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Global dynamics of two systems of exponential difference equations by Lyapunov function

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Abstract
In this paper, we study the boundedness character and persistence, existence and uniqueness of the positive equilibrium, local and global behavior, and rate of convergence of positive solutions of two systems of exponential difference equations. Furthermore, by constructing a discrete Lyapunov function, we obtain the global asymptotic stability of the unique positive equilibrium point. Some numerical examples are given to verify our theoretical results.
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Keywords: difference equations; boundedness; persistence; asymptotic behavior; Lyapunov function; rate of convergence

1 Introduction and preliminaries

Since difference equations and systems of difference equations containing exponential terms have many potential applications in biology, there are many papers dealing with such equations. See, for example the following.

El-Metwally *et al.* [1] have investigated the boundedness character, asymptotic behavior, periodicity nature of the positive solutions, and stability of the equilibrium point of the following population model:

$$x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n},$$

where the parameters α, β are positive numbers and the initial conditions are arbitrary non-negative real numbers.

Ozturk *et al.* [2] have investigated the boundedness, asymptotic behavior, periodicity, and stability of the positive solutions of the following difference equation:

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}},$$

where the parameters α, β, γ are positive numbers and the initial conditions are arbitrary non-negative numbers.

Bozkurt [3] has investigated the local and global behavior of positive solutions of the following difference equation:

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}},$$

where the parameters α, β, γ, and the initial conditions are arbitrary positive numbers.

Papaschinopoulos *et al.* [4] have investigated the boundedness, persistence, and asymptotic behavior of positive solutions of the following two directional interactive and invasive species models:

$$x_{n+1} = a + bx_{n-1}e^{-y_n}, \qquad y_{n+1} = c + dy_{n-1}e^{-x_n},$$

where the parameters a, b, c, d and the initial conditions are arbitrary positive numbers.

Papaschinopoulos *et al.* [5] have investigated the asymptotic behavior of the solutions of the following three systems of difference equations of exponential form:

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, & y_{n+1} &= \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}}, \\ x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, & y_{n+1} &= \frac{\delta + \epsilon e^{-x_n}}{\zeta + y_{n-1}}, \\ x_{n+1} &= \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, & y_{n+1} &= \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}}, \end{aligned}$$

where the parameters α, β, γ, δ, ϵ, δ are positive numbers and the initial conditions are arbitrary non-negative numbers.

Papaschinopoulos and Schinas [6] have investigated the asymptotic behavior of the positive solutions of the systems of the two difference equations:

$$\begin{aligned} x_{n+1} &= a + by_{n-1}e^{-y_n}, & y_{n+1} &= c + dx_{n-1}e^{-x_n}, \\ x_{n+1} &= a + by_{n-1}e^{-x_n}, & y_{n+1} &= c + dx_{n-1}e^{-y_n}, \end{aligned}$$

where the parameters a, b, c, d, and the initial conditions are arbitrary positive numbers.

Recently, Khan and Qureshi [7] have investigated the qualitative behavior of the following exponential type system of rational difference equations:

$$x_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha x_n + \beta x_{n-1}}, \qquad y_{n+1} = \frac{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}}}{\gamma_1 + \alpha_1 y_n + \beta_1 y_{n-1}}, \quad n = 0, 1, \ldots,$$

where α, β, γ, α_1, β_1, γ_1, and the initial conditions x_0, x_{-1}, y_0, y_{-1} are positive real numbers.

Motivated by the above studies, our aim in this paper is to investigate the qualitative behavior of positive solutions of the following two systems of exponential rational difference equations:

$$x_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}, \qquad y_{n+1} = \frac{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1}$$

and

$$x_{n+1} = \frac{\alpha e^{-x_n} + \beta e^{-x_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}, \qquad y_{n+1} = \frac{\alpha_1 e^{-y_n} + \beta_1 e^{-y_{n-1}}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}, \quad n = 0, 1, \ldots, \tag{2}$$

where the parameters α, β, γ, α_1, β_1, γ_1, and the initial conditions are positive real numbers.

More precisely, we investigate the boundedness character, persistence, existence and uniqueness of positive steady state, local asymptotic stability and global behavior of the unique positive equilibrium point, and rate of convergence of positive solutions of systems (1) and (2) which converge to its unique positive equilibrium point. For basic theory and applications of difference equations we refer the reader [8–16] and references therein.

Let us consider the four-dimensional discrete dynamical system of the form

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \qquad y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}), \quad n = 0, 1, \ldots, \tag{3}$$

where $f : I^2 \times J^2 \to I$ and $g : I^2 \times J^2 \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (3) is uniquely determined by the initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-1, 0\}$. Along with system (3) we consider the corresponding vector map $F = (f, x_n, g, y_n)$. An equilibrium point of (3) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{y}, \bar{y}), \qquad \bar{y} = g(\bar{x}, \bar{x}, \bar{y}, \bar{y}).$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1 Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (3).

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-1, 0\}$, $\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\| \cdot \|$ is the usual Euclidean norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called a global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) An equilibrium point $(\bar{x}, \bar{y})$ is called an asymptotic global attractor if it is a global attractor and stable.

Definition 2 Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map

$$F = (f, x_n, g, y_n),$$

where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (3) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$$

and F_J is the Jacobian matrix of system (3) about the equilibrium point $(\bar{x}, \bar{y})$.

Lemma 1 [17] *Consider the system $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$, where $\bar{X}$ is a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If any of the eigenvalue has a modulus greater than one, then $\bar{X}$ is unstable.*

The following result gives the rate of convergence of solutions of a system of difference equations:

$$X_{n+1} = \bigl(A + B(n)\bigr)X_n, \tag{4}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\bigl\|B(n)\bigr\| \to 0 \tag{5}$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\bigl\|(x, y)\bigr\| = \sqrt{x^2 + y^2}.$$

Proposition 1 (Perron's theorem) [18] *Suppose that condition* (5) *holds. If X_n is a solution of* (4), *then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \bigl(\|X_n\|\bigr)^{1/n}$$

or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

2 On the system $x_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}$, $y_{n+1} = \frac{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}$

In this section, we shall investigate the asymptotic behavior of system (1). Let $(\bar{x}, \bar{y})$ be the equilibrium point of system (1) then

$$\bar{x} = \frac{(\alpha + \beta)e^{-\bar{y}}}{\gamma + (\alpha + \beta)\bar{y}}, \qquad \bar{y} = \frac{(\alpha_1 + \beta_1)e^{-\bar{x}}}{\gamma_1 + (\alpha_1 + \beta_1)\bar{x}}.$$

To construct the corresponding linearized form of system (1), we consider the following transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \mapsto (f, f_1, g, g_1), \tag{6}$$

where

$$f = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}, \qquad f_1 = x_n, \qquad g = \frac{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}, \qquad g_1 = y_n.$$

The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (6) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & 0 & -\frac{\alpha(e^{-\bar{y}}+\bar{x})}{\gamma+(\alpha+\beta)\bar{y}} & -\frac{\beta(e^{-\bar{y}}+\bar{x})}{\gamma+(\alpha+\beta)\bar{y}} \\ 1 & 0 & 0 & 0 \\ -\frac{\alpha_1(e^{-\bar{x}}+\bar{y})}{\gamma_1+(\alpha_1+\beta_1)\bar{x}} & -\frac{\beta_1(e^{-\bar{x}}+\bar{y})}{\gamma_1+(\alpha_1+\beta_1)\bar{x}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

2.1 Boundedness and persistence

The following theorem shows that every positive solution $\{(x_n, y_n)\}$ of system (1) is bounded and persists.

Theorem 1 *Every positive solution* $\{(x_n, y_n)\}$ *of system* (1) *is bounded and persists.*

Proof Let $\{(x_n, y_n)\}$ be an arbitrary solution of (1). From (1), we have

$$x_n \leq \frac{\alpha + \beta}{\gamma} = U_1, \qquad y_n \leq \frac{\alpha_1 + \beta_1}{\gamma_1} = U_2, \quad n = 0, 1, 2, \ldots. \tag{7}$$

In addition from (1) and (7), we have

$$x_n \geq \frac{(\alpha + \beta) e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma + (\alpha + \beta)\frac{\alpha_1+\beta_1}{\gamma_1}} = L_1, \qquad y_n \geq \frac{(\alpha_1 + \beta_1) e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1 + (\alpha_1 + \beta_1)\frac{\alpha+\beta}{\gamma}} = L_2, \quad n = 2, 3, \ldots. \tag{8}$$

Hence, from (7) and (8), we get

$$L_1 \leq x_n \leq U_1, \qquad L_2 \leq y_n \leq U_2, \quad n = 3, 4, \ldots.$$

So the proof is complete. □

2.2 Existence of invariant set for solutions

Theorem 2 *Let* $\{(x_n, y_n)\}$ *be a positive solution of system* (1). *Then* $[L_1, U_1] \times [L_2, U_2]$ *is an invariant set for system* (1).

