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Existence of solutions for nonlinear singular fractional differential equations with fractional derivative condition

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Abstract

In this paper, we investigate a boundary value problem for singular fractional differential equations with a fractional derivative condition. The existence and uniqueness of solutions are obtained by means of the fixed point theorem. Some examples are presented to illustrate our main results.

Keywords: boundary value problem; singular fractional differential equation; Caputo fractional derivative; fixed point theory

1 Introduction

Differential equations of fractional order have recently been addressed by many researchers of various fields of science and engineering such as physics, chemistry, biology, economics, control theory, and biophysics; see [1, 2]. On the other hand, fractional differential equations also serve as an excellent tool for the description of memory and hereditary properties of various materials and processes. With these advantages, the model of fractional order become more and more practical and realistic than the classical of integer order, such effects in the latter are not taken into account. As a result, the subject of fractional differential equations is gaining much attention and importance.

Recently, much attention has been focused on the study of the existence and uniqueness of solutions for boundary value problem of fractional differential equations with nonlocal boundary conditions by the use of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory, the upper and lower solution method, *etc.*); see [3–17].

In [18], Agarwal *et al.* investigated the existence of solutions for the singular fractional boundary value problems

$$\begin{cases} D^\alpha u(t) + f(t, u(t), D^\mu u(t)) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $0 < \mu \leq \alpha - 1$ are real numbers, D^α is the standard Riemann-Liouville fractional derivative, f satisfies the Caratheodory conditions on $[0, 1] \times (0, \infty) \times R$, f is positive, and $f(t, x, y)$ is singular at $x = 0$.

In [19], Yan *et al.* studied the existence and uniqueness of solutions for a class of fractional differential equations with integral boundary conditions

$$\begin{cases} {}^C D_{0+}^\alpha x(t) + f(t, x(t), {}^C D_{0+}^\beta x(t)) = 0, & t \in [0, 1], \\ x(0) + x'(0) = y(x), & \int_0^1 x(t) dt = m, \\ x''(0) = x'''(0) = \dots = x^{(n-1)}(0) = 0, \end{cases}$$

where ${}^C D_{0+}^\alpha, {}^C D_{0+}^\beta$ are the Caputo fractional derivatives, $f : [0, 1] \times R \times R \rightarrow R$ is a continuous function, $y : [0, 1] \rightarrow R$ is a continuous function, and $m \in R, n - 1 < \alpha < n (n \geq 2), 0 < \beta < 1$ is a real number.

In [20], Guezane-Lakoud and Bensebaa discussed the existence and uniqueness of solutions for a fractional boundary value problem with a fractional derivative condition,

$$\begin{cases} {}^C D_{0+}^q u(t) + f(t, u(t), {}^C D_{0+}^\sigma u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, & u'(1) = {}^C D_{0+}^\sigma u(1), \end{cases}$$

where $f : [0, 1] \times R \times R \rightarrow R$ is a given function, $2 < q < 3, 0 < \sigma < 1$, and ${}^C D_{0+}^q$ represents the standard Caputo fractional derivative of order q .

Motivated by all the works above, this paper deals with the existence and uniqueness of solutions for the singular fractional boundary value problem with a fractional derivative condition,

$$\begin{cases} {}^C D_{0+}^q u(t) + f(t, u(t), {}^C D_{0+}^\sigma u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u'(1) = {}^C D_{0+}^\sigma u(1), \end{cases} \tag{1.1}$$

where $2 < q < 3, 0 < \sigma < 1, f : (0, 1] \times R \times R \rightarrow R$ is continuous, $f(t, x, y)$ may be singular at $t = 0, {}^C D_{0+}^q$ is the standard Caputo derivative.

The paper is organized as follows. In Section 2, we shall introduce some definitions and lemmas to prove our main results. In Section 3, we establish some criteria for the existence for the boundary value problem (1.1) by using the Banach fixed point theorem and the Schauder fixed point theorem. Finally, we present two examples to illustrate our main results.

2 Preliminaries and lemmas

In this section, we present definitions and some fundamental facts from fractional calculus which can be found in [21].

