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The pth moment asymptotic stability and exponential stability of stochastic functional differential equations with polynomial growth condition

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Abstract

This paper mainly discusses the pth moment asymptotic stability and the exponential stability of nonlinear stochastic functional differential equations (SFDEs) satisfying the local Lipschitz condition but not the linear growth condition. These new conditions assume that the coefficients of SFDEs are polynomials or dominated by polynomials. We establish some sufficient conditions for the pth moment asymptotic stability and the exponential stability of nonlinear SFDEs by applying some novel techniques. Some nontrivial examples are provided to illustrate our results.

MSC: 93E03; 93D20

Keywords: stochastic functional differential equations; polynomial growth condition; pth moment asymptotic stability; pth moment exponential stability; almost sure exponential stability

1 Introduction

Stochastic modeling plays an important role in describing the dynamical systems in many branches of science and industries. Since the introduction of Itô's formula, stochastic functional differential equations (SFDEs) have been used successfully to model such systems in many application fields such as biology, engineering, economics and finance (*e.g.*, [1, 2]). The stability of SFDEs, including stochastic delay differential equations (SDDEs), is the most fundamental problem. It has received lots of attention over past years, and many important results have been established by many researchers (*e.g.*, [1, 3–13]). In particular, by exploiting the positivity of the solution, Appleby and Reynolds [14] studied the exponential and non-exponential convergence rates of linear convolution Itô-Volterra equations. Then Appleby and Rodkina [15] discussed the stochastic stability and stochastic asymptotic stability of a nonlinear Volterra difference equation. Mao [7] considered the exponential stability of linear SDDEs, and he also extended it to the case of semi-linear uncertain SDDEs [9]. Mao [8] investigated Razumikhin-type theorems on the exponential stability of SFDEs whose coefficients satisfy the local Lipschitz condition and the linear growth condition, which was used to study the exponential stability of stochastic interval systems [4].

On the one hand, most of the stability criteria concentrate on the exponential stability. In many practical uses, however, stabilizing a system exponentially fast is not economical and sometimes unfeasible. Therefore, it is also significant to study the asymptotic stability for SFDEs further. For example, Mao [10] obtained useful criteria on the almost sure asymptotic stability for SDDEs. By using Razumikhin technique, Huang and Deng [3] studied the general pth moment asymptotic stability of SFDEs.

On the other hand, most of the stability criteria above require the coefficients of corresponding systems to satisfy the linear growth condition or the one-side linear growth condition. However, many stochastic systems do not satisfy these conditions. Therefore, it is necessary to study the stability for stochastic systems without the linear growth condition or the one-side linear growth condition. Recently, Shen *et al.* [16] used the LaSalle technique to study the almost sure asymptotic stability of SFDEs without satisfying the linear growth condition. Then Luo *et al.* [6] established new criteria on the asymptotic stability and the boundedness of SFDEs where the linear growth condition was no longer needed. Liu *et al.* [5] studied the asymptotic stability of nonlinear SDDEs with the polynomial growth condition.

If the coefficients of SFDEs are polynomials or dominated by polynomials, the existing methods cannot be used directly (details can be found in examples in Section 6). Thus, it is natural to ask the following questions: What happens if SFDEs obey the polynomial growth condition? Is there a unique global solution? If yes, under what conditions do SFDEs have the properties of the pth moment asymptotic stability and the exponential stability? In the following sections, we shall answer these questions. Our aims are to establish some new criteria on the pth moment asymptotic stability and the exponential stability of SFDEs with the polynomial growth condition and the local Lipschitz condition.

The organization of this paper is as follows. In Section 2, we give some necessary notations and lemmas. In Section 3, we discuss the existence of the global solution and the pth moment asymptotic stability of SFDEs. We find that the existence result depends only on constants κ and $\bar{\kappa}$ but not on other constants. In Section 4, we give sufficient conditions for the pth moment exponential stability and the almost sure exponential stability of SFDEs. To make our theory more applicable, Section 5 discusses generalized theory on the asymptotic stability of SFDEs. To show the applications of our results, some illustrative examples are given in the final section.

2 Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfying the usual conditions, and let $B(t)$ be an m-dimensional Brownian motion defined on the probability space. Let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of all continuous R^n-valued functions φ on $[-\tau, 0]$ with the norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. Let $C = C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ be the family of all bounded, $\mathscr{F}_0$-measurable, $C([-\tau, 0]; R^n)$-valued, $\mathscr{F}_t$-adapted stochastic processes. Let η_i be probability measures on $[-\tau, 0]$ which satisfy $\int_{-\tau}^0 d\eta_i(\theta) = 1$ ($i = 1, 2, 3, 4$). Let $L^1(R_+; R_+)$ be the family of all functions $\xi : R_+ \to R_+$ such that $\int_0^{+\infty} \xi(t) \, dt < \infty$. We assume that $\mathbf{x}(t)$ is a continuous R-valued stochastic process on $t \in [-\tau, \infty)$, $\mathbf{x}_t = \{\mathbf{x}(t + \theta) : -\tau \leq \theta \leq 0\}$ for all $t \geq 0$, which is regarded as a $C([-\tau, 0]; R^n)$-valued stochastic process.

Consider an n-dimensional SFDE

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}_t, t) \, dt + \mathbf{g}(\mathbf{x}_t, t) \, dB(t) \tag{1}$$

on $t \geq 0$ with initial data $\{\mathbf{x}(\theta) : -\tau \leq \theta \leq 0\} = \boldsymbol{\zeta} \in C_{\mathscr{F}_0}^b([-\tau, 0]; R^n)$, where

$$\mathbf{f} : C([-\tau, 0]; R^n) \times R_+ \to R^n, \qquad \mathbf{g} : C([-\tau, 0]; R^n) \times R_+ \to R^{n \times m}.$$

Assume, furthermore, that $\mathbf{f}(0, t) = 0$ and $\mathbf{g}(0, t) = 0$, so system (1) has the solution $\mathbf{x}(t) = 0$. The solution is called a trial solution or an equilibrium solution.

It is said that $\mathbf{f}, \mathbf{g}$ satisfy the following local Lipschitz condition.

Assumption 2.1 For each integer $k \geq 1$, there is a positive d_k such that

$$\big|\mathbf{f}(\boldsymbol{\varphi}, t) - \mathbf{f}(\boldsymbol{\psi}, t)\big| \vee \big|\mathbf{g}(\boldsymbol{\varphi}, t) - \mathbf{g}(\boldsymbol{\psi}, t)\big| \leq d_k \|\boldsymbol{\varphi} - \boldsymbol{\psi}\|$$

for all $\boldsymbol{\varphi}, \boldsymbol{\psi} \in C([-\tau, 0]; R^n)$, $t \in R_+$ with $\|\boldsymbol{\varphi}\| \vee \|\boldsymbol{\psi}\| \leq k$.

Replacing the linear growth condition or the one-side linear growth condition, we impose the following polynomial growth condition.

Assumption 2.2 There exist constants $\kappa, \bar{\kappa}, \hat{\kappa}, \underline{\kappa}, \gamma, \bar{\gamma}, \hat{\gamma}, \underline{\gamma} \geq 0$, probability measures η_i on $[-\tau, 0]$, $i = 1, 2, 3, 4$, and positive numbers $n_1 > 1$, $n_2 > 1$ satisfying $n_1 + 1 > 2n_2$, and bounded functions $\xi_1(t), \xi_2(t) \in L^1(R_+; R_+)$ such that

$$\boldsymbol{\varphi}(0)^T \mathbf{f}(\boldsymbol{\varphi}, t) \leq -\kappa \big|\boldsymbol{\varphi}(0)\big|^{n_1+1} + \bar{\kappa} \int_{-\tau}^0 \big|\boldsymbol{\varphi}(\theta)\big|^{n_1+1} \, d\eta_1(\theta) - \hat{\kappa} \big|\boldsymbol{\varphi}(0)\big|^2$$

$$+ \underline{\kappa} \int_{-\tau}^0 \big|\boldsymbol{\varphi}(\theta)\big|^2 \, d\eta_2(\theta) + \xi_1(t),$$

$$\big|\mathbf{g}(\boldsymbol{\varphi}, t)\big| \leq \gamma \big|\boldsymbol{\varphi}(0)\big|^{n_2} + \bar{\gamma} \int_{-\tau}^0 \big|\boldsymbol{\varphi}(\theta)\big|^{n_2} \, d\eta_3(\theta) + \hat{\gamma} \big|\boldsymbol{\varphi}(0)\big| + \underline{\gamma} \int_{-\tau}^0 \big|\boldsymbol{\varphi}(\theta)\big| \, d\eta_4(\theta) + \xi_2(t)$$

for all $\boldsymbol{\varphi} \in C([-\tau, 0]; R^n)$, $t \in R_+$.

Remark 1 In this paper, the probability measures η_i, $i = 1, 2, 3, 4$, can be weakened to any right-continuous nondecreasing functions (see [17]). Compared with [5], Assumption 2.2 in this paper is a generalization of the case of hypothesis (H2) of [5].

Let $C^{2,1}(R^n \times [-\tau, +\infty); R_+)$ denote the family of all continuous nonnegative functions $V(x, t)$ on $R^n \times [-\tau, +\infty)$ which are continuously twice differentiable in x and once differentiable in t. For each $V \in C^{2,1}(R^n \times [-\tau, +\infty); R_+)$, denote an operator $\mathscr{L}V$ from $C([-\tau, 0]; R^n) \times R_+$ to R by

$$\mathscr{L}V(\boldsymbol{\varphi}, t) = V_t(\boldsymbol{\varphi}(0), t) + V_x(\boldsymbol{\varphi}(0), t)\mathbf{f}(\boldsymbol{\varphi}, t) + \frac{1}{2} \operatorname{trace}\big[\mathbf{g}^T(\boldsymbol{\varphi}, t) V_{xx}(\boldsymbol{\varphi}(0), t)\mathbf{g}(\boldsymbol{\varphi}, t)\big],$$

where

$$V_t(\mathbf{x}, t) = \frac{\partial V(\mathbf{x}, t)}{\partial t}, \qquad V_{xx}(\mathbf{x}, t) = \left(\frac{\partial^2 V(\mathbf{x}, t)}{\partial x_i x_j}\right)_{n \times n},$$

$$V_x(\mathbf{x}, t) = \left(\frac{\partial V(\mathbf{x}, t)}{\partial x_1}, \ldots, \frac{\partial V(\mathbf{x}, t)}{\partial x_n}\right).$$

To obtain our main theorems, we recall a number of lemmas.

Lemma 2.1 (Barbalat lemma [18]) *If $h(t)$ is a uniformly continuous function on $[0, \infty)$, and $h(t) \in L^1(R_+; R_+)$, then $\lim_{t \to \infty} h(t) = 0$.*

Lemma 2.2 (*cf.* [18]) *If $h(t)$ is a bounded function on $[0, \infty)$, and $h(t) \in L^1(R_+; R_+)$, then for any $\beta \geq 1$, $\int_0^{+\infty} h^\beta(t) \, dt < \infty$.*

Lemma 2.3 (*cf.* [5]) *Assume $a, b, q > 0$, $b \geq q$, $\alpha > \beta > 0$. If the following condition holds,*

$$\frac{a}{b} > (\alpha - \beta) \beta^{\frac{\beta}{\alpha - \beta}} \alpha^{\frac{-\alpha}{\alpha - \beta}},$$

then there exists $\bar{a} \in (0, a)$ satisfying, for all $t \geq 0$,

$$a + bt^\alpha - qt^\beta > \bar{a}.$$

Lemma 2.4 (*cf.* [12]) *Assume $\alpha, \beta > 0$. For any $h(t) \in C(R^n; R)$, if $\limsup_{|t| \to \infty} (h(t)/|t|^\alpha) = 0$, then there exists a constant H satisfying*

$$\sup_{t \in R^n} \left\{ -\beta |t|^\alpha + h(t) \right\} < H.$$

Lemma 2.5 *Let $A_1(t)$ and $A_2(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A_1(0) = A_2(0) = 0$ a.s., let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s., and let ζ be a nonnegative $\mathscr{F}_0$-measurable random variable such that $E\zeta < \infty$. Define $X(t) = \zeta + A_1(t) - A_2(t) + M(t)$ for $t \geq 0$. If $X(t)$ is nonnegative, then*

$$\left\{ \lim_{t \to \infty} A_1(t) < \infty \right\} \subset \left\{ \lim_{t \to \infty} X(t) < \infty \right\} \cap \left\{ \lim_{t \to \infty} A_2(t) < \infty \right\} \quad a.s.,$$

where $C \subset D$ a.s. means $\mathbb{P}(C \cap D^c) = 0$. In particular, if $\lim_{t \to \infty} A_1(t) < \infty$ a.s., then with probability one,

$$\lim_{t \to \infty} X(t) < \infty, \qquad \lim_{t \to \infty} A_2(t) < \infty, \qquad -\infty < \lim_{t \to \infty} M(t) < \infty.$$

This lemma is called nonnegative semi-martingale convergence theorem (see [19]), which will play an important role in this paper.

3 pth moment asymptotic stability of SFDEs

The classical theory is not used directly to system (1), so it is necessary to establish the following existence-and-uniqueness result.

Lemma 3.1 *If Assumptions 2.1, 2.2 and $\kappa > \bar{\kappa}$ hold, then for any initial data $\zeta \in C$, there is a unique global solution $\mathbf{x}(t, \zeta)$ of system (1) on $t \geq -\tau$.*

Remark 2 To ensure that a stochastic differential equation has a unique global solution for any given initial data, the coefficients of this equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (see [1, 2, 20]) or the linear growth condition and some non-Lipschitz condition (see [21]), which shows that the linear growth condition plays an important role. In this paper, however, we only require the

coefficients to satisfy the polynomial growth condition and the local Lipschitz condition. The solution of system (1) may explode at a finite time, so it is important to examine the existence and uniqueness of the global solution of system (1). Since it is not so easy to see this fact directly, we give the proof in the appendix. The fact that we write down our Lemma 3.1 here is to keep our paper completely based on Assumptions 2.1 and 2.2.

Compared with the non-explosion of the global solution (Lemma 3.1), the asymptotic boundedness in the sense of pth moment is more interesting. We also state the following result of asymptotic boundedness and give the proof in the Appendix.

Lemma 3.2 *If Assumptions* 2.1, 2.2 *and* $\kappa > \bar{\kappa}$ *hold, then for any* $p \geq 0$ *and any initial data* $\boldsymbol{\zeta} \in C$, *there exists a constant* $M_p > 0$ *such that the global solution* $\mathbf{x}(t, \boldsymbol{\zeta})$ *of system* (1) *has the property*

$$\sup_{-\tau \leq t < +\infty} E|\mathbf{x}(t, \boldsymbol{\zeta})|^p \leq M_p.$$

To study the pth moment asymptotic stability, we prove the following lemma.