Proof For any positive solution $\{(x_n, y_n)\}$ of system (1) with initial conditions $x_0, x_{-1} \in [L_1, U_1]$, and $y_0, y_{-1} \in [L_2, U_2]$, we have

$$x_1 = \frac{\alpha e^{-y_0} + \beta e^{-y_{-1}}}{\gamma + \alpha y_0 + \beta y_{-1}} \leq \frac{\alpha + \beta}{\gamma}$$

and

$$x_1 = \frac{\alpha e^{-y_0} + \beta e^{-y_{-1}}}{\gamma + \alpha y_0 + \beta y_{-1}} \geq \frac{(\alpha + \beta) e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma + (\alpha + \beta)\frac{\alpha_1+\beta_1}{\gamma_1}}.$$

Moreover,

$$y_1 = \frac{\alpha_1 e^{-x_0} + \beta_1 e^{-x_{-1}}}{\gamma_1 + \alpha_1 x_0 + \beta_1 x_{-1}} \leq \frac{\alpha_1 + \beta_1}{\gamma_1}$$

and

$$y_1 = \frac{\alpha_1 e^{-x_0} + \beta_1 e^{-x_{-1}}}{\gamma_1 + \alpha_1 x_0 + \beta_1 x_{-1}} \geq \frac{(\alpha_1 + \beta_1) e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1 + (\alpha_1 + \beta_1)\frac{\alpha+\beta}{\gamma}}.$$

Hence, $x_1 \in [L_1, U_1]$ and $y_1 \in [L_2, U_2]$. Similarly, one can show that if $x_k \in [L_1, U_1]$ and $y_k \in [L_2, U_2]$, then $x_{k+1} \in [L_1, U_1]$ and $y_{k+1} \in [L_2, U_2]$. □

2.3 Existence and uniqueness of the positive equilibrium and local stability

Theorem 3 *Suppose that*

$$\eta < \bigl(\gamma \bigl(\gamma_1 + (\alpha_1 + \beta_1)L_1\bigr) + (\alpha + \beta)(\alpha_1 + \beta_1)e^{-L_1}\bigr)^2 \bigl(\gamma_1 + (\alpha_1 + \beta_1)L_1\bigr), \tag{9}$$

where

$$\begin{aligned}\eta = {} & (\alpha + \beta)(\alpha_1 + \beta_1) e^{-\frac{(\alpha_1+\beta_1)e^{-L_1}}{\gamma_1+(\alpha_1+\beta_1)L_1} - L_1} \bigl((\gamma + \alpha + \beta)\bigl(\gamma_1 + (\alpha_1 + \beta_1)U_1\bigr) \\ & + (\alpha + \beta)(\alpha_1 + \beta_1)e^{-L_1}\bigr)\bigl(\gamma_1 + (1 + U_1)(\alpha_1 + \beta_1)\bigr).\end{aligned}$$

Then system (1) *has a unique positive equilibrium point* $(\bar{x}, \bar{y})$ *in* $[L_1, U_1] \times [L_2, U_2]$.

Proof Consider the following system of equations:

$$x = \frac{(\alpha + \beta)e^{-y}}{\gamma + (\alpha + \beta)y}, \qquad y = \frac{(\alpha_1 + \beta_1)e^{-x}}{\gamma_1 + (\alpha_1 + \beta_1)x}.$$

Let $F(x) = \frac{(\alpha+\beta)e^{-f(x)}}{\gamma+(\alpha+\beta)f(x)} - x$, where $f(x) = \frac{(\alpha_1+\beta_1)e^{-x}}{\gamma_1+(\alpha_1+\beta_1)x}$ and $x \in [L_1, U_1]$. Then it follows that $F(L_1) = \frac{(\alpha+\beta)e^{-f(L_1)}}{\gamma+(\alpha+\beta)f(L_1)} - L_1$. Now, $F(L_1) > 0$ if and only if

$$(\alpha + \beta)e^{-\frac{(\alpha_1+\beta_1)e^{-L_1}}{\gamma_1+(\alpha_1+\beta_1)L_1}} > L_1 \left(\gamma + \frac{(\alpha + \beta)(\alpha_1 + \beta_1)e^{-L_1}}{\gamma_1 + (\alpha_1 + \beta_1)L_1}\right).$$

Furthermore, we have $F(U_1) = \frac{(\alpha+\beta)e^{-f(U_1)}}{\gamma+(\alpha+\beta)f(U_1)} - U_1$ where $f(U_1) = \frac{(\alpha_1+\beta_1)e^{-U_1}}{\gamma_1+(\alpha_1+\beta_1)U_1}$. It is easy to see that $F(U_1) < 0$ if and only if

$$(\alpha + \beta)e^{-\frac{(\alpha_1+\beta_1)e^{-U_1}}{\gamma_1+(\alpha_1+\beta_1)U_1}} < U_1 \left(\gamma + \frac{(\alpha + \beta)(\alpha_1 + \beta_1)e^{-U_1}}{\gamma_1 + (\alpha_1 + \beta_1)U_1}\right).$$

Hence, $F(x)$ has at least one positive solution in $[L_1, U_1]$. Furthermore, assume that condition (9) is satisfied, then one has

$$\frac{dF(x)}{dx} < -1 + \frac{\eta}{(\gamma(\gamma_1 + (\alpha_1 + \beta_1)L_1) + (\alpha + \beta)(\alpha_1 + \beta_1)e^{-L_1})^2(\gamma_1 + (\alpha_1 + \beta_1)L_1)} < 0.$$

Hence, $F(x) = 0$ has a unique positive solution in $[L_1, U_1]$. This completes the proof. □

Theorem 4 *Assume that*

$$(\alpha+\beta)(\alpha_1+\beta_1)\left(e^{-L_2}+U_1\right)\left(e^{-L_1}+U_2\right) < \left(\gamma+(\alpha+\beta)L_2\right)\left(\gamma_1+(\alpha_1+\beta_1)L_1\right).$$

Then the unique positive equilibrium point $(\bar{x},\bar{y})$ *in* $[L_1,U_1]\times[L_2,U_2]$ *of system* (1) *is locally asymptotically stable.*

Proof The characteristic polynomial of the Jacobian matrix $F_J(\bar{x},\bar{y})$ about the equilibrium point $(\bar{x},\bar{y})$ is given by

$$\begin{aligned} P(\lambda) = {} & \lambda^4 - \frac{\alpha\alpha_1(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})}\lambda^2 - \frac{(\alpha\beta_1+\beta\alpha_1)(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})}\lambda \\ & - \frac{\beta\beta_1(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})}. \end{aligned}$$

Let $\Phi(\lambda)=\lambda^2$ and

$$\begin{aligned} \Psi(\lambda) = {} & \frac{\alpha\alpha_1(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})}\lambda^2 + \frac{(\alpha\beta_1+\beta\alpha_1)(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})}\lambda \\ & + \frac{\beta\beta_1(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})}. \end{aligned}$$

Assume that $(\alpha+\beta)(\alpha_1+\beta_1)(e^{-L_2}+U_1)(e^{-L_1}+U_2) < (\gamma+(\alpha+\beta)L_2)(\gamma_1+(\alpha_1+\beta_1)L_1)$. Then one has

$$\begin{aligned} \left|\Psi(\lambda)\right| \le {} & \frac{\alpha\alpha_1(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})} + \frac{(\alpha\beta_1+\beta\alpha_1)(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})} \\ & + \frac{\beta\beta_1(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})} \\ = {} & (\alpha\alpha_1+\alpha\beta_1+\beta\alpha_1+\beta\beta_1)\left(\frac{(e^{-\bar{y}}+\bar{x})(e^{-\bar{x}}+\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})}\right) \\ < {} & \frac{(\alpha+\beta)(\alpha_1+\beta_1)(e^{-L_2}+U_1)(e^{-L_1}+U_2)}{(\gamma+(\alpha+\beta)L_2)(\gamma_1+(\alpha_1+\beta_1)L_1)} < 1. \end{aligned}$$

Then, by Rouche's theorem, $\Phi(\lambda)$ and $\Phi(\lambda)-\Psi(\lambda)$ have the same number of zeroes in an open unit disk $|\lambda|<1$. Hence, the unique positive equilibrium point $(\bar{x},\bar{y})$ in $[L_1,U_1]\times[L_2,U_2]$ of system (1) is locally asymptotically stable. □

2.4 Global character

Theorem 5 *If*

$$(\alpha+\beta)e^{-L_2} < \bar{x}\left(\gamma+(\alpha+\beta)L_2\right) \quad \textit{and} \quad (\alpha_1+\beta_1)e^{-L_1} < \bar{y}\left(\gamma_1+(\alpha_1+\beta_1)L_1\right), \tag{10}$$

then the unique positive equilibrium point $(\bar{x},\bar{y})$ *of system* (1) *is globally asymptotically stable.*

Proof Arranging as in [19], we consider the following discrete time analog of the Lyapunov function:

$$V_n = \bar{x}\left(\frac{x_n}{\bar{x}} - 1 - \ln\frac{x_n}{\bar{x}}\right) + \bar{y}\left(\frac{y_n}{\bar{y}} - 1 - \ln\frac{y_n}{\bar{y}}\right).$$