Let $E = \{x : x \in C[0, 1], {}^C D_{0+}^\sigma x \in C[0, 1]\}, 0 < \sigma < 1$, endowed with the norm

$$\|x\| = \max \left\{ \max_{t \in [0, 1]} |x(t)|, \max_{t \in [0, 1]} |{}^C D_{0+}^\sigma x(t)| \right\},$$

then $(E, \|\cdot\|)$ is a Banach space.

Definition 2.1 [21] If $g \in C[a, b]$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined as

$$I_{a+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s) ds.$$

Definition 2.2 [21] Let $\alpha > 0$, $n = [\alpha] + 1$. If $g \in AC^n([a, b])$, then the Caputo fractional derivative of order α defined by

$${}^C D_{a^+}^\alpha g(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n - \alpha - 1} g^{(n)}(s) ds$$

exists almost everywhere on $[a, b]$ ($[\alpha]$ denotes the integer part of the real number α).

Lemma 2.1 [21] Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$. Then the following relations hold:

$${}^C D_{a^+}^\alpha t^{\beta - 1} = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} t^{\beta - \alpha - 1}, \quad \beta > n$$

and

$${}^C D_{a^+}^\alpha t^k = 0, \quad k = 0, 1, 2, \dots, n - 1.$$

Lemma 2.2 [21] For $\alpha > 0$, $g(t) \in C[a, b]$, the homogeneous fractional differential equation ${}^C D_{a^+}^\alpha g(t) = 0$ has a solution

$$g(t) = C_1 + C_2 t + C_3 t^2 + \dots + C_n t^{n-1},$$

where C_i , $i = 1, 2, \dots, n$, and $n = [\alpha] + 1$.

Lemma 2.3 [21] Let $p, q \geq 0$, and $f \in L_1([a, b])$. Then

$$\begin{aligned} I_{a^+}^p I_{a^+}^q f(t) &= I_{a^+}^{p+q} f(t) = I_{a^+}^q I_{a^+}^p f(t), \\ {}^C D_{a^+}^q I_{a^+}^q f(t) &= f(t), \quad \forall t \in [a, b]. \end{aligned}$$

Lemma 2.4 [21] Let $\beta > \alpha > 0$, and $f \in L_1([a, b])$. Then for all $t \in [a, b]$ we have

$${}^C D_{a^+}^\alpha I_{a^+}^\beta f(t) = I_{a^+}^{\beta - \alpha} f(t).$$

Lemma 2.5 (Schauder fixed point theorem) Let (E, d) be a complete metric space, let U be a closed convex subset of E , and let $A : U \rightarrow U$ be a mapping such that the set $\{Au : u \in U\}$ is relatively compact in E . Then A has at least one fixed point.

Lemma 2.6 For $y \in C[0, 1]$ and $2 < q < 3$, $0 < \sigma < 1$, the unique solution of

$$\begin{cases} {}^C D_{0^+}^q u(t) = y(t), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u'(1) = {}^C D_{0^+}^\sigma u(1), \end{cases}$$

is given by

$$u(t) = \int_0^1 G(t, s) y(s) ds,$$

where

$$G(t, s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{\Gamma(3-\sigma)t^2(1-s)^{q-2}}{2\Gamma(3-\sigma)-\Gamma(3)} \left(\frac{(1-s)^{1-\sigma}}{\Gamma(q-\sigma)} - \frac{1}{\Gamma(q-1)} \right), & 0 \leq s \leq t \leq 1, \\ \frac{\Gamma(3-\sigma)t^2(1-s)^{q-2}}{2\Gamma(3-\sigma)-\Gamma(3)} \left(\frac{(1-s)^{1-\sigma}}{\Gamma(q-\sigma)} - \frac{1}{\Gamma(q-1)} \right), & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.1)$$

Proof By Lemma 2.2, we get

$$u(t) = I_{0+}^q y(t) + C_1 + C_2 t + C_3 t^2,$$

for some $C_i \in R, i = 1, 2, 3$. So, we have

$$u'(t) = I_{0+}^{q-1} y(t) + C_2 + 2C_3 t.$$

From the conditions $u(0) = u'(0) = 0$, we obtain $C_1 = C_2 = 0$. Hence,

$${}^C D_{0+}^\sigma u(t) = I_{0+}^{q-\sigma} y(t) + C_3 {}^C D_{0+}^\sigma t^2 = I_{0+}^{q-\sigma} y(t) + C_3 \frac{\Gamma(3)}{\Gamma(3-\sigma)} t^{2-\sigma}.$$