Lemma 3.3 *If Assumptions* 2.1, 2.2 *and* $\kappa > \bar{\kappa}$ *hold, then for any* $p \geq 2$ *and any initial data* $\boldsymbol{\zeta} \in C$, $E|\mathbf{x}(t, \boldsymbol{\zeta})|^p$ *is uniformly continuous on* $[0, \infty)$, *where* $\mathbf{x}(t, \boldsymbol{\zeta})$ *is the global solution of system* (1).

Proof For the sake of simplicity, write $\mathbf{x}(t) = \mathbf{x}(t, \boldsymbol{\zeta})$, $\mathbf{x}_t = \mathbf{x}_t^{\boldsymbol{\zeta}}$. From Lemma 3.2, we know that for any $p \geq 2$, $\sup_{-\tau \leq t < \infty} E|\mathbf{x}(t)|^p \leq M_p$. Using Itô's formula, we compute that, for any $0 < s < t < \infty$,

$$\left| E|\mathbf{x}(t)|^p - E|\mathbf{x}(s)|^p \right|$$

$$= \frac{p}{2} \left| E \int_s^t |\mathbf{x}(r)|^{p-2} \left(2\mathbf{x}^T(r)\mathbf{f}(\mathbf{x}_r, r) + |\mathbf{g}(\mathbf{x}_r, r)|^2 + (p-2)|\mathbf{x}(r)|^{-2} \right.\right.$$

$$\left.\left. \times |\mathbf{x}^T(r)\mathbf{g}(\mathbf{x}_r, r)|^2 \right) dr \right|$$

$$\leq \frac{p}{2} E \int_s^t \left[|\mathbf{x}(r)|^{p-2} \left(|2\mathbf{x}^T(r)\mathbf{f}(\mathbf{x}_r, r)| + (p-1)|\mathbf{g}(\mathbf{x}_r, r)|^2 \right) \right] dr$$

$$\leq \frac{p}{2} \int_s^t E \left(2|\mathbf{x}(r)|^{p-2}|\mathbf{x}^T(r)\mathbf{f}(\mathbf{x}_r, r)| + (p-1)|\mathbf{x}(r)|^{p-2}|\mathbf{g}(\mathbf{x}_r, r)|^2 \right) dr. \tag{2}$$

From Assumption 2.2, through simple computation we get that

$$|\mathbf{x}(r)|^{p-2}|\mathbf{x}^T(r)\mathbf{f}(\mathbf{x}_r, r)|$$

$$\leq \frac{p-2}{p-1}|\mathbf{x}(r)|^{p-1} + \frac{1}{p-1}|\mathbf{x}^T(r)\mathbf{f}(\mathbf{x}_r, r)|^{p-1}$$

$$\leq \frac{p-2}{p-1}|\mathbf{x}(r)|^{p-1} + \frac{5^{p-2}}{p-1} \left[\kappa^{p-1}|\mathbf{x}(r)|^{(n_1+1)(p-1)} \right.$$

$$\left. + \bar{\kappa}^{p-1} \left(\int_{-\tau}^0 |\mathbf{x}(r+\theta)|^{(n_1+1)} d\eta_1(\theta) \right)^{p-1} + \hat{\kappa}^{p-1}|\mathbf{x}(r)|^{2(p-1)} \right.$$

$$
+ \underline{\kappa}^{p-1} \left(\int_{-\tau}^{0} |\mathbf{x}(r+\theta)|^2 \, d\eta_2(\theta) \right)^{p-1} + \xi_1^{p-1}(r) \bigg],
$$

$$
|\mathbf{x}(r)|^{p-2} |\mathbf{g}(\mathbf{x}_r, r)|^2
$$

$$
\leq \frac{p-2}{p} |\mathbf{x}(r)|^p + \frac{2}{p} |\mathbf{g}(\mathbf{x}_r, r)|^p
$$

$$
\leq \frac{p-2}{p} |\mathbf{x}(r)|^p + \frac{2}{p} 5^{p-1} \left[\gamma^p |\mathbf{x}(r)|^{p n_2} + \bar{\gamma}^p \left(\int_{-\tau}^{0} |\mathbf{x}(r+\theta)|^{n_2} \, d\eta_3(\theta) \right)^p \right.
$$

$$
\left. + \hat{\gamma}^p |\mathbf{x}(r)|^p + \underline{\gamma}^p \left(\int_{-\tau}^{0} |\mathbf{x}(r+\theta)| \, d\eta_4(\theta) \right)^p + \xi_2^p(r) \right], \tag{3}
$$

where we have used the elemental inequalities: for any $c, b, \alpha, \beta \geq 0$, $\alpha + \beta > 0$, $c^\alpha b^\beta \leq \frac{\alpha}{\alpha+\beta} c^{\alpha+\beta} + \frac{\beta}{\alpha+\beta} b^{\alpha+\beta}$; for any $c_i \geq 0$, $i = 1, 2, \ldots, d$, $\theta \geq 1$, $(\sum_{i=1}^{d} c_i)^\theta \leq d^{\theta-1} \sum_{i=1}^{d} c_i^\theta$.

By virtue of the boundedness of $\xi_1(t)$, $\xi_2(t)$, assuming $\xi_1(t) \vee \xi_2(t) \leq \Psi$ and substituting (3) into (2), we have

$$
\big| E|\mathbf{x}(t)|^p - E|\mathbf{x}(s)|^p \big|
$$

$$
\leq \frac{p}{2} \int_s^t 2 \frac{p-2}{p-1} E|\mathbf{x}(r)|^{p-1} + 2 \frac{5^{p-2}}{p-1} \left[\kappa^{p-1} E|\mathbf{x}(r)|^{(n_1+1)(p-1)} + \hat{\kappa}^{p-1} E|\mathbf{x}(r)|^{2(p-1)} \right.
$$

$$
+ \bar{\kappa}^{p-1} E \left(\int_{-\tau}^{0} |\mathbf{x}(r+\theta)|^{(n_1+1)} \, d\eta_1(\theta) \right)^{p-1} + \underline{\kappa}^{p-1} E \left(\int_{-\tau}^{0} |\mathbf{x}(r+\theta)|^2 \, d\eta_2(\theta) \right)^{p-1}
$$

$$
\left. + \xi_1^{p-1}(r) \right] + \frac{p-2}{p}(p-1) E|\mathbf{x}(r)|^p + \frac{2}{p} 5^{p-1}(p-1) \left[\bar{\gamma}^p E \left(\int_{-\tau}^{0} |\mathbf{x}(r+\theta)|^{n_2} \, d\eta_3(\theta) \right)^p \right.
$$

$$
\left. + \gamma^p E|\mathbf{x}(r)|^{p n_2} + \hat{\gamma}^p E|\mathbf{x}(r)|^p + \underline{\gamma}^p E \left(\int_{-\tau}^{0} |\mathbf{x}(r+\theta)| \, d\eta_4(\theta) \right)^p + \xi_2^p(r) \right] dr
$$

$$
\leq \frac{p}{2} \int_s^t 2 \frac{p-2}{p-1} E|\mathbf{x}(r)|^{p-1} + 2 \frac{5^{p-2}}{p-1} \left[\kappa^{p-1} E|\mathbf{x}(r)|^{(n_1+1)(p-1)} + \hat{\kappa}^{p-1} E|\mathbf{x}(r)|^{2(p-1)} \right.
$$

$$
+ \bar{\kappa}^{p-1} \int_{-\tau}^{0} E|\mathbf{x}(r+\theta)|^{(n_1+1)(p-1)} \, d\eta_1(\theta) + \underline{\kappa}^{p-1} \int_{-\tau}^{0} E|\mathbf{x}(r+\theta)|^{2(p-1)} \, d\eta_2(\theta)
$$

$$
\left. + \xi_1^{p-1}(r) \right] + \frac{p-2}{p}(p-1) E|\mathbf{x}(r)|^p + \frac{2}{p} 5^{p-1}(p-1) \left[\bar{\gamma}^p \int_{-\tau}^{0} E|\mathbf{x}(r+\theta)|^{p n_2} \, d\eta_3(\theta) \right.
$$

$$
\left. + \gamma^p E|\mathbf{x}(r)|^{p n_2} + \hat{\gamma}^p E|\mathbf{x}(r)|^p + \underline{\gamma}^p \int_{-\tau}^{0} E|\mathbf{x}(r+\theta)|^p \, d\eta_4(\theta) + \xi_2^p(r) \right] dr
$$

$$
\leq \frac{p}{2} \left[2 \frac{p-2}{p-1} M_{p-1} + 2 \frac{5^{p-2}}{p-1} \left(\kappa^{p-1} M_{(n_1+1)(p-1)} + \bar{\kappa}^{p-1} M_{(n_1+1)(p-1)} \right. \right.
$$

$$
\left. + \hat{\kappa}^{p-1} M_{2(p-1)} + \underline{\kappa}^{p-1} M_{2(p-1)} + \Psi^{p-1} \right) + \frac{p-2}{p}(p-1) M_p + \frac{2}{p} 5^{p-1}(p-1)
$$

$$
\left. \times \left(\gamma^p M_{p n_2} + \bar{\gamma}^p M_{p n_2} + \hat{\gamma}^p M_p + \underline{\gamma}^p M_p + \Psi^p \right) \right] (t-s),
$$

which completes the proof of the uniform continuity of $E|\mathbf{x}(t)|^p$. □

In this section, our aim is to study the pth moment asymptotic stability of system (1). The following theorem establishes a new sufficient condition for the pth moment asymptotic stability.

Theorem 3.1 *If Assumptions* 2.1, 2.2 *and the following condition* (4) *hold,*

$$\frac{(\gamma + \bar{\gamma})^2}{2(\kappa - \bar{\kappa})} + \frac{(\hat{\gamma} + \underline{\gamma})^2}{2(\hat{\kappa} - \underline{\kappa}) - 2L(\kappa - \bar{\kappa})} < 1, \qquad (\hat{\kappa} - \underline{\kappa}) - L(\kappa - \bar{\kappa}) > 0, \qquad \kappa - \bar{\kappa} > 0, \qquad (4)$$

then for any $p \in (0, p_0)$ *and any initial data* $\boldsymbol{\zeta} \in C$, *there is a unique global solution* $\mathbf{x}(t, \boldsymbol{\zeta})$ *of system* (1) *on* $t \geq -\tau$, *and* $\mathbf{x}(t, \boldsymbol{\zeta})$ *is pth moment asymptotically stable, namely*

$$\lim_{t \to \infty} E|\mathbf{x}(t, \boldsymbol{\zeta})|^p = 0, \qquad (5)$$

where $L = (n_1 - 2n_2 + 1)(2n_2 - 2)^{\frac{2n_2-2}{n_1-2n_2+1}}(n_1 - 1)^{\frac{1-n_1}{n_1-2n_2+1}}$, $p_0 = 1 + \left[\frac{(\gamma + \bar{\gamma})^2}{2(\kappa - \bar{\kappa})} + \frac{(\hat{\gamma} + \underline{\gamma})^2}{2(\hat{\kappa} - \underline{\kappa}) - 2L(\kappa - \bar{\kappa})}\right]^{-1}$.

Proof From Lemma 3.1 and condition (4), there exists a unique global solution. For the sake of simplicity, write $\mathbf{x}(t) = \mathbf{x}(t, \boldsymbol{\zeta})$. Moreover, from the definition of p_0 and condition (4), we have $p_0 > 2$. For any $p \in [2, p_0)$, applying Itô's formula to $V(\mathbf{x}, t) = |\mathbf{x}(t)|^p$ and using the elemental inequalities (for any $c, b \in R$, $0 < \theta < 1$, $(c + b)^2 \leq \frac{c^2}{1-\theta} + \frac{b^2}{\theta}$), we obtain

$$\mathscr{L}V \leq \frac{p}{2}|\mathbf{x}(t)|^{p-2}\left[-2\kappa|\mathbf{x}(t)|^{n_1+1} + 2\bar{\kappa}\int_{-\tau}^{0}|\mathbf{x}(t+\theta)|^{n_1+1}\,d\eta_1(\theta)\right.$$

$$- 2\hat{\kappa}|\mathbf{x}(t)|^2 + 2\underline{\kappa}\int_{-\tau}^{0}|\mathbf{x}(t+\theta)|^2\,d\eta_2(\theta)$$

$$+ \frac{p-1}{\rho_1}\left(\frac{\gamma^2}{\delta_1}|\mathbf{x}(t)|^{2n_2} + \frac{\bar{\gamma}^2}{1-\delta_1}\int_{-\tau}^{0}|\mathbf{x}(t+\theta)|^{2n_2}\,d\eta_3(\theta)\right)$$

$$+ \frac{p-1}{\rho_2(1-\rho_1)}\left(\frac{\hat{\gamma}^2}{\delta_2}|\mathbf{x}(t)|^2 + \frac{\underline{\gamma}^2}{1-\delta_2}\int_{-\tau}^{0}|\mathbf{x}(t+\theta)|^2\,d\eta_4(\theta)\right) + 2\xi_1(t)$$

$$\left. + \frac{p-1}{(1-\rho_1)(1-\rho_2)}\xi_2^2(t)\right]$$

$$\leq \frac{p}{2}\left[-2(\kappa-\bar{\kappa})|\mathbf{x}(t)|^{p+n_1-1} - \left(2\hat{\kappa} - 2\underline{\kappa} - \frac{p-1}{\rho_2(1-\rho_1)}\left(\frac{\hat{\gamma}^2}{\delta_2} + \frac{\underline{\gamma}^2}{1-\delta_2}\right) - 2\frac{p-2}{p}\varepsilon_1^p\right.\right.$$

$$\left. - \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\frac{p-2}{p}\varepsilon_2^p\right)|\mathbf{x}(t)|^p + \frac{p-1}{\rho_1}\left(\frac{\gamma^2}{\delta_1} + \frac{\bar{\gamma}^2}{1-\delta_1}\right)|\mathbf{x}(t)|^{p+2n_2-2}\right] + 2\underline{\kappa}J_2$$

$$+ \frac{p-1}{\rho_2(1-\rho_1)}\frac{\underline{\gamma}^2}{1-\delta_2}J_4 + \frac{p}{2}\frac{p-1}{\rho_1}\frac{\bar{\gamma}^2}{1-\delta_1}\frac{2n_2}{p+2n_2-2}J_3 + p\bar{\kappa}\frac{n_1+1}{p+n_1-1}J_1$$

$$+ 2\varepsilon_1^{-\frac{p(p-2)}{2}}\xi_1^{\frac{p}{2}}(t) + \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\xi_2^p(t), \qquad (6)$$