The nonnegativity of V_n follows from the following inequality:

$$x - 1 - \ln x \geq 0, \quad \forall x > 0.$$

Furthermore, we have

$$-\ln\left(\frac{x_{n+1}}{x_n}\right) = \ln\left(1 - \left(1 - \frac{x_n}{x_{n+1}}\right)\right) \leq -\frac{x_{n+1} - x_n}{x_{n+1}},$$
$$-\ln\left(\frac{y_{n+1}}{y_n}\right) = \ln\left(1 - \left(1 - \frac{y_n}{y_{n+1}}\right)\right) \leq -\frac{y_{n+1} - y_n}{y_{n+1}}.$$

Assume that (10) holds true, then it follows that

$$\begin{aligned}
V_{n+1} - V_n &= \bar{x}\left(\frac{x_{n+1}}{\bar{x}} - 1 - \ln\frac{x_{n+1}}{\bar{x}}\right) + \bar{y}\left(\frac{y_{n+1}}{\bar{y}} - 1 - \ln\frac{y_{n+1}}{\bar{y}}\right) - \bar{x}\left(\frac{x_n}{\bar{x}} - 1 - \ln\frac{x_n}{\bar{x}}\right) \\
&\quad - \bar{y}\left(\frac{y_n}{\bar{y}} - 1 - \ln\frac{y_n}{\bar{y}}\right) \\
&\leq (x_{n+1} - x_n)\left(1 - \frac{\bar{x}}{x_{n+1}}\right) + (y_{n+1} - y_n)\left(1 - \frac{\bar{y}}{y_{n+1}}\right) \\
&= (x_{n+1} - x_n)\left(\frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}} - \bar{x}(\gamma + \alpha y_n + \beta y_{n-1})}{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}\right) \\
&\quad + (y_{n+1} - y_n)\left(\frac{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}} - \bar{y}(\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1})}{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}}}\right) \\
&\leq (U_1 - L_1)\left(\frac{(\alpha + \beta)e^{-L_2} - \bar{x}(\gamma + (\alpha + \beta)L_2)}{(\alpha + \beta)e^{-L_2}}\right) \\
&\quad + (U_2 - L_2)\left(\frac{(\alpha_1 + \beta_1)e^{-L_1} - \bar{y}(\gamma_1 + (\alpha_1 + \beta_1)L_1)}{(\alpha_1 + \beta_1)e^{-L_1}}\right) \leq 0,
\end{aligned}$$

for all $n \geq 0$. Thus V_n is a non-increasing non-negative sequence. It follows that $\lim_{n\to\infty} V_n \geq 0$. Hence, we obtain $\lim_{n\to\infty}(V_{n+1} - V_n) = 0$. Then it follows that $\lim_{n\to\infty} x_{n+1} = \bar{x}$ and $\lim_{n\to\infty} y_{n+1} = \bar{y}$. Furthermore, $V_n \leq V_0$ for all $n \geq 0$, which shows that $(\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]$ is uniformly stable. Hence, the unique positive equilibrium point $(\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]$ of system (1) is globally asymptotically stable. □

2.5 Rate of convergence

In this section, we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of system (1).

Let $\{(x_n, y_n)\}$ be any solution of system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$. To find the error terms, note that

$$\begin{aligned}
x_{n+1} - \bar{x} &= \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{(\alpha + \beta)e^{-\bar{y}}}{\gamma + (\alpha + \beta)\bar{y}} \\
&= -\frac{\alpha e^{-y_n}(e^{y_n - \bar{y}} - 1)}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\beta e^{-y_{n-1}}(e^{y_{n-1} - \bar{y}} - 1)}{\gamma + \alpha y_n + \beta y_{n-1}}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\alpha \bar{x}(y_n - \bar{y})}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\beta \bar{x}(y_{n-1} - \bar{y})}{\gamma + \alpha y_n + \beta y_{n-1}} \\
= &- \frac{\alpha e^{-y_n}(y_n - \bar{y} + O_1((y_n - \bar{y})^2))}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\beta e^{-y_{n-1}}(y_{n-1} - \bar{y} + O_2((y_{n-1} - \bar{y})^2))}{\gamma + \alpha y_n + \beta y_{n-1}} \\
&- \frac{\alpha \bar{x}(y_n - \bar{y})}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\beta \bar{x}(y_{n-1} - \bar{y})}{\gamma + \alpha y_n + \beta y_{n-1}}.
\end{aligned}
$$

So,

$$
\begin{aligned}
x_{n+1} - \bar{x} = &- \frac{\alpha(e^{-y_n} + \bar{x})}{\gamma + \alpha y_n + \beta y_{n-1}}(y_n - \bar{y}) - \frac{\beta(e^{-y_{n-1}} + \bar{x})}{\gamma + \alpha y_n + \beta y_{n-1}}(y_{n-1} - \bar{y}) + O_1\big((y_n - \bar{y})^2\big) \\
&+ O_2\big((y_{n-1} - \bar{y})^2\big).
\end{aligned} \tag{11}
$$

Similarly,

$$
\begin{aligned}
y_{n+1} - \bar{y} = &- \frac{\alpha_1(e^{-x_n} + \bar{y})}{\gamma_1 + \alpha_1 x_n + \beta x_{n-1}}(x_n - \bar{x}) - \frac{\beta_1(e^{-x_{n-1}} + \bar{y})}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}(x_{n-1} - \bar{x}) + O_3\big((x_n - \bar{x})^2\big) \\
&+ O_4\big((x_{n-1} - \bar{x})^2\big).
\end{aligned} \tag{12}
$$

From (11) and (12), we have

$$
\begin{aligned}
x_{n+1} - \bar{x} &\approx - \frac{\alpha(e^{-y_n} + \bar{x})}{\gamma + \alpha y_n + \beta y_{n-1}}(y_n - \bar{y}) - \frac{\beta(e^{-y_{n-1}} + \bar{x})}{\gamma + \alpha y_n + \beta y_{n-1}}(y_{n-1} - \bar{y}), \\
y_{n+1} - \bar{y} &\approx - \frac{\alpha_1(e^{-x_n} + \bar{y})}{\gamma_1 + \alpha_1 x_n + \beta x_{n-1}}(x_n - \bar{x}) - \frac{\beta_1(e^{-x_{n-1}} + \bar{y})}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}(x_{n-1} - \bar{x}).
\end{aligned} \tag{13}
$$

Let $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$. Then system (13) can be represented as

$$
e_{n+1}^1 \approx a_n e_n^2 + b_n e_{n-1}^2, \qquad e_{n+1}^2 \approx c_n e_n^1 + d_n e_{n-1}^1,
$$

where

$$
\begin{aligned}
a_n &= - \frac{\alpha(e^{-y_n} + \bar{x})}{\gamma + \alpha y_n + \beta y_{n-1}}, \qquad b_n = - \frac{\beta(e^{-y_{n-1}} + \bar{x})}{\gamma + \alpha y_n + \beta y_{n-1}}, \\
c_n &= - \frac{\alpha_1(e^{-x_n} + \bar{y})}{\gamma_1 + \alpha_1 x_n + \beta x_{n-1}}, \qquad d_n = - \frac{\beta_1(e^{-x_{n-1}} + \bar{y})}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= - \frac{\alpha(e^{-\bar{y}} + \bar{x})}{\gamma + (\alpha + \beta)\bar{y}}, \qquad \lim_{n\to\infty} b_n = - \frac{\beta(e^{-\bar{y}} + \bar{x})}{\gamma + (\alpha + \beta)\bar{y}}, \\
\lim_{n\to\infty} c_n &= - \frac{\alpha_1(e^{-\bar{x}} + \bar{y})}{\gamma_1 + (\alpha_1 + \beta_1)\bar{x}}, \qquad \lim_{n\to\infty} d_n = - \frac{\beta_1(e^{-\bar{x}} + \bar{y})}{\gamma_1 + (\alpha_1 + \beta_1)\bar{x}}.
\end{aligned}
$$

So, the limiting system of the error terms can be written as

$$
\begin{pmatrix} e_{n+1}^1 \\ e_n^1 \\ e_{n+1}^2 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{\alpha(e^{-\bar{y}}+\bar{x})}{\gamma+(\alpha+\beta)\bar{y}} & -\frac{\beta(e^{-\bar{y}}+\bar{x})}{\gamma+(\alpha+\beta)\bar{y}} \\ 1 & 0 & 0 & 0 \\ -\frac{\alpha_1(e^{-\bar{x}}+\bar{y})}{\gamma_1+(\alpha_1+\beta_1)\bar{x}} & -\frac{\beta_1(e^{-\bar{x}}+\bar{y})}{\gamma_1+(\alpha_1+\beta_1)\bar{x}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-2}^2 \end{pmatrix},
$$

which is similar to the linearized system of (1) about the equilibrium point $(\bar{x},\bar{y})$. Using Proposition 1, one has the following result.