The condition $u'(1) = {}^C D_{0+}^\sigma u(1)$ implies that

$$C_3 = \frac{\Gamma(3-\sigma)}{2\Gamma(3-\sigma)-\Gamma(3)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) y(s) ds.$$

Therefore, $u(t)$ can be written as

$$\begin{aligned} u(t) &= I_{0+}^q y(t) + \frac{\Gamma(3-\sigma)t^2}{2\Gamma(3-\sigma)-\Gamma(3)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) y(s) ds \\ &= \int_0^1 G(t, s) y(s) ds, \end{aligned}$$

where $G(t, s)$ is defined by (2.1). The proof is complete. □

3 Existence and uniqueness results

Define the operator $T : E \rightarrow E$ by

$$Tu(t) = \int_0^1 G(t, s) f(s, u(s), {}^C D_{0+}^\sigma u(s)) ds, \quad \forall t \in [0, 1].$$

Denote

$$\begin{aligned} B_1 &= \frac{B(q, 1-\delta)}{\Gamma(q)} + \frac{\Gamma(3-\sigma)}{2\Gamma(3-\sigma)-\Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right), \\ B_2 &= \frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{\Gamma(3)}{2\Gamma(3-\sigma)-\Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right). \end{aligned}$$

Lemma 3.1 *Let $2 < q < 3, 0 < \delta < 1, F : (0, 1] \rightarrow R$ is continuous, and $\lim_{t \rightarrow 0+} F(t) = \infty$. Suppose that $t^\delta F(t)$ is continuous on $[0, 1]$. Then the function $u(t) = \int_0^1 G(t, s) F(s) ds$ is continuous on $[0, 1]$.*

Proof By the continuity of $t^\delta F(t)$ and $u(t) = \int_0^1 G(t,s)s^{-\delta} s^\delta F(s) ds$. It is easy to know that $u(t) = 0$. Now we separate the process into three cases.

Case 1. For $t_0 = 0$ and $\forall t \in (0, 1]$. Because of the continuity of $t^\delta F(t)$, there exists a constant $M > 0$ such that $|t^\delta F(t)| \leq M, t \in [0, 1]$, then

$$\begin{aligned} |u(t) - u(0)| &= \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} s^{-\delta} s^\delta F(s) ds + \frac{\Gamma(3-\sigma)t^2}{2\Gamma(3-\sigma) - \Gamma(3)} \right. \\ &\quad \left. \times \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) s^{-\delta} s^\delta F(s) ds \right| \\ &\leq M \int_0^t \frac{(t-s)^{q-1} s^{-\delta}}{\Gamma(q)} ds + \frac{\Gamma(3-\sigma)t^2 M}{2\Gamma(3-\sigma) - \Gamma(3)} \\ &\quad \times \int_0^1 \left(\frac{(1-s)^{q-\sigma-1} s^{-\delta}}{\Gamma(q-\sigma)} + \frac{(1-s)^{q-2} s^{-\delta}}{\Gamma(q-1)} \right) ds \\ &= \frac{Mt^{q-\delta}}{\Gamma(q)} \int_0^1 (1-w)^{q-1} w^{-\delta} dw + \frac{\Gamma(3-\sigma)t^2 M}{2\Gamma(3-\sigma) - \Gamma(3)} \\ &\quad \times \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right) \\ &= \frac{Mt^{q-\delta}}{\Gamma(q)} B(q, 1-\delta) + \frac{\Gamma(3-\sigma)t^2 M}{2\Gamma(3-\sigma) - \Gamma(3)} \\ &\quad \times \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right) \\ &\rightarrow 0, \quad \text{as } t \rightarrow 0, \end{aligned}$$

where B denotes the beta function.