where $\rho_1, \rho_2, \delta_1, \delta_2 \in (0,1)$, $J_1 = \int_{-\tau}^{0}|\mathbf{x}(t+\theta)|^{p+n_1-1}\,d\eta_1(\theta) - |\mathbf{x}(t)|^{p+n_1-1}$, $J_2 = \int_{-\tau}^{0}|\mathbf{x}(t+\theta)|^p\,d\eta_2(\theta) - |\mathbf{x}(t)|^p$, $J_3 = \int_{-\tau}^{0}|\mathbf{x}(t+\theta)|^{p+2n_2-2}\,d\eta_3(\theta) - |\mathbf{x}(t)|^{p+2n_2-2}$, $J_4 = \int_{-\tau}^{0}|\mathbf{x}(t+\theta)|^p\,d\eta_4(\theta) - |\mathbf{x}(t)|^p$. Setting $\delta_1 = \frac{\gamma}{\gamma+\bar{\gamma}}$, $\delta_2 = \frac{\hat{\gamma}}{\hat{\gamma}+\underline{\gamma}}$, we have

$$\mathscr{L}V \leq \frac{p}{2}\left[-2(\kappa-\bar{\kappa})|\mathbf{x}(t)|^{p+n_1-1} + \frac{(p-1)(\gamma+\bar{\gamma})^2}{\rho_1}|\mathbf{x}(t)|^{p+2n_2-2} - \left(2\hat{\kappa} - 2\underline{\kappa}\right.\right.$$

$$\left.\left. - \frac{(p-1)(\hat{\gamma}+\underline{\gamma})^2}{\rho_2(1-\rho_1)} - 2\frac{p-2}{p}\varepsilon_1^p - \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\frac{p-2}{p}\varepsilon_2^p\right)|\mathbf{x}(t)|^p\right] + 2\underline{\kappa}J_2$$

$$+ p\bar{\kappa}\frac{n_1 + 1}{p + n_1 - 1}J_1 + \frac{p}{2}\frac{(p-1)(\bar{\gamma} + \gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p + 2n_2 - 2}J_3 + \frac{(p-1)(\gamma + \hat{\gamma})\gamma}{\rho_2(1 - \rho_1)}J_4$$

$$+ 2\varepsilon_1^{-\frac{p(p-2)}{2}}\xi_1^{\frac{p}{2}}(t) + \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\xi_2^p(t). \tag{7}$$

Let $G(|\mathbf{x}(t)|) = (2(\hat{\kappa} - \underline{\kappa}) - \frac{(\hat{\gamma}+\gamma)^2}{1-\rho_1}(p-1)) + 2(\kappa - \bar{\kappa})|\mathbf{x}(t)|^{n_1-1} - \frac{p-1}{\rho_1}(\gamma + \bar{\gamma})^2|\mathbf{x}(t)|^{2n_2-2}$. Because $p \in [2, p_0)$, then $(p-1)\frac{(\gamma+\bar{\gamma})^2}{2(\kappa-\bar{\kappa})} + (p-1)\frac{(\hat{\gamma}+\gamma)^2}{2(\hat{\kappa}-\underline{\kappa})-2L(\kappa-\bar{\kappa})} < 1$. So there exists $u_0 > 1$ such that $(p-1)\frac{(\gamma+\bar{\gamma})^2}{2(\kappa-\bar{\kappa})}u_0 + (p-1)\frac{(\hat{\gamma}+\gamma)^2}{2(\hat{\kappa}-\underline{\kappa})-2L(\kappa-\bar{\kappa})} = 1$. Let $\rho_1(u) = (p-1)\frac{(\gamma+\bar{\gamma})^2}{2(\kappa-\bar{\kappa})}u$, $u \in [1, u_0)$, satisfying $2(\kappa - \bar{\kappa}) \geq \frac{p-1}{\rho_1(u)}(\gamma + \bar{\gamma})^2$, $2(\hat{\kappa} - \underline{\kappa}) - \frac{(\hat{\gamma}+\gamma)^2}{1-\rho_1(u)}(p-1) > 2(\kappa - \bar{\kappa})L$. Using Lemma 2.3, we get that there exists a constant $\bar{a} > 0$ satisfying $\inf_{t\geq 0} G(|\mathbf{x}(t)|) > \bar{a}$. So we choose ρ_2 which is sufficiently close to 1 and sufficiently small ε_1, ε_2 such that

$$\left(2(\hat{\kappa} - \underline{\kappa}) - \frac{(p-1)(\hat{\gamma} + \underline{\gamma})^2}{(1-\rho_1)\rho_2} - \frac{2(p-2)}{p}\varepsilon_1^p - \frac{(p-2)}{p}\frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^p\right)$$

$$- \frac{p-1}{\rho_1}(\gamma + \bar{\gamma})^2|\mathbf{x}(t)|^{2n_2-2} + 2(\kappa - \bar{\kappa})|\mathbf{x}(t)|^{n_1-1} > \bar{a}.$$

Thus, we have

$$\mathscr{L}V \leq -\frac{p}{2}\bar{a}|\mathbf{x}(t)|^p + p\bar{\kappa}\frac{n_1 + 1}{p + n_1 - 1}J_1 + 2\underline{\kappa}J_2 + \frac{p}{2}\frac{(p-1)(\bar{\gamma} + \gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p + 2n_2 - 2}J_3$$

$$+ \frac{(p-1)(\underline{\gamma} + \hat{\gamma})\underline{\gamma}}{1-\rho_1}J_4 + 2\varepsilon_1^{-\frac{p(p-2)}{2}}\xi_1^{\frac{p}{2}}(t) + \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\xi_2^p(t).$$

By virtue of the fact that $\int_0^t(\int_{-\tau}^0 |\mathbf{x}(s+\theta)|^{w_i}\,d\eta_i(\theta) - |\mathbf{x}(s)|^{w_i})\,ds \leq \int_{-\tau}^0 |\mathbf{x}(s)|^{w_i}\,ds$ for $w_1 = p + n_1 - 1$, $w_3 = p + 2n_2 - 2$, $w_2 = w_4 = p$, respectively, we have

$$E|\mathbf{x}(t)|^p \leq E|\mathbf{x}(0)|^p - \frac{p}{2}\bar{a}\int_0^t E|\mathbf{x}(s)|^p\,ds + p\bar{\kappa}\frac{n_1 + 1}{p + n_1 - 1}\int_{-\tau}^0 E|\mathbf{x}(s)|^{p+n_1-1}\,ds$$

$$+ \frac{p}{2}\frac{(p-1)(\bar{\gamma} + \gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p + 2n_2 - 2}\int_{-\tau}^0 E|\mathbf{x}(s)|^{p+2n_2-2}\,ds$$

$$+ \left(2\underline{\kappa} + \frac{(p-1)(\underline{\gamma} + \hat{\gamma})\underline{\gamma}}{1-\rho_1}\right)$$

$$\times \int_{-\tau}^0 E|\mathbf{x}(s)|^p\,ds + \int_0^t\left[\frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\xi_2^p(s) + 2\varepsilon_1^{-\frac{p(p-2)}{2}}\xi_1^{\frac{p}{2}}(s)\right]ds. \tag{8}$$

From (8) and Lemma 2.2, we get that $\int_0^\infty E|\mathbf{x}(s)|^p\,ds < \infty$. Combining with Lemma 3.3, we claim that $E|\mathbf{x}(t)|^p$ is uniformly continuous on $[0, \infty)$ for any $p \in [2, p_0)$. Due to Lemma 2.1, we have

$$\lim_{t\to\infty} E|\mathbf{x}(t)|^p = 0. \tag{9}$$

For any $p \in (0, 2)$, using Hölder's inequality, we get

$$\lim_{t\to\infty} E|\mathbf{x}(t)|^p \leq \lim_{t\to\infty}\left(E|\mathbf{x}(t)|^2\right)^{\frac{p}{2}} = 0. \tag{10}$$

$\square$

Remark 3 Clearly, the key of the proof is the positive lower-boundedness of the function $G(|\mathbf{x}(t)|)$, which depends on condition (4) and the definition of p_0.

4 Exponential stability of SFDEs

To study exponential stability, we slightly modify the polynomial growth condition (Assumption 2.2) as follows.

Assumption 4.1 There exist constants $\kappa, \bar{\kappa}, \hat{\kappa}, \underline{\kappa}, \kappa_0, \gamma, \bar{\gamma}, \hat{\gamma}, \underline{\gamma}, \gamma_0 \geq 0$, $\alpha_1, \alpha_2 > 0$, probability measures η_i on $[-\tau, 0]$, $i = 1, 2, 3, 4$, and positive numbers $n_1 > 1$, $n_2 > 1$ satisfying $n_1 + 1 > 2n_2$ such that

$$\boldsymbol{\varphi}(0)^T \mathbf{f}(\boldsymbol{\varphi}, t) \leq -\kappa \left|\boldsymbol{\varphi}(0)\right|^{n_1+1} + \bar{\kappa} \int_{-\tau}^{0} \left|\boldsymbol{\varphi}(\theta)\right|^{n_1+1} d\eta_1(\theta) - \hat{\kappa} \left|\boldsymbol{\varphi}(0)\right|^2$$

$$+ \underline{\kappa} \int_{-\tau}^{0} \left|\boldsymbol{\varphi}(\theta)\right|^2 d\eta_2(\theta) + \kappa_0 e^{-\alpha_1 t},$$

$$\left|\mathbf{g}(\boldsymbol{\varphi}, t)\right| \leq \gamma \left|\boldsymbol{\varphi}(0)\right|^{n_2} + \bar{\gamma} \int_{-\tau}^{0} \left|\boldsymbol{\varphi}(\theta)\right|^{n_2} d\eta_3(\theta) + \hat{\gamma} \left|\boldsymbol{\varphi}(0)\right| + \underline{\gamma} \int_{-\tau}^{0} \left|\boldsymbol{\varphi}(\theta)\right| d\eta_4(\theta) + \gamma_0 e^{-\alpha_2 t}$$

for all $\boldsymbol{\varphi} \in C([-\tau, 0]; R^n)$, $t \in R_+$.

Remark 4 It is obvious that Assumption 4.1 is only a special case of Assumption 2.2.

In this section, our aim is to study the exponential stability of system (1). We have the following theorem.

Theorem 4.1 *If Assumptions* 2.1, 4.1 *and condition* (4) *hold, then for any initial data* $\boldsymbol{\zeta} \in C$, *there is a unique global solution* $\mathbf{x}(t, \boldsymbol{\zeta})$ *of system* (1) *on* $t \geq -\tau$, *and* $\mathbf{x}(t, \boldsymbol{\zeta})$ *is almost surely exponentially stable and pth moment exponentially stable for any* $p \in (0, p_0)$, *namely*

$$\limsup_{t \to \infty} \frac{\log(|\mathbf{x}(t, \boldsymbol{\zeta})|)}{t} \leq -\frac{\varepsilon_p}{p}, \quad p \in [2, p_0) \ a.s., \tag{11}$$

$$\limsup_{t \to \infty} \frac{\log(E|\mathbf{x}(t, \boldsymbol{\zeta})|^p)}{t} \leq -\varepsilon_p, \quad p \in [2, p_0),$$

$$\limsup_{t \to \infty} \frac{\log(E|\mathbf{x}(t, \boldsymbol{\zeta})|^p)}{t} \leq -\frac{p}{2}\varepsilon_2, \quad p \in (0, 2), \tag{12}$$

where p_0 *is the same as defined in Theorem* 3.1, *and* ε_p *is a positive constant which only depends on p but not on* $\boldsymbol{\zeta}$.

Proof From condition (4) and the definition of p_0, we obtain $(p-1)\frac{(\gamma+\bar{\gamma})^2}{2(\kappa-\bar{\kappa})} + (p-1) \times \frac{(\hat{\gamma}+\underline{\gamma})^2}{2(\hat{\kappa}-L\kappa)-2(\underline{\kappa}-L\bar{\kappa})} < 1$ for any $p \in [2, p_0)$. Then there exists at least a sufficiently small positive constant ε satisfying $(p-1)\frac{(\gamma+\bar{\gamma})(\gamma+\bar{\gamma}e^{\varepsilon\tau})}{2(\kappa-\bar{\kappa}e^{\varepsilon\tau})} + (p-1)\frac{(\hat{\gamma}+\underline{\gamma})(\hat{\gamma}+\underline{\gamma}e^{\varepsilon\tau})}{2(\hat{\kappa}-L\kappa)-\frac{2}{p}\varepsilon-2(\underline{\kappa}-L\bar{\kappa})e^{\varepsilon\tau}} < 1$, $(\hat{\kappa}-L\kappa)-\frac{\varepsilon}{p}-(\underline{\kappa}-L\bar{\kappa})e^{\varepsilon\tau} > 0$, $\kappa - \bar{\kappa}e^{\varepsilon\tau} > 0$, $\varepsilon < \frac{p\alpha_1}{2}$, $\varepsilon < p\alpha_2$. So, by the continuity, define

$$\varepsilon_p = \sup\left\{\varepsilon > 0 : \frac{(\gamma+\bar{\gamma})(\gamma+\bar{\gamma}e^{\varepsilon\tau})}{2(\kappa-\bar{\kappa}e^{\varepsilon\tau})} + \frac{(\hat{\gamma}+\underline{\gamma})(\hat{\gamma}+\underline{\gamma}e^{\varepsilon\tau})}{2(\hat{\kappa}-L\kappa)-\frac{2}{p}\varepsilon-2(\underline{\kappa}-L\bar{\kappa})e^{\varepsilon\tau}} < \frac{1}{p-1},\right.$$

$$\left.(\hat{\kappa}-L\kappa)-\frac{\varepsilon}{p}-(\underline{\kappa}-L\bar{\kappa})e^{\varepsilon\tau} > 0, \kappa - \bar{\kappa}e^{\varepsilon\tau} > 0, \varepsilon < \frac{p\alpha_1}{2}, \varepsilon < p\alpha_2\right\}.$$