Theorem 6 *Assume that* $\{(x_n,y_n)\}$ *be a positive solution of system* (1) *such that* $\lim_{n\to\infty} x_n = \bar{x}$, *and* $\lim_{n\to\infty} y_n = \bar{y}$, *where* $\bar{x}$ *in* $[L_1,U_1]$ *and* $\bar{y}$ *in* $[L_2,U_2]$. *Then the error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-2}^2 \end{pmatrix}$$

of every solution of (1) *satisfies both of the following asymptotic relations*:

$$\lim_{n\to\infty}\left(\|e_n\|\right)^{\frac{1}{n}} = \left|\lambda_{1,2,3,4}F_J(\bar{x},\bar{y})\right|, \qquad \lim_{n\to\infty}\frac{\|e_{n+1}\|}{\|e_n\|} = \left|\lambda_{1,2,3,4}F_J(\bar{x},\bar{y})\right|,$$

where $\lambda_{1,2,3,4}F_J(\bar{x},\bar{y})$ *are the characteristic roots of Jacobian matrix* $F_J(\bar{x},\bar{y})$.

3 On the system $x_{n+1} = \frac{\alpha e^{-x_n}+\beta e^{-x_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}}$, $y_{n+1} = \frac{\alpha_1 e^{-y_n}+\beta_1 e^{-y_{n-1}}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}$

In this section, we shall investigate the asymptotic behavior of system (2). Let $(\bar{x},\bar{y})$ be the equilibrium point of system (2), then

$$\bar{x} = \frac{(\alpha+\beta)e^{-\bar{x}}}{\gamma+(\alpha+\beta)\bar{y}}, \qquad \bar{y} = \frac{(\alpha_1+\beta_1)e^{-\bar{y}}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}.$$

To construct the corresponding linearized form of system (2), we consider the following transformation:

$$(x_n,x_{n-1},y_n,y_{n-1}) \mapsto (f,f_1,g,g_1), \tag{14}$$

where

$$f = \frac{\alpha e^{-x_n}+\beta e^{-x_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}}, \qquad f_1 = x_n, \qquad g = \frac{\alpha_1 e^{-y_n}+\beta_1 e^{-y_{n-1}}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}, \qquad g_1 = y_n.$$

The Jacobian matrix about the fixed point $(\bar{x},\bar{y})$ under transformation (14) is given by

$$F_J(\bar{x},\bar{y}) = \begin{pmatrix} A & B & C & D \\ 1 & 0 & 0 & 0 \\ A_1 & B_1 & C_1 & D_1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $A = -\frac{\alpha e^{-\bar{x}}}{\gamma+(\alpha+\beta)\bar{y}}$, $B = -\frac{\beta e^{-\bar{x}}}{\gamma+(\alpha+\beta)\bar{y}}$, $C = -\frac{\alpha\bar{x}}{\gamma+(\alpha+\beta)\bar{y}}$, $D = -\frac{\beta\bar{x}}{\gamma+(\alpha+\beta)\bar{y}}$, $A_1 = -\frac{\alpha_1\bar{y}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}$, $B_1 = -\frac{\beta_1\bar{y}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}$, $C_1 = -\frac{\alpha_1 e^{-\bar{y}}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}$, $D_1 = -\frac{\beta_1 e^{-\bar{y}}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}$.

3.1 Boundedness and persistence

Theorem 7 *Every positive solution* $\{(x_n,y_n)\}$ *of system* (2) *is bounded and persists.*

Proof Let $\{(x_n, y_n)\}$ be an arbitrary solution of (2), then

$$x_n \leq \frac{\alpha + \beta}{\gamma} = U_1, \qquad y_n \leq \frac{\alpha_1 + \beta_1}{\gamma_1} = U_2, \quad n = 0, 1, \ldots. \tag{15}$$

From (2) and (15), we have

$$x_n \geq \frac{(\alpha + \beta)e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma + (\alpha + \beta)\frac{\alpha_1+\beta_1}{\gamma_1}} = L_1, \qquad y_n \geq \frac{(\alpha_1 + \beta_1)e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma_1 + (\alpha_1 + \beta_1)\frac{\alpha+\beta}{\gamma}} = L_2, \quad n = 2, 3, \ldots. \tag{16}$$

Hence, from (15) and (16), we get

$$L_1 \leq x_n \leq U_1, \qquad L_2 \leq y_n \leq U_2, \quad n = 3, 4, \ldots.$$

This proves the statement. □

Theorem 8 *Let $\{(x_n, y_n)\}$ be a positive solution of system* (2). *Then $[L_1, U_1] \times [L_2, U_2]$ is an invariant set for system* (2).

Proof Follows by induction. □

3.2 Existence and uniqueness and local stability

The following theorem shows the existence and uniqueness of the positive equilibrium point of system (2).

Theorem 9 *If*

$$(U_1 + 1)e^{-(L_1 + \frac{e^{-L_1}}{L_1} - \frac{\gamma}{\alpha+\beta})}\left(\frac{e^{-L_1}}{L_1} - \frac{\gamma}{\alpha + \beta} + 1\right) < L_1^2\left(\frac{e^{-L_1}}{L_1} - \frac{\gamma}{\alpha + \beta}\right)^2, \tag{17}$$

then system (2) *has a unique positive equilibrium point $(\bar{x}, \bar{y})$ in $[L_1, U_1] \times [L_2, U_2]$.*

Proof Consider the following system of algebraic equations:

$$x = \frac{(\alpha + \beta)e^{-x}}{\gamma + (\alpha + \beta)y}, \qquad y = \frac{(\alpha_1 + \beta_1)e^{-y}}{\gamma_1 + (\alpha_1 + \beta_1)x}. \tag{18}$$

Assume that $(x, y) \in [L_1, U_1] \times [L_2, U_2]$, then it follows from (18) that

$$y = \frac{e^{-x}}{x} - \frac{\gamma}{\alpha + \beta}, \qquad x = \frac{e^{-y}}{y} - \frac{\gamma_1}{\alpha_1 + \beta_1}.$$

Defining

$$F(x) = \frac{e^{-h(x)}}{h(x)} - \frac{\gamma_1}{\alpha_1 + \beta_1} - x,$$

where $h(x) = \frac{e^{-x}}{x} - \frac{\gamma}{\alpha+\beta}$, $x \in [L_1, U_1]$. It is easy to see that $F(L_1) = \frac{e^{-h(L_1)}}{h(L_1)} - \frac{\gamma_1}{\alpha_1+\beta_1} - L_1 > 0$ if and only if $e^{-(\frac{e^{-L_1}}{L_1} - \frac{\gamma}{\alpha+\beta})} > (\frac{e^{-L_1}}{L_1} - \frac{\gamma}{\alpha+\beta})(L_1 + \frac{\gamma_1}{\alpha_1+\beta_1})$. Also, $F(U_1) = \frac{e^{-h(U_1)}}{h(U_1)} - \frac{\gamma_1}{\alpha_1+\beta_1} - U_1 < 0$ if and only if $e^{-(\frac{e^{-U_1}}{U_1}-\frac{\gamma}{\alpha+\beta})} < (\frac{e^{-U_1}}{U_1}-\frac{\gamma}{\alpha+\beta})(U_1+\frac{\gamma_1}{\alpha_1+\beta_1})$. Hence, $F(x)$ has at least one positive solution in $[L_1, U_1]$. Furthermore, assume that condition (17) is satisfied, then one has

$$\begin{aligned} F'(x) &= \frac{(x+1)e^{-(x+\frac{e^{-x}}{x}-\frac{\gamma}{\alpha+\beta})}(\frac{e^{-x}}{x}-\frac{\gamma}{\alpha+\beta}+1)}{x^2(\frac{e^{-x}}{x}-\frac{\gamma}{\alpha+\beta})^2} - 1 \\ &\le \frac{(U_1+1)e^{-(L_1+\frac{e^{-L_1}}{L_1}-\frac{\gamma}{\alpha+\beta})}(\frac{e^{-L_1}}{L_1}-\frac{\gamma}{\alpha+\beta}+1)}{L_1^2(\frac{e^{-L_1}}{L_1}-\frac{\gamma}{\alpha+\beta})^2} - 1 < 0. \end{aligned}$$

Hence, $F(x) = 0$ has a unique positive solution in $[L_1, U_1]$. This completes the proof. □

Theorem 10 *If*

$$\begin{aligned} &(\alpha+\beta)(\alpha_1+\beta_1)\bigl(e^{-L_1-L_2}+U_1U_2\bigr) \\ &\quad < (1-U_1-U_2)\bigl(\gamma+(\alpha+\beta)L_2\bigr)\bigl(\gamma_1+(\alpha_1+\beta_1)L_1\bigr), \end{aligned} \tag{19}$$

then the unique positive equilibrium point $(\bar{x}, \bar{y})$ *of system* (2) *is locally asymptotically stable.*