Case 2. For $t_0 \in (0, 1)$ and $\forall t \in (t_0, 1]$, then

$$\begin{aligned} |u(t) - u(t_0)| &= \left| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} s^{-\delta} s^\delta F(s) ds - \int_0^{t_0} \frac{(t_0-s)^{q-1}}{\Gamma(q)} s^{-\delta} s^\delta F(s) ds \right. \\ &\quad + \frac{\Gamma(3-\sigma)t^2}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) s^{-\delta} s^\delta F(s) ds \\ &\quad \left. - \frac{\Gamma(3-\sigma)t_0^2}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) s^{-\delta} s^\delta F(s) ds \right| \\ &\leq \left| \int_0^{t_0} \frac{(t-s)^{q-1} - (t_0-s)^{q-1}}{\Gamma(q)} s^{-\delta} s^\delta F(s) ds + \int_{t_0}^t \frac{(t-s)^{q-1}}{\Gamma(q)} s^{-\delta} s^\delta F(s) ds \right| \\ &\quad + \frac{\Gamma(3-\sigma)(t^2 - t_0^2)}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left| \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) s^{-\delta} s^\delta F(s) \right| ds \\ &\leq \frac{M}{\Gamma(q)} \left(\int_0^{t_0} [(t-s)^{q-1} - (t_0-s)^{q-1}] s^{-\delta} ds + \int_{t_0}^t (t-s)^{q-1} s^{-\delta} ds \right) \\ &\quad + \frac{\Gamma(3-\sigma)(t^2 - t_0^2)M}{2\Gamma(3-\sigma) - \Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right) \\ &\leq \frac{M(t^{q-\delta} - t_0^{q-\delta})B(q, 1-\delta)}{\Gamma(q)} + \frac{\Gamma(3-\sigma)(t^2 - t_0^2)M}{2\Gamma(3-\sigma) - \Gamma(3)} \\ &\quad \times \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right) \\ &\rightarrow 0, \quad \text{as } t \rightarrow t_0. \end{aligned}$$

Case 3. For $t_0 \in (0, 1]$ and $\forall t \in [0, t_0]$. The proof is similar to Case 2, here we just leave it out. This completes the proof. \square

Lemma 3.2 *Let $2 < q < 3$, $0 < \delta < 1$, $f : (0, 1] \times R \times R \rightarrow R$ is continuous, and $\lim_{t \rightarrow 0^+} f(t, \cdot, \cdot) = \infty$. Suppose that $t^\delta f(t, \cdot, \cdot)$ is continuous on $[0, 1] \times R \times R$. Then*

$${}^C D_{0^+}^\sigma Tu(t) = {}^C D_{0^+}^\sigma \left(\int_0^1 G(t, s) f(s, u(s), {}^C D_{0^+}^\sigma u(s)) ds \right)$$

is continuous on $[0, 1]$.

Proof From $u \in E$ we obtain $u(t) \in C[0, 1]$ and ${}^C D_{0^+}^\sigma u(t) \in C[0, 1]$. Hence, there exist two constants $L_1 > 0$ and $L_2 > 0$ such that $|u(t)| \leq L_1$, $|{}^C D_{0^+}^\sigma u(t)| \leq L_2$, for $t \in [0, 1]$. Since $t^\delta f(t, \cdot, \cdot)$ is continuous on $[0, 1] \times R \times R$, we have

$$\begin{aligned} M_0 &= \max_{t \in [0, 1]} |t^\delta f(t, u, v)|, \quad \text{for } -L_1 \leq u \leq L_1, -L_2 \leq v \leq L_2, \\ |{}^C D_{0^+}^\sigma Tu(t)| &= \left| {}^C D_{0^+}^\sigma \left(I_{0^+}^q f(t, u(t), {}^C D_{0^+}^\sigma u(t)) \right) + \frac{\Gamma(3 - \sigma)t^2}{2\Gamma(3 - \sigma) - \Gamma(3)} \right. \\ &\quad \times \left. \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) f(s, u(s), {}^C D_{0^+}^\sigma u(s)) ds \right| \\ &= \left| I_{0^+}^{q-\sigma} f(t, u(t), {}^C D_{0^+}^\sigma u(t)) + \frac{\Gamma(3)t^{2-\sigma}}{2\Gamma(3 - \sigma) - \Gamma(3)} \right. \\ &\quad \times \left. \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) f(s, u(s), {}^C D_{0^+}^\sigma u(s)) ds \right| \\ &\leq M_0 \int_0^t \frac{(t-s)^{q-\sigma-1} t^{-\sigma}}{\Gamma(q-\sigma)} ds + \frac{M_0 \Gamma(3)t^{2-\sigma}}{2\Gamma(3 - \sigma) - \Gamma(3)} \\ &\quad \times \int_0^1 \left(\frac{(1-s)^{q-\sigma-1} s^{-\delta}}{\Gamma(q-\sigma)} ds + \frac{(1-s)^{q-2} s^{-\delta}}{\Gamma(q-1)} \right) ds \\ &= \frac{M_0 t^{q-\sigma-\delta} B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{M_0 \Gamma(3)t^{2-\sigma}}{2\Gamma(3 - \sigma) - \Gamma(3)} \\ &\quad \times \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right). \end{aligned} \tag{3.1}$$