From Lemma 3.1 and condition (4), there exists a unique global solution. For the sake of simplicity, write $\mathbf{x}(t) = \mathbf{x}(t, \zeta)$. For any $p \in [2, p_0)$, applying Itô's formula to $V(\mathbf{x}, t) = e^{\varepsilon t} |\mathbf{x}(t)|^p$, $\varepsilon \in (0, \varepsilon_p]$ and using the elemental inequalities, we have

$$
\mathscr{L} |\mathbf{x}(t)|^p + \varepsilon |\mathbf{x}(t)|^p
$$

$$
\leq \varepsilon |\mathbf{x}(t)|^p + \frac{p}{2} |\mathbf{x}(t)|^{p-2} \left[-2\kappa |\mathbf{x}(t)|^{n_1+1} - 2\hat{\kappa} |\mathbf{x}(t)|^2 + 2\bar{\kappa} \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{n_1+1} \, d\eta_1(\theta) \right.
$$

$$
+ 2\underline{\kappa} \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^2 \, d\eta_2(\theta) + \frac{p-1}{\rho_1} \left(\frac{\gamma^2}{\delta_1} |\mathbf{x}(t)|^{2n_2} + \frac{\bar{\gamma}^2}{1-\delta_1} \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{2n_2} \, d\eta_3(\theta) \right)
$$

$$
+ \frac{p-1}{\rho_2(1-\rho_1)} \left(\frac{\hat{\gamma}^2}{\delta_2} |\mathbf{x}(t)|^2 + \frac{\underline{\gamma}^2}{1-\delta_2} \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^2 \, d\eta_4(\theta) \right)
$$

$$
\left. + 2\kappa_0 e^{-\alpha_1 t} + \frac{p-1}{(1-\rho_1)(1-\rho_2)} \gamma_0^2 e^{-2\alpha_2 t} \right]
$$

$$
\leq \frac{p}{2} \left[-2 \left(\kappa - \bar{\kappa} \left(\frac{p-2}{p+n_1-1} + \frac{n_1+1}{p+n_1-1} e^{\varepsilon\tau} \right) \right) |\mathbf{x}(t)|^{p+n_1-1} \right.
$$

$$
- \left(2\hat{\kappa} - 2\underline{\kappa} \left(\frac{p-2}{p} + \frac{2}{p} e^{\varepsilon\tau} \right) - \frac{p-1}{\rho_2(1-\rho_1)} \left(\frac{\hat{\gamma}^2}{\delta_2} + \frac{\underline{\gamma}^2}{1-\delta_2} \left(\frac{p-2}{p} + \frac{2}{p} e^{\varepsilon\tau} \right) \right) \right.
$$

$$
\left. - 2\frac{p-2}{p} \varepsilon_1^p - \frac{p-1}{1-\rho_1} \frac{1}{1-\rho_2} \frac{p-2}{p} \varepsilon_2^p - \frac{2}{p}\varepsilon \right) |\mathbf{x}(t)|^p
$$

$$
\left. + \frac{p-1}{\rho_1} \left(\frac{\gamma^2}{\delta_1} + \frac{\bar{\gamma}^2}{1-\delta_1} \left(\frac{p-2}{p+2n_2-2} + \frac{2n_2}{p+2n_2-2} e^{\varepsilon\tau} \right) \right) |\mathbf{x}(t)|^{p+2n_2-2} \right]
$$

$$
+ 2\underline{\kappa} J_2 + \frac{p-1}{\rho_2(1-\rho_1)} \frac{\underline{\gamma}^2}{1-\delta_2} J_4 + \frac{p}{2} \frac{p-1}{\rho_1} \frac{\bar{\gamma}^2}{1-\delta_1} \frac{2n_2}{p+2n_2-2} J_3 + p\bar{\kappa} \frac{n_1+1}{p+n_1-1} J_1
$$

$$
+ 2\varepsilon_1^{-\frac{p(p-2)}{2}} \left(\kappa_0^{\frac{p}{2}} e^{-\frac{p}{2}\alpha_1 t} \right) + \frac{p-1}{1-\rho_1} \frac{1}{1-\rho_2} \varepsilon_2^{-\frac{p(p-2)}{2}} \left(\gamma_0^p e^{-p\alpha_2 t} \right)
$$

$$
\leq \frac{p}{2} \left[-2 \left(\kappa - \bar{\kappa} e^{\varepsilon\tau} \right) |\mathbf{x}(t)|^{p+n_1-1} - \left(2\hat{\kappa} - 2\underline{\kappa} e^{\varepsilon\tau} - \frac{p-1}{\rho_2(1-\rho_1)} \left(\frac{\hat{\gamma}^2}{\delta_2} + \frac{\underline{\gamma}^2}{1-\delta_2} e^{\varepsilon\tau} \right) \right.\right.
$$

$$
\left. - 2\frac{p-2}{p} \varepsilon_1^p - \frac{p-1}{1-\rho_1} \frac{1}{1-\rho_2} \frac{p-2}{p} \varepsilon_2^p - \frac{2}{p}\varepsilon \right) |\mathbf{x}(t)|^p + \frac{p-1}{\rho_1} \left(\frac{\gamma^2}{\delta_1} + \frac{\bar{\gamma}^2}{1-\delta_1} e^{\varepsilon\tau} \right)
$$

$$
\left. \times |\mathbf{x}(t)|^{p+2n_2-2} \right] + 2\underline{\kappa} J_2 + \frac{p-1}{\rho_2(1-\rho_1)} \frac{\underline{\gamma}^2}{1-\delta_2} J_4 + \frac{p}{2} \frac{p-1}{\rho_1} \frac{\bar{\gamma}^2}{1-\delta_1} \frac{2n_2}{p+2n_2-2} J_3
$$

$$
+ p\bar{\kappa} \frac{n_1+1}{p+n_1-1} J_1 + 2\varepsilon_1^{-\frac{p(p-2)}{2}} \left(\kappa_0^{\frac{p}{2}} e^{-\frac{p}{2}\alpha_1 t} \right) + \frac{p-1}{1-\rho_1} \frac{1}{1-\rho_2} \varepsilon_2^{-\frac{p(p-2)}{2}} \left(\gamma_0^p e^{-p\alpha_2 t} \right), \quad (13)
$$

where $\rho_1, \rho_2, \delta_1, \delta_2 \in (0,1)$, $J_1 = \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{p+n_1-1} \, d\eta_1(\theta) - e^{\varepsilon\tau} |\mathbf{x}(t)|^{p+n_1-1}$, $J_2 = \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^p \, d\eta_2(\theta) - e^{\varepsilon\tau} |\mathbf{x}(t)|^p$, $J_3 = \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{p+2n_2-2} \, d\eta_3(\theta) - e^{\varepsilon\tau} |\mathbf{x}(t)|^{p+2n_2-2}$, $J_4 = \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^p \, d\eta_4(\theta) - e^{\varepsilon\tau} |\mathbf{x}(t)|^p$. Setting $\delta_1 = \frac{\gamma}{\gamma+\bar{\gamma}}$, $\delta_2 = \frac{\hat{\gamma}}{\hat{\gamma}+\underline{\gamma}}$, we arrive at

$$
\mathscr{L} |\mathbf{x}(t)|^p + \varepsilon |\mathbf{x}(t)|^p
$$

$$
\leq \frac{p}{2} \left[-2 \left(\kappa - \bar{\kappa} e^{\varepsilon\tau} \right) |\mathbf{x}(t)|^{p+n_1-1} - \left(2\hat{\kappa} - 2\underline{\kappa} e^{\varepsilon\tau} - \frac{(p-1)(\hat{\gamma}+\underline{\gamma})(\hat{\gamma}+\underline{\gamma} e^{\varepsilon\tau})}{\rho_2(1-\rho_1)} \right.\right.
$$

$$
\left. - 2\frac{p-2}{p} \varepsilon_1^p - \frac{p-1}{1-\rho_1} \frac{1}{1-\rho_2} \frac{p-2}{p} \varepsilon_2^p - \frac{2}{p}\varepsilon \right) |\mathbf{x}(t)|^p
$$

$$+ \frac{(p-1)(\gamma + \bar{\gamma})(\gamma + \bar{\gamma}e^{\varepsilon\tau})}{\rho_1}\big|\mathbf{x}(t)\big|^{p+2n_2-2}\bigg] + p\bar{\kappa}\frac{n_1+1}{p+n_1-1}J_1 + (2\underline{\kappa})J_2$$

$$+ \frac{p}{2}\frac{(p-1)(\bar{\gamma} + \gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p+2n_2-2}J_3 + \frac{(p-1)(\underline{\gamma} + \hat{\gamma})\underline{\gamma}}{\rho_2(1-\rho_1)}J_4 + 2\varepsilon_1^{-\frac{p(p-2)}{2}}\kappa_0^{\frac{p}{2}}e^{-\frac{p}{2}\alpha_1 t}$$

$$+ \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\gamma_0^p e^{-p\alpha_2 t}. \tag{14}$$

Let $H(|\mathbf{x}(t)|) = (2\hat{\kappa} - 2\underline{\kappa}e^{\varepsilon\tau} - \frac{(p-1)((\hat{\gamma}+\underline{\gamma})(\hat{\gamma}+\gamma e^{\varepsilon\tau}))}{(1-\rho_1)} - \frac{2}{p}\varepsilon) + 2(\kappa - \bar{\kappa}e^{\varepsilon\tau})|\mathbf{x}(t)|^{n_1-1} - \frac{(p-1)(\gamma+\bar{\gamma})(\gamma+\bar{\gamma}e^{\varepsilon\tau})}{\rho_1} \times |\mathbf{x}(t)|^{2n_2-2}$. By the same technique as the function $G(|\mathbf{x}(t)|)$ in Theorem 3.1, we get that there exists $\bar{\bar{a}} > 0$ satisfying $\inf_{t\geq 0} H(|\mathbf{x}(t)|) > \bar{\bar{a}}$. So we choose ρ_2 which is sufficiently close to 1 and sufficiently small ε_1, ε_2 such that

$$\left(2\hat{\kappa} - 2\underline{\kappa}e^{\varepsilon\tau} - \frac{(p-1)((\hat{\gamma}+\underline{\gamma})(\hat{\gamma}+\gamma e^{\varepsilon\tau}))}{\rho_2(1-\rho_1)} - 2\frac{p-2}{p}\varepsilon_1^p - \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\frac{p-2}{p}\varepsilon_2^p - \frac{2}{p}\varepsilon\right)$$

$$+ 2\big(\kappa - \bar{\kappa}e^{\varepsilon\tau}\big)\big|\mathbf{x}(t)\big|^{n_1-1} - \frac{(p-1)(\gamma+\bar{\gamma})(\gamma+\bar{\gamma}e^{\varepsilon\tau})}{\rho_1}\big|\mathbf{x}(t)\big|^{2n_2-2} > \bar{\bar{a}}.$$

Hence, we have

$$\mathscr{L}\big|\mathbf{x}(t)\big|^p + \varepsilon\big|\mathbf{x}(t)\big|^p$$

$$\leq -\frac{p}{2}\bar{\bar{a}}\big|\mathbf{x}(t)\big|^p + p\bar{\kappa}\frac{n_1+1}{p+n_1-1}J_1 + 2\underline{\kappa}J_2 + \frac{p}{2}\frac{(p-1)(\bar{\gamma}+\gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p+2n_2-2}J_3$$

$$+ \frac{(p-1)(\underline{\gamma}+\hat{\gamma})\underline{\gamma}}{1-\rho_1}J_4 + 2\varepsilon_1^{-\frac{p(p-2)}{2}}\kappa_0^{\frac{p}{2}}e^{-\frac{p}{2}\alpha_1 t} + \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\gamma_0^p e^{-p\alpha_2 t}.$$

By virtue of the fact that

$$\int_0^t e^{\varepsilon s}J_i\,\mathrm{d}s = \int_0^t e^{\varepsilon s}\left(\int_{-\tau}^0 \big|\mathbf{x}(s+\theta)\big|^{w_i}\,\mathrm{d}\eta_i(\theta) - e^{\varepsilon\tau}\big|\mathbf{x}(s)\big|^{w_i}\right)\mathrm{d}s \leq e^{\varepsilon\tau}\int_{-\tau}^0\big|\mathbf{x}(s)\big|^{w_i}\,\mathrm{d}s$$

for $w_1 = p + n_1 - 1$, $w_3 = p + 2n_2 - 2$, $w_2 = w_4 = p$, respectively, and using Itô's formula to $V(\mathbf{x}, t) = e^{\varepsilon t}|\mathbf{x}(t)|^p$, $\varepsilon \in (0, \varepsilon_p]$, we have

$$e^{\varepsilon t}\big|\mathbf{x}(t)\big|^p = \big|\mathbf{x}(0)\big|^p + \int_0^t e^{\varepsilon s}\big(\mathscr{L}\big|\mathbf{x}(s)\big|^p + \varepsilon\big|\mathbf{x}(s)\big|^p\big)\,\mathrm{d}s + M(t)$$

$$\leq \big|\mathbf{x}(0)\big|^p - \frac{p}{2}\bar{\bar{a}}\int_0^t e^{\varepsilon s}\big|\mathbf{x}(s)\big|^p\,\mathrm{d}s + \frac{p}{2}\frac{(p-1)(\bar{\gamma}+\gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p+2n_2-2}\int_0^t e^{\varepsilon s}J_3\,\mathrm{d}s$$

$$+ p\bar{\kappa}\frac{n_1+1}{p+n_1-1}\int_0^t e^{\varepsilon s}J_1\,\mathrm{d}s + 2\underline{\kappa}\int_0^t e^{\varepsilon s}J_2\,\mathrm{d}s + \frac{(p-1)(\underline{\gamma}+\hat{\gamma})\underline{\gamma}}{1-\rho_1}\int_0^t e^{\varepsilon s}J_4\,\mathrm{d}s$$

$$+ 2\varepsilon_1^{-\frac{p(p-2)}{2}}\kappa_0^{\frac{p}{2}}\int_0^t e^{(\varepsilon-\frac{p}{2}\alpha_1)s}\,\mathrm{d}s$$

$$+ \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\gamma_0^p\int_0^t e^{(\varepsilon-p\alpha_2)s}\,\mathrm{d}s + M(t)$$

$$\leq \big|\mathbf{x}(0)\big|^p - \frac{p}{2}\bar{\bar{a}}\int_0^t e^{\varepsilon s}\big|\mathbf{x}(s)\big|^p\,\mathrm{d}s$$

$$+ \frac{p}{2}\frac{(p-1)(\bar{\gamma}+\gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p+2n_2-2}e^{\varepsilon\tau}\int_{-\tau}^0\big|\mathbf{x}(s)\big|^{p+2n_2-2}\,\mathrm{d}s$$

$$+ p\bar{\kappa}\frac{n_1 + 1}{p + n_1 - 1}e^{\varepsilon\tau}\int_{-\tau}^{0}\big|\mathbf{x}(s)\big|^{p+n_1-1}\,\mathrm{d}s$$

$$+ \left(2\underline{\kappa} + \frac{(p-1)(\underline{\gamma} + \hat{\gamma})\underline{\gamma}}{1 - \rho_1}\right)e^{\varepsilon\tau}\int_{-\tau}^{0}\big|\mathbf{x}(s)\big|^{p}\,\mathrm{d}s$$

$$+ 2\varepsilon_1^{-\frac{p(p-2)}{2}}\kappa_0^{\frac{p}{2}}\frac{1}{(\frac{p}{2}\alpha_1 - \varepsilon)} + \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\gamma_0^{p}\frac{1}{(p\alpha_2 - \varepsilon)} + M(t), \qquad (15)$$

where $M(t) = \int_{0}^{t} p e^{\varepsilon s}|\mathbf{x}(s)|^{p-1}g(\mathbf{x}_s, s)\,\mathrm{d}B(s)$ is a local martingale with the initial value $M(0) = 0$. Applying the nonnegative semi-martingale convergence theorem (see Lemma 2.5), we obtain that

$$\limsup_{t\to\infty} e^{\varepsilon t}\big|\mathbf{x}(t)\big|^{p} < \infty \quad \text{a.s.} \qquad (16)$$

Hence, there exists a finite positive random variable $\bar{\zeta}$ such that

$$\sup_{0\le t<\infty} e^{\varepsilon t}\big|\mathbf{x}(t)\big|^{p} < \bar{\zeta} \quad \text{a.s.} \qquad (17)$$

So, letting $\varepsilon \to \varepsilon_p$ in the inequality above, we claim that

$$\limsup_{t\to\infty}\frac{\log(|\mathbf{x}(t)|)}{t} \le -\frac{\varepsilon_p}{p}, \quad p \in [2, p_0) \text{ a.s.},$$

which is the required result (11).