Proof The characteristic equation of the Jacobian matrix $F_J(\bar{x}, \bar{y})$ about the equilibrium point $(\bar{x}, \bar{y})$ is given by

$$\lambda^4 - p_4\lambda^3 + p_3\lambda^2 + p_2\lambda + p_1 = 0,$$

where $p_4 = A + C_1$, $p_3 = AC_1 - B - A_1C - D_1$, $p_2 = AD_1 - A_1D + BC_1 - B_1C$, $p_1 = BD_1 - B_1D$. Assuming condition (19) one has

$$\begin{aligned} \sum_{i=1}^{4}|p_i| &= \frac{(\alpha+\beta)e^{-\bar{x}}}{\gamma+(\alpha+\beta)\bar{y}} + \frac{(\alpha_1+\beta_1)e^{-\bar{y}}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}} \\ &\quad + \frac{(\alpha\alpha_1+\alpha\beta_1+\alpha_1\beta+\beta\beta_1)e^{-\bar{x}-\bar{y}}+(\alpha\alpha_1+\alpha\beta_1+\alpha_1\beta+\beta\beta_1)\bar{x}\bar{y}}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})} \\ &= \bar{x}+\bar{y}+\frac{(\alpha+\beta)(\alpha_1+\beta_1)(e^{-\bar{x}-\bar{y}}+\bar{x}\bar{y})}{(\gamma+(\alpha+\beta)\bar{y})(\gamma_1+(\alpha_1+\beta_1)\bar{x})} \\ &< U_1+U_2+\frac{(\alpha+\beta)(\alpha_1+\beta_1)(e^{-L_1-L_2}+U_1U_2)}{(\gamma+(\alpha+\beta)L_2)(\gamma_1+(\alpha_1+\beta_1)L_1)} < 1. \end{aligned} \tag{20}$$

Therefore, inequality (20) and Remark 1.3.1 of reference [20] implies that the unique positive equilibrium point $(\bar{x}, \bar{y})$ of system (2) is locally asymptotically stable. This completes the proof. □

3.3 Global character

Theorem 11 *If*

$$(\alpha+\beta)e^{-L_1} < \bar{x}\bigl(\gamma+(\alpha+\beta)L_2\bigr) \quad \textit{and} \quad (\alpha_1+\beta_1)e^{-L_2} < \bar{y}\bigl(\gamma_1+(\alpha_1+\beta_1)L_1\bigr), \tag{21}$$

then the unique positive equilibrium point $(\bar{x}, \bar{y})$ *of system* (2) *is globally asymptotically stable.*

Proof Using arrangements for the proof of Theorem 5 and assume that (21) holds true, then

$$\begin{aligned} V_{n+1} - V_n \le{}& (U_1 - L_1)\left(\frac{(\alpha+\beta)e^{-L_1} - \bar{x}(\gamma + (\alpha+\beta)L_2)}{(\alpha+\beta)e^{-L_1}}\right) \\ &+ (U_2 - L_2)\left(\frac{(\alpha_1+\beta_1)e^{-L_2} - \bar{y}(\gamma_1 + (\alpha_1+\beta_1)L_1)}{(\alpha_1+\beta_1)e^{-L_2}}\right) \le 0, \end{aligned}$$

for all $n \ge 0$ so that $V_n \ge 0$ is a non-increasing sequence. It follows that $\lim_{n\to\infty} V_n \ge 0$. Hence, we obtain $\lim_{n\to\infty}(V_{n+1} - V_n) = 0$. It follows that $\lim_{n\to\infty} x_{n+1} = \bar{x}$ and $\lim_{n\to\infty} y_{n+1} = \bar{y}$. Furthermore, $V_n \le V_0$ for all $n \ge 0$, which implies that $(\bar{x},\bar{y}) \in [L_1, U_1] \times [L_2, U_2]$ is uniform stable. Hence, the unique positive equilibrium point $(\bar{x},\bar{y}) \in [L_1, U_1] \times [L_2, U_2]$ of system (2) is globally asymptotically stable. □

3.4 Rate of convergence

In this section we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of system (2).

Let $\{(x_n, y_n)\}$ be any solution of system (2) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$. To find the error terms,

$$\begin{aligned} x_{n+1} - \bar{x} &= \frac{\alpha e^{-x_n} + \beta e^{-x_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{(\alpha+\beta)e^{-\bar{x}}}{\gamma + (\alpha+\beta)\bar{y}} \\ &= -\frac{\alpha e^{-x_n}(e^{x_n - \bar{x}} - 1)}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\beta e^{-x_{n-1}(e^{x_{n-1}-\bar{x}}-1)}}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\alpha\bar{x}(y_n - \bar{y})}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\beta\bar{x}(y_{n-1} - \bar{y})}{\gamma + \alpha y_n + \beta y_{n-1}} \\ &= -\frac{\alpha e^{-x_n}(x_n - \bar{x} + O_1((x_n - \bar{x})^2))}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\beta e^{-x_{n-1}}(x_{n-1} - \bar{x} + O_2((x_{n-1} - \bar{x})^2))}{\gamma + \alpha y_n + \beta y_{n-1}} \\ &\quad - \frac{\alpha\bar{x}(y_n - \bar{y})}{\gamma + \alpha y_n + \beta y_{n-1}} - \frac{\beta\bar{x}(y_{n-1} - \bar{y})}{\gamma + \alpha y_n + \beta y_{n-1}}. \end{aligned}$$

So,

$$\begin{aligned} x_{n+1} - \bar{x} ={}& -\frac{\alpha e^{-x_n}}{\gamma + \alpha y_n + \beta y_{n-1}}(x_n - \bar{x}) - \frac{\beta e^{-x_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}(x_{n-1} - \bar{x}) \\ &- \frac{\alpha\bar{x}}{\gamma + \alpha y_n + \beta y_{n-1}}(y_n - \bar{y}) - \frac{\beta\bar{x}}{\gamma + \alpha y_n + \beta y_{n-1}}(y_{n-1} - \bar{y}) \\ &+ O_1\big((x_n - \bar{x})^2\big) + O_2\big((x_{n-1} - \bar{x})^2\big). \end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned} y_{n+1} - \bar{y} ={}& -\frac{\alpha_1\bar{y}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}(x_n - \bar{x}) - \frac{\beta_1\bar{y}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}(x_{n-1} - \bar{x}) \\ &- \frac{\alpha_1 e^{-y_n}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}(y_n - \bar{y}) - \frac{\beta_1 e^{-y_{n-1}}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}(y_{n-1} - \bar{y}) \\ &+ O_3\big((y_n - \bar{y})^2\big) + O_4\big((y_{n-1} - \bar{y})^2\big). \end{aligned} \tag{23}$$

From (22) and (23), we have

$$\left.\begin{aligned} x_{n+1}-\bar{x} &\approx -\tfrac{\alpha e^{-x_n}}{\gamma+\alpha y_n+\beta y_{n-1}}(x_n-\bar{x})-\tfrac{\beta e^{-x_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}}(x_{n-1}-\bar{x})\\ &\quad -\tfrac{\alpha\bar{x}}{\gamma+\alpha y_n+\beta y_{n-1}}(y_n-\bar{y})-\tfrac{\beta\bar{x}}{\gamma+\alpha y_n+\beta y_{n-1}}(y_{n-1}-\bar{y}),\\ y_{n+1}-\bar{y} &\approx -\tfrac{\alpha_1\bar{y}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}(x_n-\bar{x})-\tfrac{\beta_1\bar{y}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}(x_{n-1}-\bar{x})\\ &\quad -\tfrac{\alpha_1 e^{-y_n}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}(y_n-\bar{y})-\tfrac{\beta_1 e^{-y_{n-1}}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}(y_{n-1}-\bar{y}). \end{aligned}\right\} \tag{24}$$

Let $e_n^1 = x_n - \bar{x}$, and $e_n^2 = y_n - \bar{y}$. Then system (24) can be represented as

$$e_{n+1}^1 \approx a_n e_n^1 + b_n e_{n-1}^1 + c_n e_n^2 + d_n e_{n-1}^2, \qquad e_{n+1}^2 \approx e_n e_n^1 + f_n e_{n-1}^1 + g_n e_n^2 + h_n e_{n-1}^2,$$

where

$$\begin{aligned} a_n &= -\frac{\alpha e^{-x_n}}{\gamma+\alpha y_n+\beta y_{n-1}}, & b_n &= -\frac{\beta e^{-x_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}}, & c_n &= -\frac{\alpha\bar{x}}{\gamma+\alpha y_n+\beta y_{n-1}},\\ d_n &= -\frac{\beta\bar{x}}{\gamma+\alpha y_n+\beta y_{n-1}}, & e_n &= -\frac{\alpha_1\bar{y}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}, & f_n &= -\frac{\beta_1\bar{y}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}},\\ g_n &= -\frac{\alpha_1 e^{-y_n}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}, & h_n &= -\frac{\beta_1 e^{-y_{n-1}}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}}. \end{aligned}$$