Observing that $t^{q-\sigma-\delta}$, $t^{2-\sigma}$ are continuous on $[0, 1]$, we can show ${}^C D_{0^+}^\sigma Tu(t)$ is continuous on $[0, 1]$ by using the same method as in Lemma 3.1. The proof is completed. \square

Lemma 3.3 *Let $2 < q < 3$, $0 < \delta < 1$, $f : (0, 1] \times R \times R \rightarrow R$ is continuous, and $\lim_{t \rightarrow 0^+} f(t, \cdot, \cdot) = \infty$. Assume that $t^\delta f(t, \cdot, \cdot)$ is continuous on $[0, 1] \times R \times R$. Then the operator $T : E \rightarrow E$ is completely continuous.*

Proof For $\forall u \in E$, $Tu(t) = \int_0^1 G(t, s) f(s, u(s), {}^C D_{0^+}^\sigma u(s)) ds$, by Lemma 3.1 and Lemma 3.2, we have $T : E \rightarrow E$. Now we separate the proof into three steps.

Step 1. Proof of $T : E \rightarrow E$ is continuous.

Let $u_0 \in E$ and $\|u_0\| = C_0$. If $u \in E$ and $\|u - u_0\| < 1$, then $\|u\| < 1 + C_0 = C$. By the continuity of $t^\delta f(t, u(t), {}^C D_{0^+}^\sigma u(t))$, we know that $t^\delta f(t, u(t), {}^C D_{0^+}^\sigma u(t))$ is uniformly continuous

on $[0, 1] \times [-C, C] \times [-C, C]$. Thus for $\forall \varepsilon > 0$, there exists $\eta > 0$ ($\eta < 1$), such that

$$|t^\delta f(t, u(t), {}^C D_{0+}^\sigma u(t)) - t^\delta f(t, u_0(t), {}^C D_{0+}^\sigma u_0(t))| < \varepsilon, \quad \text{for all } t \in [0, 1], \quad (3.2)$$

$u \in E$, with $\|u - u_0\| < \eta$.

It follows from (3.2) that

$$\begin{aligned} & |Tu(t) - Tu_0(t)| \\ & \leq \int_0^1 |G(t, s)s^{-\delta}| |s^\delta f(s, u(s), {}^C D_{0+}^\sigma u(s)) - s^\delta f(s, u_0(s), {}^C D_{0+}^\sigma u_0(s))| ds \\ & < \varepsilon \int_0^1 |G(t, s)s^{-\delta}| ds \\ & \leq \varepsilon \left(\int_0^t \frac{(t-s)^{q-1}s^{-\delta}}{\Gamma(q)} ds + \frac{\Gamma(3-\sigma)t^2}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left| \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) s^{-\delta} \right| ds \right) \\ & \leq \varepsilon \left[\frac{t^{q-\delta} B(q, 1-\delta)}{\Gamma(q)} + \frac{\Gamma(3-\sigma)t^2}{2\Gamma(3-\sigma) - \Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right) \right] \\ & \leq \varepsilon B_1. \end{aligned} \quad (3.3)$$