Next, we prove the result (12). From (15), we get

$$E e^{\varepsilon t}\big|\mathbf{x}(t)\big|^{p} = E\big|\mathbf{x}(0)\big|^{p} + E\int_{0}^{t}e^{\varepsilon s}\big(\mathscr{L}\big|\mathbf{x}(s)\big|^{p} + \varepsilon\big|\mathbf{x}(s)\big|^{p}\big)\,\mathrm{d}s$$

$$\le E\big|\mathbf{x}(0)\big|^{p} - \frac{p}{2}\bar{\bar{a}}E\int_{0}^{t}e^{\varepsilon s}\big|\mathbf{x}(s)\big|^{p}\,\mathrm{d}s + \frac{p}{2}\frac{(p-1)(\bar{\gamma} + \gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p + 2n_2 - 2}e^{\varepsilon\tau}$$

$$\times \int_{-\tau}^{0}E\big|\mathbf{x}(s)\big|^{p+2n_2-2}\,\mathrm{d}s + p\bar{\kappa}\frac{n_1 + 1}{p + n_1 - 1}e^{\varepsilon\tau}\int_{-\tau}^{0}E\big|\mathbf{x}(s)\big|^{p+n_1-1}\,\mathrm{d}s$$

$$+ \left(2\underline{\kappa} + \frac{(p-1)(\underline{\gamma} + \hat{\gamma})\underline{\gamma}}{1 - \rho_1}\right)e^{\varepsilon\tau}\int_{-\tau}^{0}E\big|\mathbf{x}(s)\big|^{p}\,\mathrm{d}s + 2\varepsilon_1^{-\frac{p(p-2)}{2}}\kappa_0^{\frac{p}{2}}\frac{1}{(\frac{p}{2}\alpha_1 - \varepsilon)}$$

$$+ \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\gamma_0^{p}\frac{1}{(p\alpha_2 - \varepsilon)}$$

$$\le c_1, \qquad (18)$$

where $c_1 = \frac{p}{2}\frac{(p-1)(\bar{\gamma}+\gamma)\bar{\gamma}}{\rho_1}\frac{2n_2}{p+2n_2-2}e^{\varepsilon\tau}\int_{-\tau}^{0}E|\mathbf{x}(s)|^{p+2n_2-2}\,\mathrm{d}s + p\bar{\kappa}\frac{n_1+1}{p+n_1-1}e^{\varepsilon\tau}\int_{-\tau}^{0}E|\mathbf{x}(s)|^{p+n_1-1}\,\mathrm{d}s + (2\underline{\kappa} + \frac{(p-1)(\underline{\gamma}+\hat{\gamma})\underline{\gamma}}{1-\rho_1})e^{\varepsilon\tau}\int_{-\tau}^{0}E|\mathbf{x}(s)|^{p}\,\mathrm{d}s + 2\varepsilon_1^{-\frac{p(p-2)}{2}}\kappa_0^{\frac{p}{2}}\frac{1}{(\frac{p}{2}\alpha_1-\varepsilon)} + \frac{p-1}{1-\rho_1}\frac{1}{1-\rho_2}\varepsilon_2^{-\frac{p(p-2)}{2}}\gamma_0^{p}\frac{1}{(p\alpha_2-\varepsilon)} + E|\mathbf{x}(0)|^{p}$. This implies

$$E\big|\mathbf{x}(t)\big|^{p} \le c_1 e^{-\varepsilon t}, \quad \forall t > 0. \qquad (19)$$

So, letting $\varepsilon \to \varepsilon_p$ in the inequality above, we claim that, for $p \in [2, p_0)$,

$$\limsup_{t\to\infty}\frac{\log(E|\mathbf{x}(t)|^{p})}{t} \le -\varepsilon_p.$$

For any $p \in (0,2)$, using Hölder's inequality, we obtain

$$E|\mathbf{x}(t)|^p \le \left(E|\mathbf{x}(t)|^2\right)^{\frac{p}{2}} \le c_2^{\frac{p}{2}} e^{-\frac{p}{2}\varepsilon_2 t}, \tag{20}$$

where $c_2 = E|\mathbf{x}(0)|^2 + \frac{(\bar{\gamma}+\gamma)\bar{\gamma}}{\rho_1} e^{\varepsilon\tau} \int_{-\tau}^{0} E|\mathbf{x}(s)|^{2n_2}\,ds + (2\underline{\kappa} + \frac{(\gamma+\hat{\gamma})\gamma}{1-\rho_1})e^{\varepsilon\tau}\int_{-\tau}^{0} E|\mathbf{x}(s)|^2\,ds + 2\kappa_0 \frac{1}{(\alpha_1-\varepsilon)} + 2\bar{\kappa}e^{\varepsilon\tau}\int_{-\tau}^{0} E|\mathbf{x}(s)|^{n_1+1}\,ds + \frac{1}{1-\rho_1}\frac{1}{1-\rho_2}\gamma_0^2 \frac{1}{(2\alpha_2-\varepsilon)}$, which yields the assertion, for $p \in (0,2)$,

$$\limsup_{t\to\infty} \frac{\log(E|\mathbf{x}(t)|^p)}{t} \le -\frac{p}{2}\varepsilon_2,$$

which is the required result (12). □

Remark 5 Since Assumption 4.1 implies Assumption 2.2, the conditions of Theorem 4.1 can guarantee the pth moment asymptotic stability.

As a special case of Assumption 4.1, we obtain the following corollary directly from Theorem 4.1.

Corollary 4.1 *If Assumptions* 2.1, 2.2 *and condition* (4) *hold, and the bounded functions* $\xi_1(t)$, $\xi_2(t)$ *in Assumption* 2.2 *are* 0, *then for any* $p \in (0,p_0)$ *and any initial data* $\boldsymbol{\zeta} \in C$, *there is a unique global solution* $\mathbf{x}(t,\boldsymbol{\zeta})$ *of system* (1) *on* $t \ge -\tau$, *and* $\mathbf{x}(t,\boldsymbol{\zeta})$ *is almost surely exponentially stable and pth moment exponentially stable, namely*

$$\limsup_{t\to\infty} \frac{\log(|\mathbf{x}(t,\boldsymbol{\zeta})|)}{t} \le -\frac{\varepsilon_p'}{p}, \quad p \in [2,p_0)\ a.s.,$$

$$\limsup_{t\to\infty} \frac{\log(E|\mathbf{x}(t,\boldsymbol{\zeta})|^p)}{t} \le -\varepsilon_p', \quad p \in [2,p_0),$$

$$\limsup_{t\to\infty} \frac{\log(E|\mathbf{x}(t,\boldsymbol{\zeta})|^p)}{t} \le -\frac{p}{2}\varepsilon_2', \quad p \in (0,2),$$

where p_0 *is the same as defined in Theorem* 3.1, *and* ε_p' *is a positive constant which only depends on p but not on* ζ.

5 Generalized theory on asymptotic stability of SFDEs

Under Assumptions 2.2 and 4.1, we can see from the proofs of Theorems 3.1 and 4.1 that the positive lower-boundedness of the functions $G(|\mathbf{x}(t)|)$ and $H(|\mathbf{x}(t)|)$ play important roles, while condition (4) and the definition of p_0 are mainly used to ensure the positive lower-boundedness of the functions $G(|\mathbf{x}(t)|)$ and $H(|\mathbf{x}(t)|)$. So we wonder whether there exist more general assumptions and new conditions to guarantee these results. This is equivalent to finding more general assumptions and new conditions such that system (1) can still have the previous results, namely there exists a unique global solution almost surely, and the solution is asymptotically stable and, moreover, exponentially stable.

To make our theory more applicable, we replace the polynomial growth condition (Assumption 2.2) by the following general assumption.

Assumption 5.1 There are two functions $V \in C^{2,1}(R^n \times [-\tau, \infty); R_+)$, $W \in C(R^n \times [-\tau, \infty); R_+)$ and three probability measures η_i on $[-\tau, 0]$ with $\int_{-\tau}^{0} d\eta_i(\theta) = 1$ $(i = 1,2,3)$, as well

as some constants $a_0, a_1, a_2, a_3, k_1, k_2, k_3 \geq 0$, and $\beta_2 > \beta_1 > 1$, such that

$$\liminf_{|x| \to \infty, 0 \leq t < \infty} V(x, t) = \infty, \tag{21}$$

while for all $(\varphi, t) \in C([-\tau, 0]; R^n) \times R_+$,

$$\begin{aligned}
\mathscr{L} V(\boldsymbol{\varphi}, t) \leq {}& a_0 - a_1 W^{\beta_2}(\boldsymbol{\varphi}(0), t) \\
&+ k_1 \int_{-\tau}^0 W^{\beta_2}(\boldsymbol{\varphi}(\theta), t + \theta) \, d\eta_1(\theta) + a_2 W^{\beta_1}(\boldsymbol{\varphi}(0), t) \\
&+ k_2 \int_{-\tau}^0 W^{\beta_1}(\boldsymbol{\varphi}(\theta), t + \theta) \, d\eta_2(\theta) - a_3 W(\boldsymbol{\varphi}(0), t) \\
&+ k_3 \int_{-\tau}^0 W(\boldsymbol{\varphi}(\theta), t + \theta) \, d\eta_3(\theta).
\end{aligned} \tag{22}$$

Remark 6 Condition (21) is known as radial unboundedness in the literature [20].

Remark 7 Compared with [22], our condition (22), different from their condition (5.4), emphasizes that $\mathscr{L} V$ is dominated by a polynomial function of W. What is more, the probability measures η_i can also be weakened to any right-continuous nondecreasing functions (see [17]).

In the same way as [22], we can prove the following lemma.

Lemma 5.1 *If Assumptions* 2.1, 5.1 *and* $a_1 > k_1$ *hold, then for any initial data* $\boldsymbol{\zeta} \in C$, *there is a unique global solution* $\mathbf{x}(t, \boldsymbol{\zeta})$ *of system* (1) *on* $t \in [-\tau, \infty)$.

Now we examine the asymptotic stability and the exponential stability of system (1).

Theorem 5.1 *If Assumptions* 2.1, 5.1 *and the following condition* (23) *hold,*

$$a_1 - k_1 \geq a_2 + k_2 > 0, \qquad a_3 - k_3 > (a_1 - k_1) L', \qquad a_0 = 0, \tag{23}$$

except that (22) *is replaced by*

$$\begin{aligned}
\mathscr{L} V(\boldsymbol{\varphi}, t) \leq {}& a_0 - a_1 V^{\beta_2}(\boldsymbol{\varphi}(0), t) \\
&+ k_1 \int_{-\tau}^0 V^{\beta_2}(\boldsymbol{\varphi}(\theta), t + \theta) \, d\eta_1(\theta) + a_2 V^{\beta_1}(\boldsymbol{\varphi}(0), t) \\
&+ k_2 \int_{-\tau}^0 V^{\beta_1}(\boldsymbol{\varphi}(\theta), t + \theta) \, d\eta_2(\theta) - a_3 V(\boldsymbol{\varphi}(0), t) \\
&+ k_3 \int_{-\tau}^0 V(\boldsymbol{\varphi}(\theta), t + \theta) \, d\eta_3(\theta),
\end{aligned} \tag{24}$$

then for any initial data $\boldsymbol{\zeta} \in C$, *there is a unique global solution* $\mathbf{x}(t, \boldsymbol{\zeta})$ *of system* (1) *on* $t \in [-\tau, \infty)$, *and* $V(\mathbf{x}(t, \boldsymbol{\zeta}), t)$ *has the following properties*:
  (i) *If* $V(\mathbf{x}(t, \boldsymbol{\zeta}), t)$ *is uniformly continuous about t on* $[0, \infty)$, *then*

$$\lim_{t \to \infty} V(\mathbf{x}(t, \boldsymbol{\zeta}), t) = 0 \quad \text{a.s.} \tag{25}$$

(ii) *If $EV(\mathbf{x}(t, \zeta), t)$ is uniformly continuous about t on $[0, \infty)$, then*

$$\lim_{t \to \infty} EV\big(\mathbf{x}(t, \zeta), t\big) = 0, \tag{26}$$

where $L' = (\beta_2 - \beta_1)(\beta_2 - 1)^{-\frac{\beta_2 - 1}{\beta_2 - \beta_1}} (\beta_1 - 1)^{\frac{\beta_1 - 1}{\beta_2 - \beta_1}}$.

Proof The proof is similar to Theorem 3.1. From Lemma 5.1 and condition (23), there exists a unique global solution. For the sake of simplicity, write $\mathbf{x}(t) = \mathbf{x}(t, \zeta)$. Applying Itô's formula to $V(\mathbf{x}(t), t)$, we have

$$\mathscr{L}V(\mathbf{x}_t, t) \leq -(a_1 - k_1) V^{\beta_2}\big(\mathbf{x}(t), t\big) + (a_2 + k_2) V^{\beta_1}\big(\mathbf{x}(t), t\big) - (a_3 - k_3) V\big(\mathbf{x}(t), t\big)$$
$$+ k_1 J_1 + k_2 J_2 + k_3 J_3, \tag{27}$$

where $J_1 = \int_{-\tau}^0 V^{\beta_2}(\mathbf{x}(t + \theta), t + \theta) \, d\eta_1(\theta) - V^{\beta_2}(\mathbf{x}(t), t)$, $J_2 = \int_{-\tau}^0 V^{\beta_1}(\mathbf{x}(t + \theta), t + \theta) \, d\eta_2(\theta) - V^{\beta_1}(\mathbf{x}(t), t)$, $J_3 = \int_{-\tau}^0 V(\mathbf{x}(t + \theta), t + \theta) \, d\eta_3(\theta) - V(\mathbf{x}(t), t)$.