Moreover,

$$\begin{aligned} \lim_{n\to\infty} a_n &= -\frac{\alpha e^{-\bar{x}}}{\gamma+(\alpha+\beta)\bar{y}}, & \lim_{n\to\infty} b_n &= -\frac{\beta e^{-\bar{x}}}{\gamma+(\alpha+\beta)\bar{y}},\\ \lim_{n\to\infty} c_n &= -\frac{\alpha\bar{x}}{\gamma+(\alpha+\beta)\bar{y}}, & \lim_{n\to\infty} d_n &= -\frac{\beta\bar{x}}{\gamma+(\alpha+\beta)\bar{y}}.\\ \lim_{n\to\infty} e_n &= -\frac{\alpha_1\bar{y}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}, & \lim_{n\to\infty} f_n &= -\frac{\beta_1\bar{y}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}},\\ \lim_{n\to\infty} g_n &= -\frac{\alpha_1 e^{-\bar{y}}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}, & \lim_{n\to\infty} h_n &= -\frac{\beta_1 e^{-\bar{y}}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}. \end{aligned}$$

So, the limiting system of the error terms can be written as

$$\begin{pmatrix} e_{n+1}^1\\ e_n^1\\ e_{n+1}^2\\ e_n^2 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha e^{-\bar{x}}}{\gamma+(\alpha+\beta)\bar{y}} & -\frac{\beta e^{-\bar{x}}}{\gamma+(\alpha+\beta)\bar{y}} & -\frac{\alpha\bar{x}}{\gamma+(\alpha+\beta)\bar{y}} & -\frac{\beta\bar{x}}{\gamma+(\alpha+\beta)\bar{y}}\\ 1 & 0 & 0 & 0\\ -\frac{\alpha_1\bar{y}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}} & -\frac{\beta_1\bar{y}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}} & -\frac{\alpha_1 e^{-\bar{y}}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}} & -\frac{\beta_1 e^{-\bar{y}}}{\gamma_1+(\alpha_1+\beta_1)\bar{x}}\\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^1\\ e_{n-1}^1\\ e_n^2\\ e_{n-2}^2 \end{pmatrix},$$

which is similar to linearized system of (2) about the equilibrium point $(\bar{x},\bar{y})$. Using Proposition 1, one has the following result.

Theorem 12 *Assume that* $\{(x_n,y_n)\}$ *is a positive solution of system* (2) *such that* $\lim_{n\to\infty} x_n = \bar{x}$, *and* $\lim_{n\to\infty} y_n = \bar{y}$, *where* $\bar{x}$ *in* $[L_1,U_1]$ *and* $\bar{y}$ *in* $[L_2,U_2]$. *Then the error vector*

$$e_n = \begin{pmatrix} e_n^1\\ e_{n-1}^1\\ e_n^2\\ e_{n-2}^2 \end{pmatrix}$$

of every solution of (2) *satisfies both of the following asymptotic relations*:

$$\lim_{n\to\infty}\left(\|e_n\|\right)^{\frac{1}{n}} = \left|\lambda_{1,2,3,4}F_J(\bar{x},\bar{y})\right|, \qquad \lim_{n\to\infty}\frac{\|e_{n+1}\|}{\|e_n\|} = \left|\lambda_{1,2,3,4}F_J(\bar{x},\bar{y})\right|,$$

where $\lambda_{1,2,3,4}F_J(\bar{x},\bar{y})$ *are the roots of the characteristic polynomial of* $F_J(\bar{x},\bar{y})$.

4 Examples

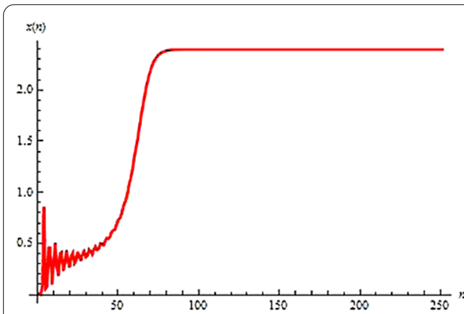
In order to verify our theoretical results and to support our theoretical discussions, we consider several interesting numerical examples. These examples represent different types of qualitative behavior of solutions of the systems of nonlinear difference equations (1) and (2). All plots in this section are drawn with Mathematica.

Example 1 Let $\alpha = 0.0005$, $\beta = 3{,}024$, $\gamma = 1{,}128$, $\alpha_1 = 1{,}005$, $\beta_1 = 1{,}025$, $\gamma_1 = 1{,}022$. Then system (1) can be written as

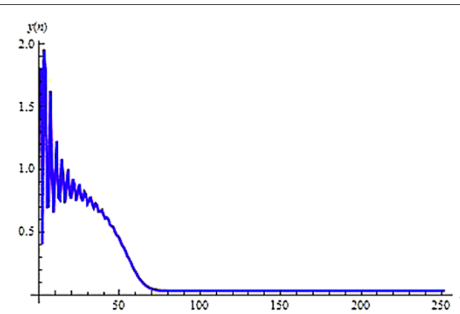
$$x_{n+1} = \frac{0.0005e^{-y_n} + 3{,}024e^{-y_{n-1}}}{1{,}128 + 0.0005y_n + 3{,}024y_{n-1}}, \qquad y_{n+1} = \frac{1{,}005e^{-x_n} + 1{,}025e^{-x_{n-1}}}{1{,}022 + 1{,}005x_n + 1{,}025x_{n-1}}, \tag{25}$$

with initial conditions $x_{-1} = 1.8$, $x_0 = 0.01$, $y_{-1} = 110.9$, $y_0 = 1.8$.

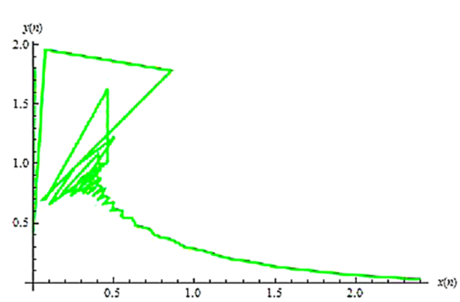
In this case the unique positive equilibrium point of system (25) is given by $(\bar{x},\bar{y}) = (2.39624, 0.0314033)$. Moreover, in Figure 1 the plot of x_n is shown in Figure 1(a), the plot of y_n is shown in Figure 1(b), and an attractor of system (25) is shown in Figure 1(c).



(a) Plot of x_n for system (25)



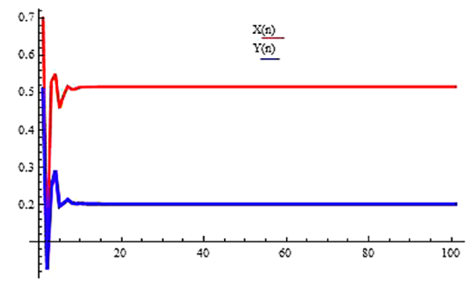
(b) Plot of y_n for system (25)

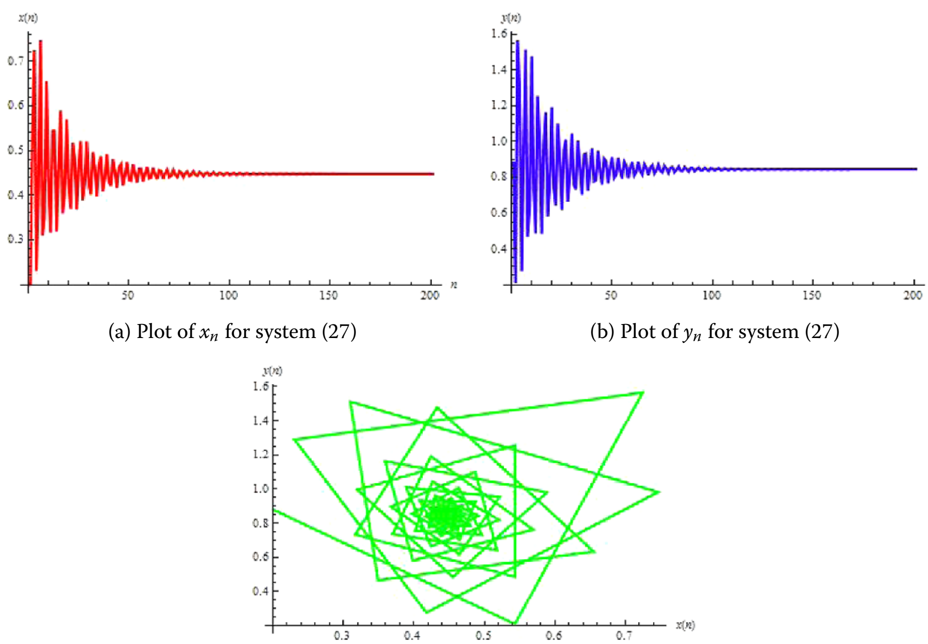


(c) An attractor of system (25)

Figure 1 Plots for system (25).

Figure 2 Plot of system (26).





(a) Plot of x_n for system (27)

(b) Plot of y_n for system (27)

(c) An attractor of system (27)

Figure 3 Plots for system (27).