On the other hand, by (3.1), we get

$$\begin{aligned} & |{}^C D_{0+}^\sigma Tu(t) - {}^C D_{0+}^\sigma Tu_0(t)| \\ & = |{}^C D_{0+}^\sigma (Tu(t) - Tu_0(t))| \\ & = \left| I_{0+}^{q-\sigma} (f(t, u(t), {}^C D_{0+}^\sigma u(t)) - f(t, u_0(t), {}^C D_{0+}^\sigma u_0(t))) \right. \\ & \quad + \frac{\Gamma(3)t^{2-\sigma}}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) \\ & \quad \times (f(s, u(s), {}^C D_{0+}^\sigma u(s)) - f(s, u_0(s), {}^C D_{0+}^\sigma u_0(s))) ds \left. \right| \\ & \leq \varepsilon \left[\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{\Gamma(3)}{2\Gamma(3-\sigma) - \Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right) \right] \\ & = \varepsilon B_2. \end{aligned} \quad (3.4)$$

Therefore, $\|Tu - Tu_0\| \rightarrow 0$ as $\|u - u_0\| \rightarrow 0$, i.e., $T : E \rightarrow E$ is continuous.

Step 2. Let $\Omega \subset E$ be bounded, then there exists a positive constant b such that $\|u\| \leq b$, $\forall u \in \Omega$. Since $t^\delta f(t, u(t), {}^C D_{0+}^\sigma u(t))$ is continuous on $[0, 1] \times [-b, b] \times [-b, b]$, we see that there exists a positive constant L such that

$$|t^\delta f(t, u(t), {}^C D_{0+}^\sigma u(t))| \leq L, \quad \forall t \in [0, 1], \forall u \in \Omega.$$

Thus, by (3.3) and (3.4), we have

$$\begin{aligned} |Tu(t)| & \leq \int_0^1 |G(t, s)s^{-\delta}| |s^\delta f(s, u(s), {}^C D_{0+}^\sigma u(s))| ds \leq L \int_0^1 |G(t, s)s^{-\delta}| ds \leq LB_1, \\ |{}^C D_{0+}^\sigma Tu(t)| & = \left| I_{0+}^{q-\sigma} f(t, u(t), {}^C D_{0+}^\sigma u(t)) + \frac{\Gamma(3)t^{2-\sigma}}{2\Gamma(3-\sigma) - \Gamma(3)} \right. \end{aligned}$$

$$\begin{aligned} & \times \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) f(s, u(s), {}^C D_{0^+}^\sigma u(s)) \, ds \Big| \\ & \leq LB_2. \end{aligned}$$

So, $T(\Omega)$ is bounded.

Step 3. We will prove that $T(\Omega)$ is equicontinuous.

For all $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and $u \in \Omega$ we have

$$\begin{aligned} & |Tu(t_2) - Tu(t_1)| \\ & = \left| \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] s^{-\delta} s^\delta f(s, u(s), {}^C D_{0^+}^\sigma u(s)) \, ds \right. \\ & \quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-\delta} s^\delta f(s, u(s), {}^C D_{0^+}^\sigma u(s)) \, ds \\ & \quad \left. + \frac{\Gamma(3-\sigma)(t_2^2 - t_1^2)}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) s^{-\delta} s^\delta f(s, u(s), {}^C D_{0^+}^\sigma u(s)) \, ds \right| \\ & \leq \frac{L}{\Gamma(q)} \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] s^{-\delta} \, ds + \frac{L}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} s^\delta \, ds \\ & \quad + \frac{L\Gamma(3-\sigma)(t_2^2 - t_1^2)}{2\Gamma(3-\sigma) - \Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right) \\ & = \frac{L(t_2^{q-\delta} - t_1^{q-\delta})B(q, 1-\delta)}{\Gamma(q)} \\ & \quad + \frac{L\Gamma(3-\sigma)(t_2^2 - t_1^2)}{2\Gamma(3-\sigma) - \Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right), \tag{3.5} \end{aligned}$$