Let $Q(V(\mathbf{x}(t), t)) = (a_1 - k_1) V^{\beta_2 - 1}(\mathbf{x}(t), t) - (a_2 + k_2) V^{\beta_1 - 1}(\mathbf{x}(t), t) + (a_3 - k_3)$. By the same technique as function $G(|\mathbf{x}(t)|)$ in Theorem 3.1, there exists a constant $\bar{Q} > 0$ satisfying $\inf_{t \geq 0} Q(V(\mathbf{x}(t), t)) > \bar{Q}$. Hence, we have

$$\mathscr{L}V(\mathbf{x}_t, t) \leq -\bar{Q} V\big(\mathbf{x}(t), t\big) + k_1 J_1 + k_2 J_2 + k_3 J_3.$$

By virtue of the fact that

$$\int_0^t J_i \, ds = \int_0^t \left(\int_{-\tau}^0 V^{w_i'}\big(\mathbf{x}(s + \theta), s + \theta\big) \, d\eta_i(\theta) - V^{w_i'}\big(\mathbf{x}(s), s\big) \right) ds$$
$$\leq \int_{-\tau}^0 V^{w_i'}\big(\mathbf{x}(s), s\big) \, ds$$

for $w_1' = \beta_2$, $w_2' = \beta_1$, $w_3' = 1$, respectively, and using Itô's formula, we have

$$V\big(\mathbf{x}(t), t\big) = V\big(\mathbf{x}(0), 0\big) + \int_0^t \mathscr{L}V(\mathbf{x}_s, s) \, ds + M_1(t)$$
$$\leq V\big(\mathbf{x}(0), 0\big) - \bar{Q} \int_0^t V\big(\mathbf{x}(s), s\big) \, ds + k_1 \int_{-\tau}^0 V^{\beta_2}\big(\mathbf{x}(s), s\big) \, ds$$
$$+ k_2 \int_{-\tau}^0 V^{\beta_1}\big(\mathbf{x}(s), s\big) \, ds + k_3 \int_{-\tau}^0 V\big(\mathbf{x}(s), s\big) \, ds + M_1(t), \tag{28}$$

where $M_1(t) = \int_0^t V_x(\mathbf{x}(s), s) g(\mathbf{x}_s, s) \, dB(s)$ is a local martingale with the initial value $M_1(0) = 0$. Applying the nonnegative semi-martingale convergence theorem (see Lemma 2.5), we obtain that

$$\limsup_{t \to \infty} V\big(\mathbf{x}(t), t\big) < \infty \quad \text{a.s.,} \qquad \int_0^\infty V\big(\mathbf{x}(s), s\big) \, ds < \infty. \tag{29}$$

Hence, from Lemma 2.1 and the uniform continuity of $V(\mathbf{x}(t), t)$, we get the required result (25)

$$\lim_{t \to \infty} V\big(\mathbf{x}(t), t\big) = 0 \quad \text{a.s.}$$

Next, we prove the result (26). From (28), we get that

$$
\begin{aligned}
EV\big(\mathbf{x}(t),t\big) &= EV\big(\mathbf{x}(0),0\big) + E \int_0^t \mathscr{L}V(\mathbf{x}_s,s)\,\mathrm{d}s \\
&\le EV\big(\mathbf{x}(0),0\big) - \bar{Q}E \int_0^t V\big(\mathbf{x}(s),s\big)\,\mathrm{d}s + k_1 E \int_{-\tau}^0 V^{\beta_2}\big(\mathbf{x}(s),s\big)\,\mathrm{d}s \\
&\quad + k_2 E \int_{-\tau}^0 V^{\beta_1}\big(\mathbf{x}(s),s\big)\,\mathrm{d}s + k_3 E \int_{-\tau}^0 V\big(\mathbf{x}(s),s\big)\,\mathrm{d}s.
\end{aligned}
\tag{30}
$$

This implies

$$
E \int_0^\infty V\big(\mathbf{x}(s),s\big)\,\mathrm{d}s = \int_0^\infty EV\big(\mathbf{x}(s),s\big)\,\mathrm{d}s < \infty.
\tag{31}
$$

Hence, from Lemma 2.1 and the uniform continuity of $EV(\mathbf{x}(t),t)$, we have

$$
\lim_{t\to\infty} EV(\mathbf{x}(t),t) = 0,
$$

which is the required result (26). $\qquad\square$

Theorem 5.2 *If Assumptions* 2.1, 5.1 *and condition* (23) *hold, except that* (22) *is replaced by* (24), *then for any initial data* $\zeta \in C$, *there is a unique global solution* $\mathbf{x}(t,\zeta)$ *of system* (1) *on* $t \in [-\tau,\infty)$, *and* $V(\mathbf{x}(t,\zeta),t)$ *has the following properties*:

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{\log(V(\mathbf{x}(t,\zeta),t))}{t} \le -\bar{\varepsilon} \quad a.s., \\
&\limsup_{t\to\infty} \frac{\log(EV(\mathbf{x}(t,\zeta),t))}{t} \le -\bar{\varepsilon},
\end{aligned}
$$

where $\bar{\varepsilon}$ *is a positive constant which does not depend on* ζ.

Proof The proof is similar to Theorem 4.1. From condition (23), we obtain that there exists at least a sufficiently small positive constant ε satisfying $a_1 - k_1 e^{\varepsilon\tau} \ge a_2 + k_2 e^{\varepsilon\tau} > 0$, $(a_3 - a_1 L') - \varepsilon - (k_3 - k_1 L') e^{\varepsilon\tau} > 0$. So, by the continuity, define

$$
\bar{\varepsilon} = \sup\big\{\varepsilon > 0 : a_1 - k_1 e^{\varepsilon\tau} \ge a_2 + k_2 e^{\varepsilon\tau} > 0, \big(a_3 - a_1 L'\big) - \varepsilon - \big(k_3 - k_1 L'\big) e^{\varepsilon\tau} > 0\big\}.
$$

From Lemma 5.1 and condition (23), there exists a unique global solution. For the sake of simplicity, write $\mathbf{x}(t) = \mathbf{x}(t,\zeta)$. Applying Itô's formula to $e^{\varepsilon t}V(\mathbf{x}(t),t)$, $\varepsilon \in (0,\bar{\varepsilon}]$, we have

$$
\begin{aligned}
&\mathscr{L}V\big(\mathbf{x}_t,t\big) + \varepsilon V\big(\mathbf{x}(t),t\big) \\
&\quad \le -a_1 V^{\beta_2}\big(\mathbf{x}(t),t\big) + k_1 \int_{-\tau}^0 V^{\beta_2}\big(\mathbf{x}(t+\theta),t+\theta\big)\,\mathrm{d}\eta_1(\theta) \\
&\qquad + a_2 V^{\beta_1}\big(\mathbf{x}(t),t\big) + k_2 \int_{-\tau}^0 V^{\beta_1}\big(\mathbf{x}(t+\theta),t+\theta\big)\,\mathrm{d}\eta_2(\theta) \\
&\qquad - (a_3 - \varepsilon)V\big(\mathbf{x}(t),t\big) + k_3 \int_{-\tau}^0 V\big(\mathbf{x}(t+\theta),t+\theta\big)\,\mathrm{d}\eta_3(\theta)
\end{aligned}
$$

$$\leq -\left(a_1 - k_1 e^{\varepsilon\tau}\right) V^{\beta_2}\left(\mathbf{x}(t), t\right) + \left(a_2 + k_2 e^{\varepsilon\tau}\right) V^{\beta_1}\left(\mathbf{x}(t), t\right)$$

$$- \left(a_3 - k_3 e^{\varepsilon\tau} - \varepsilon\right) V\left(\mathbf{x}(t), t\right) + k_1 J_1 + k_2 J_2 + k_3 J_3, \tag{32}$$

where $J_1 = \int_{-\tau}^0 V^{\beta_2}(\mathbf{x}(t+\theta), t+\theta)\, d\eta_1(\theta) - e^{\varepsilon\tau} V^{\beta_2}(\mathbf{x}(t), t)$, $J_2 = \int_{-\tau}^0 V^{\beta_1}(\mathbf{x}(t+\theta), t+\theta)\, d\eta_2(\theta) - e^{\varepsilon\tau} V^{\beta_1}(\mathbf{x}(t), t)$, $J_3 = \int_{-\tau}^0 V(\mathbf{x}(t+\theta), t+\theta)\, d\eta_3(\theta) - e^{\varepsilon\tau} V(\mathbf{x}(t), t)$.

Let $R(V(\mathbf{x}(t), t)) = (a_1 - k_1 e^{\varepsilon\tau}) V^{\beta_2 - 1}(\mathbf{x}(t), t) - (a_2 + k_2 e^{\varepsilon\tau}) V^{\beta_1 - 1}(\mathbf{x}(t), t) + (a_3 - k_3 e^{\varepsilon\tau} - \varepsilon)$. Noting the definition of $\bar{\varepsilon}$ and by the same technique as the function $G(|\mathbf{x}(t)|)$ in Theorem 3.1, there exists a constant $\bar{R} > 0$ satisfying $\inf_{t \geq 0} R(V(\mathbf{x}(t), t)) > \bar{R}$. Therefore, we have

$$\mathscr{L} V(\mathbf{x}_t, t) + \varepsilon V\left(\mathbf{x}(t), t\right) \leq -\bar{R} V\left(\mathbf{x}(t), t\right) + k_1 J_1 + k_2 J_2 + k_3 J_3.$$

By virtue of the fact that

$$\int_0^t e^{\varepsilon s} J_i\, ds = \int_0^t e^{\varepsilon s}\left(\int_{-\tau}^0 V^{w_i'}\left(\mathbf{x}(s+\theta), s+\theta\right) d\eta_i(\theta) - e^{\varepsilon\tau} V^{w_i'}\left(\mathbf{x}(s), s\right)\right) ds$$

$$\leq e^{\varepsilon\tau} \int_{-\tau}^0 V^{w_i'}\left(\mathbf{x}(s), s\right) ds$$

for $w_1' = \beta_2$, $w_2' = \beta_1$, $w_3' = 1$, respectively, and using Itô's formula, we have

$$e^{\varepsilon t} V\left(\mathbf{x}(t), t\right) = V\left(\mathbf{x}(0), 0\right) + \int_0^t e^{\varepsilon s}\left(\mathscr{L} V(\mathbf{x}_s, s) + \varepsilon V\left(\mathbf{x}(s), s\right)\right) ds + M_2(t)$$

$$\leq V\left(\mathbf{x}(0), 0\right) - \bar{R} \int_0^t e^{\varepsilon s} V\left(\mathbf{x}(s), s\right) ds + k_1 e^{\varepsilon\tau} \int_{-\tau}^0 V^{\beta_2}\left(\mathbf{x}(s), s\right) ds$$

$$+ k_2 e^{\varepsilon\tau} \int_{-\tau}^0 V^{\beta_1}\left(\mathbf{x}(s), s\right) ds + k_3 e^{\varepsilon\tau} \int_{-\tau}^0 V\left(\mathbf{x}(s), s\right) ds + M_2(t), \tag{33}$$

where $M_2(t) = \int_0^t e^{\varepsilon s} V_x(\mathbf{x}(s), s) g(\mathbf{x}_s, s)\, dB(s)$ is a local martingale with the initial value $M_2(0) = 0$. Applying the nonnegative semi-martingale convergence theorem (see Lemma 2.5), we obtain that

$$\limsup_{t \to \infty} e^{\varepsilon t} V\left(\mathbf{x}(t), t\right) < \infty \quad \text{a.s.} \tag{34}$$

By the same technique as (16) in Theorem 3.1, letting $\varepsilon \to \bar{\varepsilon}$, we claim that

$$\limsup_{t \to \infty} \frac{\log(V(\mathbf{x}(t), t))}{t} \leq -\bar{\varepsilon} \quad \text{a.s.}$$

Next, we prove the other result. From (33), we get that

$$E e^{\varepsilon t} V\left(\mathbf{x}(t), t\right) \leq E V\left(\mathbf{x}(0), 0\right) - \bar{R} E \int_0^t e^{\varepsilon s} V\left(\mathbf{x}(s), s\right) ds + k_1 e^{\varepsilon\tau} E \int_{-\tau}^0 V^{\beta_2}\left(\mathbf{x}(s), s\right) ds$$

$$+ k_2 e^{\varepsilon\tau} E \int_{-\tau}^0 V^{\beta_1}\left(\mathbf{x}(s), s\right) ds + k_3 e^{\varepsilon\tau} E \int_{-\tau}^0 V\left(\mathbf{x}(s), s\right) ds$$

$$\leq c_4, \tag{35}$$

where $c_4 = EV(\mathbf{x}(0), 0) + k_1 e^{\varepsilon \tau} E \int_{-\tau}^{0} V^{\beta_2}(\mathbf{x}(s), s)\, \mathrm{d}s + k_2 e^{\varepsilon \tau} E \int_{-\tau}^{0} V^{\beta_1}(\mathbf{x}(s), s)\, \mathrm{d}s + k_3 e^{\varepsilon \tau} E \times \int_{-\tau}^{0} V(\mathbf{x}(s), s)\, \mathrm{d}s$. This implies

$$EV(\mathbf{x}(t), t) \le c_4 e^{-\varepsilon t}, \quad \forall t > 0. \tag{36}$$

So, letting $\varepsilon \to \bar{\varepsilon}$ in the inequality above, we claim that

$$\limsup_{t \to \infty} \frac{\log(EV(\mathbf{x}(t), t))}{t} \le -\bar{\varepsilon}. \qquad \square$$

6 Examples

In this section, we shall discuss some examples to illustrate our theory.

Example 1 Consider the scalar generalized stochastic logistic differential system of population extinction as follows:

$$\mathrm{d}\mathbf{x}(t) = \mathbf{x}(t)\big(-2 - \mathbf{x}(t)\big)\,\mathrm{d}t + \mathbf{x}(t)\big(1 + \mathbf{x}^{\frac{1}{3}}(t)\big)\,\mathrm{d}B(t) \tag{37}$$

with initial data $\mathbf{x}(0) > 0$, $\mathbf{x}(0) \in C$, where $B(t)$ is a scalar Brownian motion.