Example 2 Let $\alpha = 230$, $\beta = 132$, $\gamma = 500$, $\alpha_1 = 111$, $\beta_1 = 135$, $\gamma_1 = 600$. Then system (1) can be written as

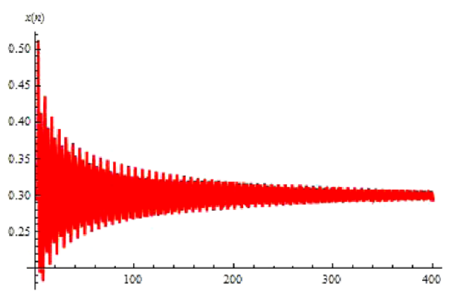
$$x_{n+1} = \frac{230e^{-y_n} + 132e^{-y_{n-1}}}{500 + 230y_n + 132y_{n-1}}, \qquad y_{n+1} = \frac{111e^{-x_n} + 135e^{-x_{n-1}}}{600 + 111x_n + 135x_{n-1}}, \quad n = 0, 1, \ldots, \tag{26}$$

with initial conditions $x_{-1} = 10.9$, $x_0 = 0.7$, $y_{-1} = 17.8$, $y_0 = 0.51$. The plot of system (26) is shown in Figure 2.

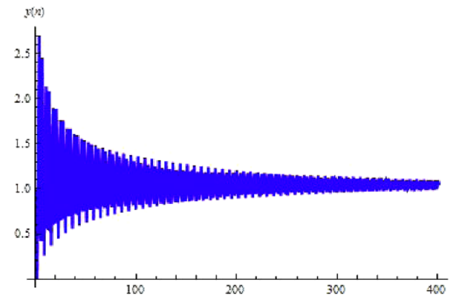
Example 3 Let $\alpha = 50$, $\beta = 1.4$, $\gamma = 30$, $\alpha_1 = 0.5$, $\beta_1 = 12.5$, $\gamma_1 = 0.8$. Then system (2) can be written as

$$x_{n+1} = \frac{50e^{-x_n} + 1.4e^{-x_{n-1}}}{30 + 50y_n + 1.4y_{n-1}}, \qquad y_{n+1} = \frac{0.5e^{-y_n} + 12.5e^{-y_{n-1}}}{0.8 + 0.5x_n + 12.5x_{n-1}}, \tag{27}$$

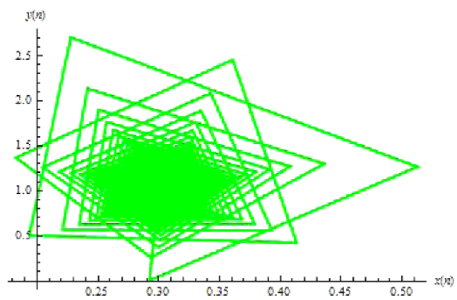
with initial conditions $x_{-1} = 0.8$, $x_0 = 0.2$, $y_{-1} = 1.8$, $y_0 = 0.88$.



(a) Plot of x_n for system (28)



(b) Plot of y_n for system (28)



(c) An attractor of system (28)

Figure 4 Plots for system (28).

In this case the unique positive equilibrium point of system (27) is given by $(\bar{x},\bar{y}) = (0.447606, 0.844287)$. Moreover, in Figure 3 the plot of x_n is shown in Figure 3(a), the plot of y_n is shown in Figure 3(b), and an attractor of system (27) is shown in Figure 3(c).

Example 4 Let $\alpha = 45$, $\beta = 1.4$, $\gamma = 66$, $\alpha_1 = 1.1$, $\beta_1 = 12.5$, $\gamma_1 = 0.5$. Then system (2) can be written as

$$x_{n+1} = \frac{45e^{-x_n} + 1.4e^{-x_{n-1}}}{66 + 45y_n + 1.4y_{n-1}}, \qquad y_{n+1} = \frac{1.1e^{-y_n} + 12.5e^{-y_{n-1}}}{0.5 + 1.1x_n + 12.5x_{n-1}}, \tag{28}$$

with initial conditions $x_{-1} = 21.7$, $x_0 = 0.3$, $y_{-1} = 2.8$, $y_0 = 0.98$.

In this case the unique positive equilibrium point of system (28) is given by $(\bar{x},\bar{y}) = (0.300252, 1.04429)$. Moreover, in Figure 4 the plot of x_n is shown in Figure 4(a), the plot of y_n is shown in Figure 4(b), and an attractor of system (28) is shown in Figure 4(c).

Example 5 Let $\alpha = 125$, $\beta = 4.5$, $\gamma = 112$, $\alpha_1 = 18$, $\beta_1 = 32$, $\gamma_1 = 0.09$. Then system (2) can be written as

$$x_{n+1} = \frac{125e^{-x_n} + 4.5e^{-x_{n-1}}}{112 + 125y_n + 4.5y_{n-1}}, \qquad y_{n+1} = \frac{18e^{-y_n} + 32e^{-y_{n-1}}}{0.09 + 18x_n + 32x_{n-1}}, \tag{29}$$

with initial conditions $x_{-1} = 20.7$, $x_0 = 0.3$, $y_{-1} = 0.9$, $y_0 = 0.2$.

In this case the unique positive equilibrium point of system (29) is given by $(\bar{x},\bar{y}) = (0.371312, 0.992951)$. Moreover, in Figure 5 the plot of x_n is shown in Figure 5(a), the plot of y_n is shown in Figure 5(b), and an attractor of system (29) is shown in Figure 5(c).

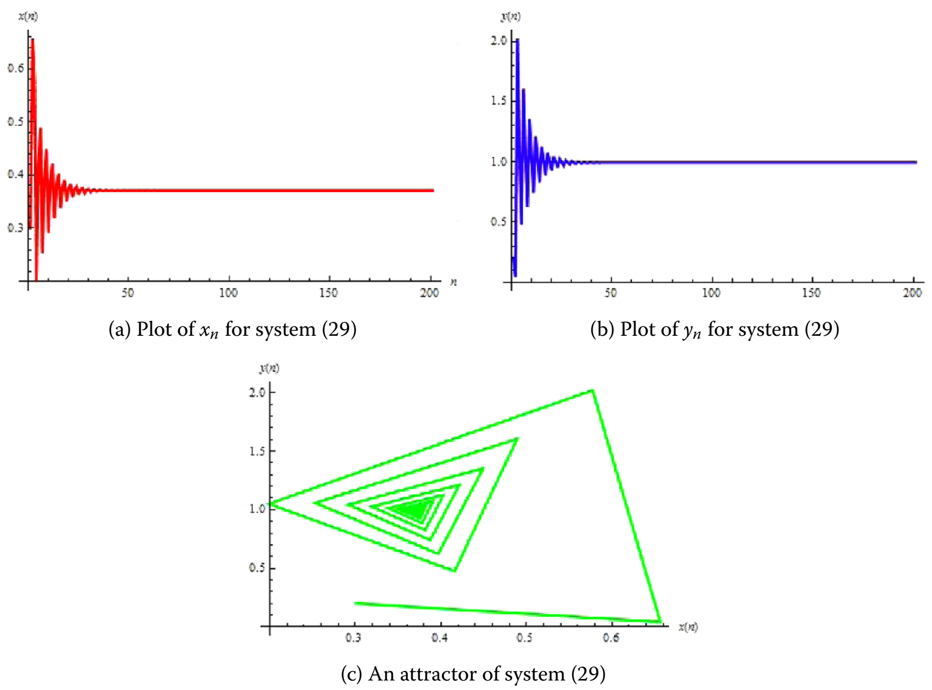


Figure 5 Plots for system (29).

Example 6 Let $\alpha = 1{,}245$, $\beta = 111$, $\gamma = 1{,}266$, $\alpha_1 = 1.1$, $\beta_1 = 32$, $\gamma_1 = 0.9$. Then system (2) can be written as

$$x_{n+1} = \frac{1{,}245e^{-x_n} + 111e^{-x_{n-1}}}{1{,}266 + 1{,}245y_n + 111y_{n-1}}, \qquad y_{n+1} = \frac{1.1e^{-y_n} + 32e^{-y_{n-1}}}{0.9 + 1.1x_n + 32x_{n-1}}, \tag{30}$$

with initial conditions $x_{-1} = 111.7$, $x_0 = 0.3$, $y_{-1} = 0.8$, $y_0 = 0.8$.

In this case the unique positive equilibrium point of system (30) is unstable. Moreover, in Figure 6 the plot of x_n is shown in Figure 6(a), the plot of y_n is shown in Figure 6(b), and a phase portrait of system (30) is shown in Figure 6(c).

