^{\infty}\hat{D}_{n,\xi,q}^{(k)}(x)\frac{(e^{\xi t}-1)^n}{\xi^n q^n n!}=e^{\frac{\xi t}{q}(x+k)}\left(\frac{\frac{\xi t}{q}}{e^{\frac{\xi t}{q}}-1}\right)^k=\sum_{n=0}^{\infty}\frac{\xi^n B_n^{(k)}(x+k)}{q^n}\frac{t^n}{n!} \tag{2.25}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\hat{D}_{n,\xi,q}^{(k)}(x)}{\xi^n q^n}\frac{1}{n!}\left(e^{\xi t}-1\right)^n&=\sum_{n=0}^{\infty}\frac{\hat{D}_{n,\xi,q}^{(k)}(x)}{\xi^n q^n}\sum_{l=n}^{\infty}S_2(l,n)\xi^l\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\xi^n\sum_{m=0}^{n}\frac{\hat{D}_{m,\xi,q}^{(k)}(x)}{\xi^m q^m}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.26}$$

By (2.25) and (2.26), we obtain the following theorem.

Theorem 2.8 *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$B_n^{(k)}(x+k)=\sum_{m=0}^{n}\hat{D}_{m,\xi,q}^{(k)}(x)\xi^{-m}q^{n-m}S_2(n,m).$$

Competing interests

The author declares that they have no competing interests.

Author's contributions

The author contributed to the manuscript and typed, read, and approved the final manuscript.

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