Similar to the proof of Theorem 2.1 of [23], using the Lyapunov function $V(\mathbf{x}) = \mathbf{x} - 1 - \log(\mathbf{x})$, we claim that the solution of system (37) is not only positive but also will not explode to infinity at any finite time; and we compute that

$$\mathbf{x}^T \mathbf{f}(\mathbf{x}_t, t) \le -\big|\mathbf{x}(t)\big|^3 - 2\big|\mathbf{x}(t)\big|^2,$$

$$\big|\mathbf{g}(\mathbf{x}_t, t)\big| \le \big|\mathbf{x}(t)\big| + \big|\mathbf{x}(t)\big|^{\frac{4}{3}}.$$

So, the parameters used in Assumption 2.2 are $\kappa = 1$, $\bar{\kappa} = 0$, $\hat{\kappa} = 2$, $\underline{\kappa} = 0$, $\gamma = 1$, $\bar{\gamma} = 0$, $\hat{\gamma} = 1$, $\underline{\gamma} = 0$, $n_1 = 2$, $n_2 = 4/3$. Through a simple computation we obtain that $L = (n_1 - 2n_2 + 1)(2n_2 - 2)^{\frac{2n_2 - 2}{n_1 - 2n_2 + 1}}(n_1 - 1)^{\frac{1 - n_1}{n_1 - 2n_2 + 1}} = \frac{4}{27}$, $p_0 = 1 + \big[\frac{(\gamma + \bar{\gamma})^2}{2(\kappa - \bar{\kappa})} + \frac{(\hat{\gamma} + \underline{\gamma})^2}{2(\hat{\kappa} - \underline{\kappa}) - 2L(\kappa - \bar{\kappa})}\big]^{-1} = 2.3$, and condition (4) is satisfied.

Through Theorem 3.1, we claim that the solution of system (37) is pth moment asymptotically stable for any $p \in (0, 2.3)$, namely

$$\lim_{t \to \infty} E\big|\mathbf{x}(t, \zeta)\big|^p = 0.$$

Through Corollary 4.1, we claim that the solution of system (37) is almost surely exponentially stable and pth moment exponentially stable, namely

$$\limsup_{t \to \infty} \frac{\log(|\mathbf{x}(t, \zeta)|)}{t} \le -\frac{\varepsilon_p'}{p}, \quad p \in [2, 2.3)\ \text{a.s.},$$

$$\limsup_{t \to \infty} \frac{\log(E|\mathbf{x}(t, \zeta)|^p)}{t} \le -\varepsilon_p', \quad p \in [2, 2.3),$$

$$\limsup_{t \to \infty} \frac{\log(E|\mathbf{x}(t, \zeta)|^p)}{t} \le -\frac{p}{2}\varepsilon_2', \quad p \in (0, 2),$$

where ε_p' is a positive constant which only depends on p but not on initial data $\mathbf{x}(0)$. Figure 1 shows a stochastic trajectory generated by the Euler scheme for time step $\triangle = 10^{-5}$ for stochastic system (37) on $[0, 1]$ with initial data $\zeta = 10$.

Figure 1 Generalized stochastic logistic differential system. The red curve shows a stochastic trajectory generated by the Euler scheme for time step $\Delta = 10^{-5}$ for one-dimensional stochastic system (37) on $[0,1]$ with initial data $\zeta = 10$ in Example 1.
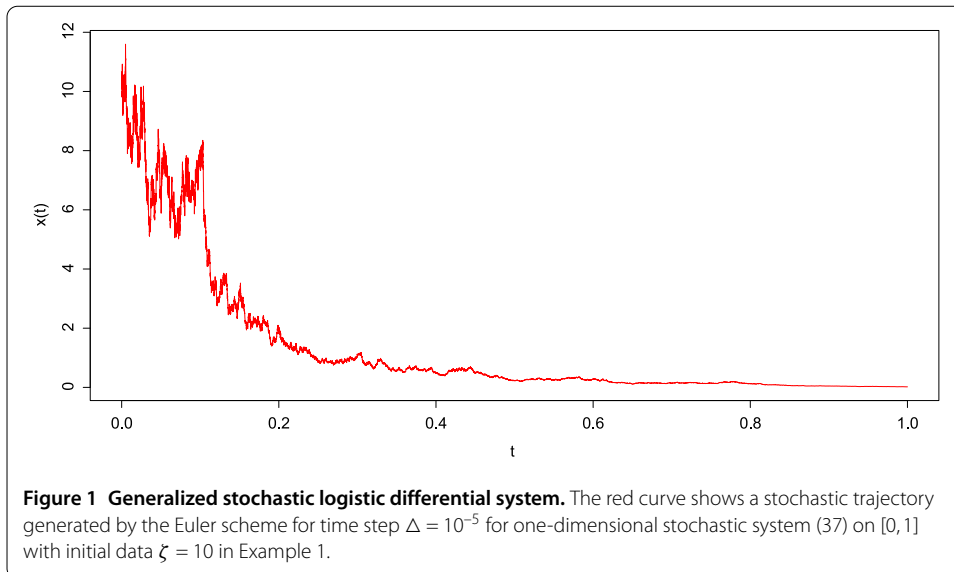
Example 2 Let us consider the scalar SFDE as follows:

$$d\mathbf{x}(t) = \left(-4\mathbf{x}^5(t) - 6\mathbf{x}(t) + 2D_1^3(\mathbf{x}_t) + 2D_2(\mathbf{x}_t) + \xi(t)\right)dt + \frac{1}{2}\left(\mathbf{x}^2(t) + \mathbf{x}(t)\right.$$

$$\left. + D_3^2(\mathbf{x}_t) + D_4(\mathbf{x}_t) + \xi(t)\right)dB(t) \tag{38}$$

with initial data $\{\mathbf{x}(\theta) : -\tau \le \theta \le 0\} = \zeta \in C$, where $B(t)$ is a scalar Brownian motion. Moreover, D_i are bounded linear operators from $C([-\tau, 0]; R)$ to R satisfying $|D_i(\mathbf{x}_t)| \le \int_{-\tau}^0 |\mathbf{x}(t + \theta)| \, d\eta_i(\theta)$, where $\eta_i(\cdot)$ are probability measures on $[-\tau, 0]$, $i = 1, 2, 3, 4$.

(1) If the function $\xi(t)$ is defined by

$$\xi(t) = \begin{cases} 1 - 4^n|t - n|, & t \in [n - \frac{1}{4^n}, n + \frac{1}{4^n}], n = 1, 2, 3, \ldots, \\ 0, & \text{others}, \end{cases}$$

then it is easy to show that $\xi(t)$ is bounded, and $\int_0^{+\infty} \xi(t) \, dt = 1$. We compute that

$$\mathbf{x}^T\mathbf{f}(\mathbf{x}_t, t) = -4|\mathbf{x}(t)|^6 + 2\mathbf{x}(t)^T D_1^3(\mathbf{x}_t) - 6|\mathbf{x}(t)|^2 + 2\mathbf{x}(t)^T D_2(\mathbf{x}_t) + \mathbf{x}(t)^T \xi(t)$$

$$\le -4|\mathbf{x}(t)|^6 + \int_{-\tau}^0 |\mathbf{x}(t + \theta)|^6 \, d\eta_1(\theta) - \frac{7}{2}|\mathbf{x}(t)|^2$$

$$+ \int_{-\tau}^0 |\mathbf{x}(t + \theta)|^2 \, d\eta_2(\theta) + \frac{1}{2}\xi^2(t),$$

$$|\mathbf{g}(\mathbf{x}_t, t)| \le \frac{1}{2}\left(|\mathbf{x}(t)|^2 + |\mathbf{x}(t)| + |D_3(\mathbf{x}_t)|^2 + |D_4(\mathbf{x}_t)| + \xi(t)\right)$$

$$\le \frac{1}{2}\left(|\mathbf{x}(t)|^2 + \int_{-\tau}^0 |\mathbf{x}(t + \theta)|^2 \, d\eta_3(\theta) + |\mathbf{x}(t)| + \int_{-\tau}^0 |\mathbf{x}(t + \theta)| \, d\eta_4(\theta) + \xi(t)\right).$$

So, we obtain that $L = (n_1 - 2n_2 + 1)(2n_2 - 2)^{\frac{2n_2 - 2}{n_1 - 2n_2 + 1}}(n_1 - 1)^{\frac{1 - n_1}{n_1 - 2n_2 + 1}} = (5 - 4 + 1)(4 - 2)^{\frac{2}{2}}(5 - 1)^{\frac{-4}{2}} = \frac{1}{4}$, $p_0 = 1 + [\frac{(\gamma + \bar{\gamma})^2}{2(\kappa - \bar{\kappa})} + \frac{(\hat{\gamma} + \gamma)^2}{2(\hat{\kappa} - \underline{\kappa}) - 2L(\kappa - \bar{\kappa})}]^{-1} = \frac{61}{19} > 2$, and condition (4) becomes $\frac{(\gamma + \bar{\gamma})^2}{2(\kappa - \bar{\kappa})} + \frac{(\hat{\gamma} + \gamma)^2}{2(\hat{\kappa} - \underline{\kappa}) - 2L(\kappa - \bar{\kappa})} = \frac{1}{2(4 - 1)} + \frac{1}{5 - \frac{3}{2}} = \frac{19}{42} < 1$, $(\hat{\kappa} - \underline{\kappa}) - L(\kappa - \bar{\kappa}) = \frac{5}{2} - \frac{3}{4} > 0$.

Through Theorem 3.1, we claim that for any initial data $\boldsymbol{\zeta} \in C$, the solution of system (38) is pth moment asymptotically stable for any $p \in (0, \frac{61}{19})$, namely

$$\lim_{t \to \infty} E|\mathbf{x}(t, \boldsymbol{\zeta})|^p = 0.$$

(2) If $\xi(t) = e^{-t}$, then it is also easy to show that $\xi(t)$ is bounded, and $\int_0^{+\infty} \xi(t)\,dt = 1$.

Through Theorem 3.1, we claim that for any initial data $\boldsymbol{\zeta} \in C$, the solution of system (38) is pth moment asymptotically stable for any $p \in (0, \frac{61}{19})$, namely

$$\lim_{t \to \infty} E|\mathbf{x}(t, \boldsymbol{\zeta})|^p = 0.$$

Through Theorem 4.1, we claim that for any initial data $\boldsymbol{\zeta} \in C$, the solution of system (38) is almost surely exponentially stable and pth moment exponentially stable, namely

$$\limsup_{t \to \infty} \frac{\log(|\mathbf{x}(t, \boldsymbol{\zeta})|)}{t} \leq -\frac{\varepsilon_p}{p}, \quad p \in \left[2, \frac{61}{19}\right) \text{ a.s.,}$$

$$\limsup_{t \to \infty} \frac{\log(E|\mathbf{x}(t, \boldsymbol{\zeta})|^p)}{t} \leq -\varepsilon_p, \quad p \in \left[2, \frac{61}{19}\right),$$

$$\limsup_{t \to \infty} \frac{\log(E|\mathbf{x}(t, \boldsymbol{\zeta})|^p)}{t} \leq -\frac{p}{2}\varepsilon_2, \quad p \in (0, 2),$$

where ε_p is defined by

$$\varepsilon_p = \sup\left\{ \varepsilon > 0 : \frac{1 + e^{\varepsilon\tau}}{4(4 - e^{\varepsilon\tau})} + \frac{1 + e^{\varepsilon\tau}}{10 - \frac{4}{p}\varepsilon - 3e^{\varepsilon\tau}} < \frac{1}{p-1}, \right.$$

$$\left. 10 - \frac{4}{p}\varepsilon - 3e^{\varepsilon\tau} > 0, 4 - e^{\varepsilon\tau} > 0, \varepsilon < p \right\}.$$

Appendix

Proof of Lemma 3.1 For any given initial data $\boldsymbol{\zeta} \in C$, by [1], Assumption 2.1 and conditions $\mathbf{f}(0, t) = 0$ and $\mathbf{g}(0, t) = 0$ guarantee a unique maximal local solution $\mathbf{x}(t, \boldsymbol{\zeta})$ to system (1) on $t \in [-\tau, \sigma_\infty)$, where σ_∞ is the explosion time. Let $k_0 > 0$ be sufficiently large satisfying $\|\zeta\| < k_0$. For each integer $k \geq k_0$, define the stopping time $\tau_k = \inf\{t \in [0, \sigma_\infty) : |x(t)| \geq k\}$. Obviously, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{t \to \infty} \tau_k$, so $\tau_\infty \leq \sigma_\infty$ a.s. If we can obtain that $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. For the sake of simplicity, write $\mathbf{x}(t) = \mathbf{x}(t, \boldsymbol{\zeta})$. Using Itô's formula to $V(\mathbf{x}, t) = |\mathbf{x}(t)|^2$, we obtain

$$\mathscr{L}V(\mathbf{x}, t) = 2\mathbf{x}^T(t)\mathbf{f}(\mathbf{x}_t, t) + |\mathbf{g}(\mathbf{x}_t, t)|^2$$

$$\leq 2\left(-\kappa|\mathbf{x}(t)|^{n_1+1} + \bar{\kappa}\int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{n_1+1}\,d\eta_1(\theta) - \hat{\kappa}|\mathbf{x}(t)|^2 \right.$$

$$+ \underline{\kappa}\int_{-\tau}^0 |\mathbf{x}(t+\theta)|^2\,d\eta_2(\theta) + \xi_1(t) \Bigg)$$

$$+ 5\left(\gamma^2|\mathbf{x}(t)|^{2n_2} + \bar{\gamma}^2\int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{2n_2}\,d\eta_3(\theta) \right.$$

$$+ \hat{\gamma}^2|\mathbf{x}(t)|^2 + \underline{\gamma}^2\int_{-\tau}^0 |\mathbf{x}(t+\theta)|^2\,d\eta_4(\theta) + \xi_2^2(t) \Bigg)$$

$$\leq -2(\kappa - \bar{\kappa})\big|\mathbf{x}(t)\big|^{n_1+1} + 5\big(\gamma^2 + \bar{\gamma}^2\big)\big|\mathbf{x}(t)\big|^{2n_2} + \big(5\big(\hat{\gamma}^2 + \underline{\gamma}^2\big) - 2(\hat{\kappa} - \underline{\kappa})\big)\big|\mathbf{x}(t)\big|^2$$
$$+ 2\bar{\kappa}\tilde{J}_1 + 2\underline{\kappa}\tilde{J}_2 + 5\bar{\gamma}^2\tilde{J}_3 + 5\underline{\gamma}^2\tilde{J}_4 + 2\xi_1(t) + 5\xi_2^2(t),$$

where $\tilde{J}_1 = \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{n_1+1}\,d\eta_1(\theta) - |\mathbf{x}(t)|^{n_1+1}$, $\tilde{J}_2 = \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^2\,d\eta_2(\theta) - |\mathbf{x}(t)|^2$, $\tilde{J}_3 = \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{2n_2}\,d\eta_3(\theta) - |\mathbf{x}(t)|^{2n_2}$, $\tilde{J}_4 = \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^2\,d\eta_4(\theta) - |\mathbf{x}(t)|^2$.