Example 7 Let $\alpha = 1{,}145$, $\beta = 201$, $\gamma = 1{,}266$, $\alpha_1 = 15$, $\beta_1 = 232$, $\gamma_1 = 3$. Then system (2) can be written as

$$x_{n+1} = \frac{1{,}145e^{-x_n} + 201e^{-x_{n-1}}}{1{,}266 + 1{,}145y_n + 201y_{n-1}}, \qquad y_{n+1} = \frac{15e^{-y_n} + 232e^{-y_{n-1}}}{3 + 15x_n + 232x_{n-1}}, \tag{31}$$

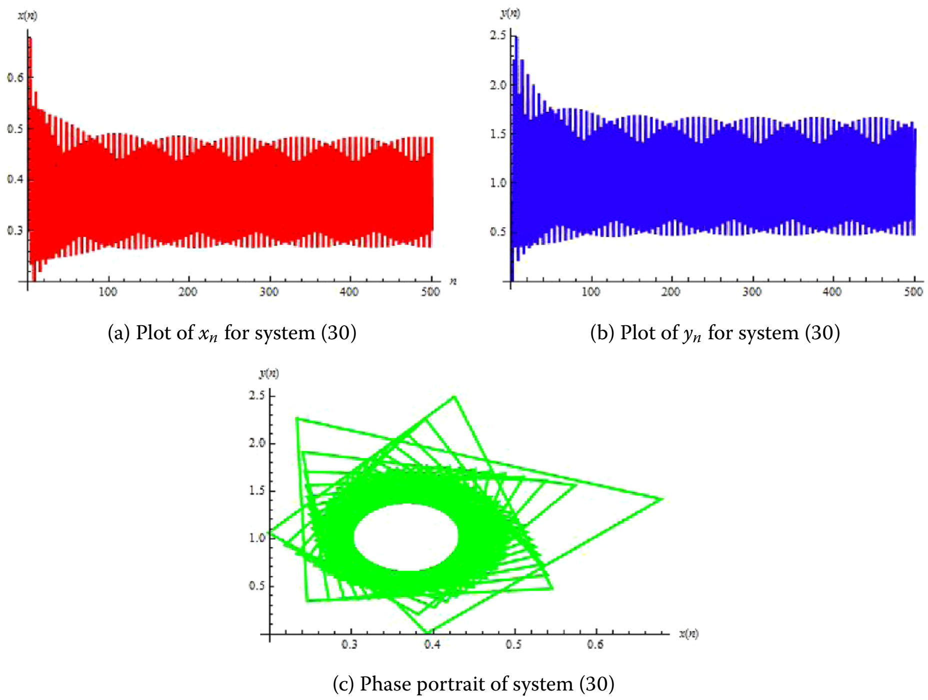
with initial conditions $x_{-1} = 0.9$, $x_0 = 0.5$, $y_{-1} = 0.001$, $y_0 = 0.8$.

In this case the unique positive equilibrium point of system (31) is unstable. Moreover, in Figure 7 the plot of x_n is shown in Figure 7(a), the plot of y_n is shown in Figure 7(b), and a phase portrait of system (31) is shown in Figure 7(c).

Example 8 Let $\alpha = 2{,}145$, $\beta = 166$, $\gamma = 2{,}566$, $\alpha_1 = 16$, $\beta_1 = 252$, $\gamma_1 = 3$. Then system (2) can be written as

$$x_{n+1} = \frac{2{,}145e^{-x_n} + 166e^{-x_{n-1}}}{2{,}566 + 2{,}145y_n + 21y_{n-1}}, \qquad y_{n+1} = \frac{16e^{-y_n} + 252e^{-y_{n-1}}}{3 + 16x_n + 252x_{n-1}}, \tag{32}$$

with initial conditions $x_{-1} = 2.9$, $x_0 = 0.3$, $y_{-1} = 0.02$, $y_0 = 1.7$.

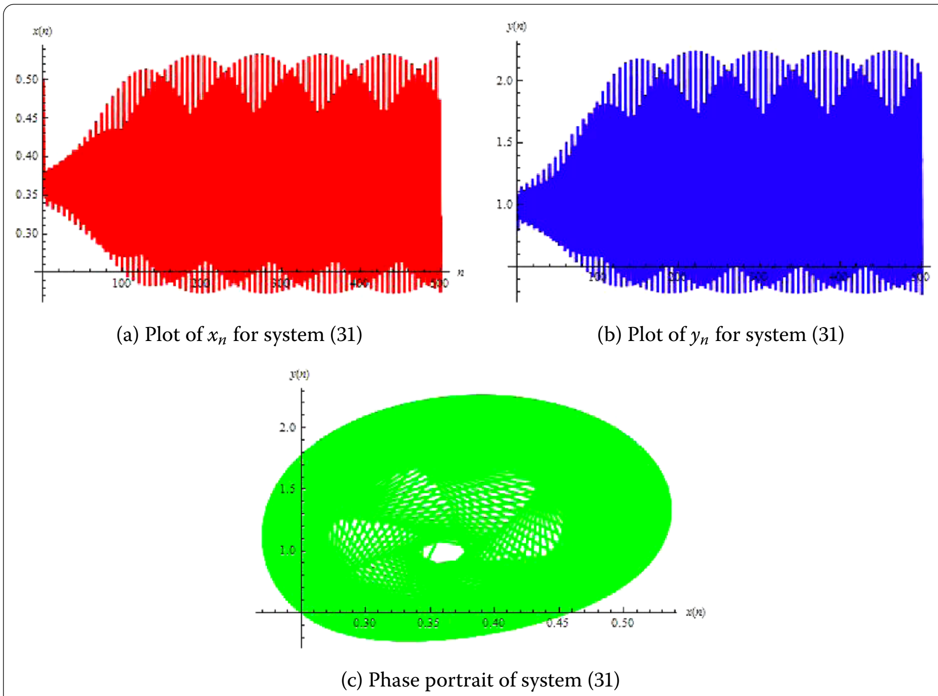


(a) Plot of x_n for system (30)

(b) Plot of y_n for system (30)

(c) Phase portrait of system (30)

Figure 6 Plots for system (30).

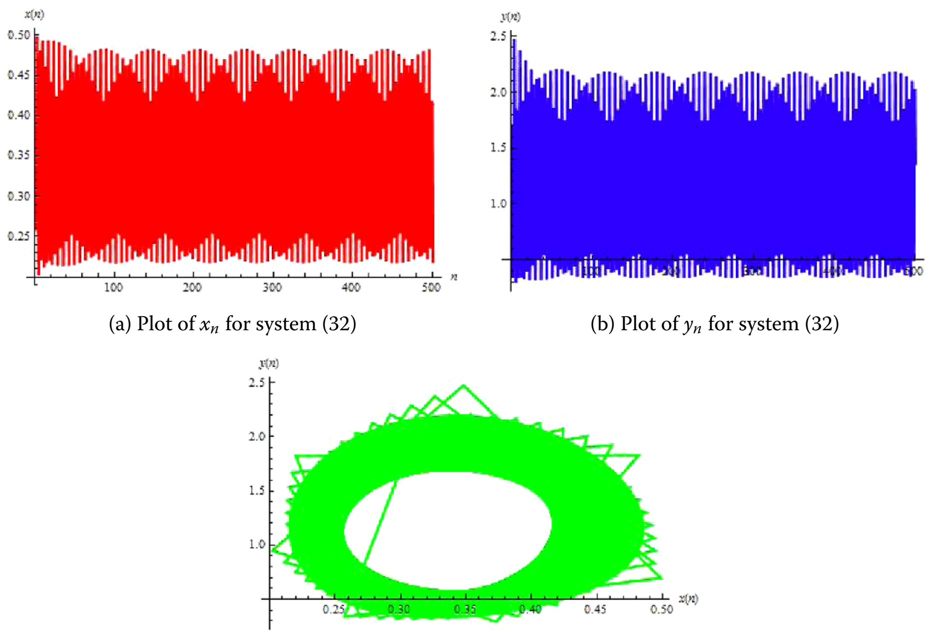


(a) Plot of x_n for system (31)

(b) Plot of y_n for system (31)

(c) Phase portrait of system (31)

Figure 7 Plots for system (31).



(a) Plot of x_n for system (32)

(b) Plot of y_n for system (32)

(c) Phase portrait of system (32)

Figure 8 Plots for system (32).

In this case the unique positive equilibrium point of system (32) is unstable. Moreover, in Figure 8 the plot of x_n is shown in Figure 8(a), the plot of y_n is shown in Figure 8(b), and a phase portrait of system (32) is shown in Figure 8(c).

5 Conclusion

This work is related to the qualitative behavior of some systems of exponential rational difference equations. We have investigated the existence and uniqueness of the positive steady state of system (1) and (2). For all positive values of the parameters the boundedness and persistence of positive solutions are proved. Moreover, we have shown that the unique positive equilibrium point of system (1) and (2) is locally as well as globally asymptotically stable under certain parametric conditions. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which parametric conditions lead to these long-term behaviors. By constructing a discrete Lyapunov function, we have obtained the global asymptotic stability of the positive equilibrium of (1) and (2). Finally, some illustrative examples are provided to support our theoretical discussion. First two examples show that the unique positive equilibrium point of system (1) is stable with different parametric values. Meanwhile Examples 3, 4, and 5 show that the unique positive equilibrium point of system (2) is stable whereas the last three examples show that the unique positive equilibrium point of system (2) is unstable with suitable parametric choices.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author carried out the proof of the main results and approved the final manuscript.

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