Noting $\kappa > \bar{\kappa}$, $n_1 + 1 > 2n_2 \geq 2$ and $|\mathbf{x}(t)| \geq 0$ for any $t \geq 0$, by Lemma 2.4, $R_1(|\mathbf{x}(t)|) = -2(\kappa - \bar{\kappa})|\mathbf{x}(t)|^{n_1+1} + 5(\gamma^2 + \bar{\gamma}^2)|\mathbf{x}(t)|^{2n_2} + (5(\hat{\gamma}^2 + \underline{\gamma}^2) - 2(\hat{\kappa} - \underline{\kappa}))|\mathbf{x}(t)|^2$, as a function of $|\mathbf{x}(t)|$, has a positive upper-boundedness, *i.e.*, there is a positive constant $\tilde{R}_1$ such that

$$R_1\big(\big|\mathbf{x}(t)\big|\big) = -2(\kappa - \bar{\kappa})\big|\mathbf{x}(t)\big|^{n_1+1} + 5\big(\gamma^2 + \bar{\gamma}^2\big)\big|\mathbf{x}(t)\big|^{2n_2} + \big(5\big(\hat{\gamma}^2 + \underline{\gamma}^2\big) - 2(\hat{\kappa} - \underline{\kappa})\big)\big|\mathbf{x}(t)\big|^2$$
$$\leq \tilde{R}_1. \tag{39}$$

(This technique has been used by many researchers, for example, [12].)

From Lemma 2.2, we have $\int_0^{+\infty}(5\xi_2^2(s) + 2\xi_1(s))\,ds < \infty$. In view of the fact that

$$\int_0^{t\wedge\tau_k} \tilde{J}_i\,ds = \int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 \big|\mathbf{x}(s+\theta)\big|^{w_i'}\,d\eta_i(\theta) - \big|\mathbf{x}(s)\big|^{w_i'}\right)ds \leq \int_{-\tau}^0 \big|\mathbf{x}(s)\big|^{w_i'}\,ds$$

for $w_1' = n_1 + 1$, $w_3' = 2n_2$, $w_2' = w_4' = 2$, we have that, for $t \geq 0$,

$$E\big|\mathbf{x}(t \wedge \tau_k)\big|^2 = E\big|\mathbf{x}(0)\big|^2 + E\int_0^{t\wedge\tau_k}\big[2\mathbf{x}^T(s)\mathbf{f}(\mathbf{x}_s,s) + \big|\mathbf{g}(\mathbf{x}_s,s)\big|\big]^2\,ds$$
$$\leq E\big|\mathbf{x}(0)\big|^2 + E\int_0^{t\wedge\tau_k}\big[\tilde{R}_1 + 2\bar{\kappa}\tilde{J}_1 + 2\underline{\kappa}\tilde{J}_2 + 5\bar{\gamma}^2\tilde{J}_3$$
$$+ 5\underline{\gamma}^2\tilde{J}_4 + 2\xi_1(s) + 5\xi_2^2(s)\big]\,ds$$
$$\leq E\big|\mathbf{x}(0)\big|^2 + \tilde{R}_1 E(t \wedge \tau_k) + 2\bar{\kappa}\int_{-\tau}^0\big|\mathbf{x}(s)\big|^{n_1+1}\,ds + 2\underline{\kappa}\int_{-\tau}^0\big|\mathbf{x}(s)\big|^2\,ds$$
$$+ 5\bar{\gamma}^2\int_{-\tau}^0\big|\mathbf{x}(s)\big|^{2n_2}\,ds + 5\underline{\gamma}^2\int_{-\tau}^0\big|\mathbf{x}(s)\big|^2\,ds + \int_0^{\infty}\big[2\xi_1(s) + 5\xi_2^2(s)\big]\,ds$$
$$\leq \bar{R}_1 + \tilde{R}_1 t,$$

where $\bar{R}_1 = E|\mathbf{x}(0)|^2 + 2\bar{\kappa}\int_{-\tau}^0 |\mathbf{x}(s)|^{n_1+1}\,ds + 2\underline{\kappa}\int_{-\tau}^0 |\mathbf{x}(s)|^2\,ds + 5\bar{\gamma}^2\int_{-\tau}^0 |\mathbf{x}(s)|^{2n_2}\,ds + 5\underline{\gamma}^2 \times \int_{-\tau}^0 |\mathbf{x}(s)|^2\,ds + \int_0^{\infty}[2\xi_1(s) + 5\xi_2^2(s)]\,ds$. Noting that

$$E\big|\mathbf{x}(t \wedge \tau_k)\big|^2 \geq E\big(\big|\mathbf{x}(t \wedge \tau_k)\big|^2 I_{\{\tau_k \leq t\}}\big) \geq k^2 P\{\tau_k \leq t\},$$

we get that

$$P\{\tau_\infty \leq t\} = \lim_{k\to\infty} P\{\tau_k \leq t\} \leq \lim_{k\to\infty}\frac{\bar{R}_1 + \tilde{R}_1 t}{k^2} = 0.$$

Since t is arbitrary, we must have that $\tau_\infty = \infty$ a.s. and this completes the proof. $\square$

Proof of Lemma 3.2 Since $\kappa > \bar{\kappa}$, the existence and uniqueness of the solution follows from Lemma 3.1, and there exists at least a sufficiently small positive constant ε satisfying $\kappa >$

$\bar{\kappa}e^{\varepsilon\tau}$. So, by the continuity, define $\varepsilon'' = \sup\{\varepsilon > 0 : \kappa > \bar{\kappa}e^{\varepsilon\tau}\}$. For the sake of simplicity, write $\mathbf{x}(t) = \mathbf{x}(t, \zeta)$, $\mathbf{x}_t = \mathbf{x}_t^\zeta$. For any $p \geq 2$, applying Itô's formula to $V(\mathbf{x}, t) = e^{\varepsilon t}|\mathbf{x}(t)|^p$, $\varepsilon \in (0, \varepsilon'']$, we arrive at

$$\mathscr{L}V(\mathbf{x}, t) = e^{\varepsilon t}\big(\mathscr{L}|\mathbf{x}(t)|^p + \varepsilon|\mathbf{x}(t)|^p\big)$$

$$\leq e^{\varepsilon t}\bigg[\frac{p}{2}|\mathbf{x}(t)|^{p-2}\big(2\mathbf{x}^T(r)\mathbf{f}(\mathbf{x}_r, r) + (p-1)|\mathbf{g}(\mathbf{x}_r, r)|^2\big) + \varepsilon|\mathbf{x}(t)|^p\bigg]$$

$$\leq e^{\varepsilon t}\frac{p}{2}|\mathbf{x}(t)|^{p-2}\bigg[2\bigg(-\kappa|\mathbf{x}(t)|^{n_1+1} + \bar{\kappa}\int_{-\tau}^0|\mathbf{x}(t+\theta)|^{n_1+1}\,\mathrm{d}\eta_1(\theta) - |\mathbf{x}(t)|^2$$

$$+\underline{\kappa}\int_{-\tau}^0|\mathbf{x}(t+\theta)|^2\,\mathrm{d}\eta_2(\theta) + \xi_1(t)\bigg) + 5(p-1)\bigg(\gamma^2|\mathbf{x}(t)|^{2n_2}$$

$$+\bar{\gamma}^2\int_{-\tau}^0|\mathbf{x}(t+\theta)|^{2n_2}\,\mathrm{d}\eta_3(\theta) + \hat{\gamma}^2|\mathbf{x}(t)|^2$$

$$+\underline{\gamma}^2\int_{-\tau}^0|\mathbf{x}(t+\theta)|^2\,\mathrm{d}\eta_4(\theta) + \xi_2^2(t)\bigg) + \frac{2}{p}\varepsilon|\mathbf{x}(t)|^2\bigg]$$

$$\leq \frac{p}{2}e^{\varepsilon t}\big[-2\big(\kappa - \bar{\kappa}e^{\varepsilon\tau}\big)|\mathbf{x}(t)|^{p+n_1-1} + o\big(|\mathbf{x}(t)|^{p+n_1-1}\big)\big] + 2\underline{\kappa}e^{\varepsilon t}J_2$$

$$+5(p-1)\underline{\gamma}^2e^{\varepsilon t}J_4 + \frac{p}{2}5(p-1)\bar{\gamma}^2\frac{2n_2}{p+2n_2-2}e^{\varepsilon t}J_3 + p\bar{\kappa}\frac{n_1+1}{p+n_1-1}e^{\varepsilon t}J_1$$

$$+2e^{\varepsilon t}\xi_1^{\frac{p}{2}}(t) + 5(p-1)e^{\varepsilon t}\xi_2^p(t),$$

where $J_1 = \int_{-\tau}^0|\mathbf{x}(t+\theta)|^{p+n_1-1}\,\mathrm{d}\eta_1(\theta) - e^{\varepsilon\tau}|\mathbf{x}(t)|^{p+n_1-1}$, $J_2 = \int_{-\tau}^0|\mathbf{x}(t+\theta)|^p\,\mathrm{d}\eta_2(\theta) - e^{\varepsilon\tau}|\mathbf{x}(t)|^p$, $J_3 = \int_{-\tau}^0|\mathbf{x}(t+\theta)|^{p+2n_2-2}\,\mathrm{d}\eta_3(\theta) - e^{\varepsilon\tau}|\mathbf{x}(t)|^{p+2n_2-2}$, $J_4 = \int_{-\tau}^0|\mathbf{x}(t+\theta)|^p\,\mathrm{d}\eta_4(\theta) - e^{\varepsilon\tau}|\mathbf{x}(t)|^p$.

Noting that $\kappa > \bar{\kappa}e^{\varepsilon\tau}$ and $|\mathbf{x}(t)| \geq 0$ for any $t \geq 0$, by Lemma 2.4 and the same technique as (39), $\bar{G}(|\mathbf{x}(t)|) = -2(\kappa - \bar{\kappa}e^{\varepsilon\tau})|\mathbf{x}(t)|^{p+n_1-1} + o(|\mathbf{x}(t)|^{p+n_1-1})$, as a function of $|\mathbf{x}(t)|$, has a positive upper-boundedness, *i.e.*, there is a positive constant $\underline{Q}$ such that

$$\bar{G}\big(|\mathbf{x}(t)|\big) = -2\big(\kappa - \bar{\kappa}e^{\varepsilon\tau}\big)|\mathbf{x}(t)|^{p+n_1-1} + o\big(|\mathbf{x}(t)|^{p+n_1-1}\big) \leq \underline{Q}. \tag{40}$$

In view of the fact that

$$\int_0^t e^{\varepsilon s}J_i\,\mathrm{d}s = \int_0^t e^{\varepsilon s}\bigg(\int_{-\tau}^0|\mathbf{x}(s+\theta)|^{w_i}\,\mathrm{d}\eta_i(\theta) - e^{\varepsilon\tau}|\mathbf{x}(s)|^{w_i}\bigg)\,\mathrm{d}s \leq e^{\varepsilon\tau}\int_{-\tau}^0|\mathbf{x}(s)|^{w_i}\,\mathrm{d}s$$

for $w_1 = p + n_1 - 1$, $w_3 = p + 2n_2 - 2$, $w_2 = w_4 = p$, we get that, for $t \geq 0$,

$$Ee^{\varepsilon t}|\mathbf{x}(t)|^p \leq E|\mathbf{x}(0)|^p + \frac{p}{2}\underline{Q}\int_0^t e^{\varepsilon s}\,\mathrm{d}s + p\bar{\kappa}\frac{n_1+1}{p+n_1-1}e^{\varepsilon\tau}\int_{-\tau}^0|\mathbf{x}(s)|^{p+n_1-1}\,\mathrm{d}s$$

$$+2\underline{\kappa}e^{\varepsilon\tau}\int_{-\tau}^0|\mathbf{x}(s)|^p\,\mathrm{d}s + \frac{p}{2}5(p-1)\bar{\gamma}^2\frac{2n_2}{p+2n_2-2}e^{\varepsilon\tau}\int_{-\tau}^0|\mathbf{x}(s)|^{p+2n_2-2}\,\mathrm{d}s$$

$$+5(p-1)\underline{\gamma}^2e^{\varepsilon\tau}\int_{-\tau}^0|\mathbf{x}(s)|^p\,\mathrm{d}s + \int_0^t e^{\varepsilon s}\big[2\xi_1^{\frac{p}{2}}(s) + 5(p-1)\xi_2^p(s)\big]\,\mathrm{d}s. \tag{41}$$

By virtue of the boundedness of $\xi_1(t)$, $\xi_2(t)$, there is a constant $\Psi > 0$ such that $\xi_1(t) \vee \xi_2(t) \leq \Psi$, which implies that

$$Ee^{\varepsilon t}|\mathbf{x}(t)|^p \leq c + \bigg[2\Psi^{\frac{p}{2}} + 5(p-1)\Psi^p + \frac{p}{2}\underline{Q}\bigg]\int_0^t e^{\varepsilon s}\,\mathrm{d}s,$$

where $c = p\bar{\kappa}\frac{n_1+1}{p+n_1-1}e^{\varepsilon\tau}\int_{-\tau}^{0}|\mathbf{x}(s)|^{p+n_1-1}\,ds + \frac{p}{2}5(p-1)\bar{\gamma}^2\frac{2n_2}{p+2n_2-2}e^{\varepsilon\tau}\int_{-\tau}^{0}|\mathbf{x}(s)|^{p+2n_2-2}\,ds + 2\underline{\kappa}e^{\varepsilon\tau} \times \int_{-\tau}^{0}|\mathbf{x}(s)|^{p}\,ds + 5(p-1)\underline{\gamma}^2e^{\varepsilon\tau}\int_{-\tau}^{0}|\mathbf{x}(s)|^{p}\,ds + E|\mathbf{x}(0)|^{p}$. This implies

$$E|\mathbf{x}(t)|^p \le ce^{-\varepsilon t} + \frac{2\Psi^{\frac{p}{2}} + 5(p-1)\Psi^p + \frac{p}{2}\underline{Q}}{\varepsilon}\left(1 - e^{-\varepsilon t}\right). \tag{42}$$

From the boundedness of initial data $\boldsymbol{\zeta} \in C$, we claim that for any $p \ge 2$, there exists a constant $M_p > 0$ such that $\sup_{-\tau \le t < +\infty} E|\mathbf{x}(t, \boldsymbol{\zeta})|^p \le M_p$. When $p \in (0, 2)$, using Hölder's inequality, we claim that

$$\sup_{-\tau \le t < \infty} E|\mathbf{x}(t)|^p \le \left(\sup_{-\tau \le t < \infty} E|\mathbf{x}(t)|^2\right)^{\frac{p}{2}} \le M_2^{\frac{p}{2}}. \qquad \square$$

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