$$\begin{aligned} & |{}^C D_{0^+}^\sigma Tu(t_2) - {}^C D_{0^+}^\sigma Tu(t_1)| \\ & = \left| \frac{1}{\Gamma(q-\sigma)} \int_0^{t_1} [(t_2-s)^{q-\sigma-1} - (t_1-s)^{q-\sigma-1}] s^{-\delta} s^\delta f(s, u(s), {}^C D_{0^+}^\sigma u(s)) \, ds \right. \\ & \quad + \frac{1}{\Gamma(q-\sigma)} \int_{t_1}^{t_2} (t_2-s)^{q-\sigma-1} s^{-\delta} s^\delta f(s, u(s), {}^C D_{0^+}^\sigma u(s)) \, ds \\ & \quad \left. + \frac{\Gamma(3)(t_2^{2-\sigma} - t_1^{2-\sigma})}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) s^{-\delta} s^\delta f(s, u(s), {}^C D_{0^+}^\sigma u(s)) \, ds \right| \\ & \leq \frac{L}{\Gamma(q-\sigma)} \int_0^{t_1} [(t_2-s)^{q-\sigma-1} - (t_1-s)^{q-\sigma-1}] s^{-\delta} \, ds \\ & \quad + \frac{L}{\Gamma(q-\sigma)} \int_{t_1}^{t_2} (t_2-s)^{q-\sigma-1} s^\delta \, ds \\ & \quad + \frac{L\Gamma(3)(t_2^{2-\sigma} - t_1^{2-\sigma})}{2\Gamma(3-\sigma) - \Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right) \\ & = \frac{L(t_2^{q-\sigma-\delta} - t_1^{q-\sigma-\delta})B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} \\ & \quad + \frac{L\Gamma(3)(t_2^{2-\sigma} - t_1^{2-\sigma})}{2\Gamma(3-\sigma) - \Gamma(3)} \left(\frac{B(q-\sigma, 1-\delta)}{\Gamma(q-\sigma)} + \frac{B(q-1, 1-\delta)}{\Gamma(q-1)} \right). \tag{3.6} \end{aligned}$$

As $t_1 \rightarrow t_2$, the right-hand sides of the inequalities (3.5) and (3.6) tend to 0, consequently $\|Tu(t_2) - Tu(t_1)\| \rightarrow 0$, i.e., $T(\Omega)$ is equicontinuous.

By means of the Arzela-Ascoli theorem, we conclude that T is completely continuous. \square

Now we are in the position to establish the main results.

Theorem 3.1 *Assume that:*

(H₁) *There exist two constants $l > 0$ and $0 < \sigma < 1$ such that*

$$t^\delta |f(t, x_1, z_1) - f(t, x_2, z_2)| \leq l(|x_1 - x_2| + |z_1 - z_2|),$$

for each $t \in [0, 1]$ and all $x_1, x_2, z_1, z_2 \in R$.

(H₂) $\theta = \max\{2lB_1, 2lB_2\} < 1$.

Then the BVP (1.1) has a unique solution.

Proof We shall use the Banach fixed point theorem. For this, we need to verify that T is a contraction. Let $u, v \in E$, then from (H₁) and (3.3)-(3.4) we obtain

$$\begin{aligned} & |Tu(t) - Tv(t)| \\ &= \left| \int_0^1 G(t, s) (f(s, u(s), {}^C D_{0+}^\sigma u(s)) - f(s, v(s), {}^C D_{0+}^\sigma v(s))) \right| \\ &\leq \int_0^1 |G(t, s) s^{-\delta}| |s^\delta (f(s, u(s), {}^C D_{0+}^\sigma u(s)) - f(s, v(s), {}^C D_{0+}^\sigma v(s)))| ds \\ &\leq l \int_0^1 |G(t, s) s^{-\delta}| |u(s) - v(s)| ds + l \int_0^1 |G(t, s) s^{-\delta}| |{}^C D_{0+}^\sigma u(s) - {}^C D_{0+}^\sigma v(s)| ds \\ &\leq 2l \|u - v\| \int_0^1 |G(t, s) s^{-\delta}| ds \\ &\leq 2l B_1 \|u - v\|, \end{aligned} \tag{3.7}$$

$$\begin{aligned} & |{}^C D_{0+}^\sigma Tu(t) - {}^C D_{0+}^\sigma Tv(t)| \\ &= \left| \int_0^t \frac{(t-s)^{q-\sigma-1} s^{-\delta}}{\Gamma(q-\sigma)} [s^\delta (f(s, u(s), {}^C D_{0+}^\sigma u(s)) - f(s, v(s), {}^C D_{0+}^\sigma v(s)))] ds \right. \\ &\quad + \frac{\Gamma(3)t^{2-\sigma}}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1} s^{-\delta}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2} s^{-\delta}}{\Gamma(q-1)} \right) \\ &\quad \left. \times s^\delta (f(s, u(s), {}^C D_{0+}^\sigma u(s)) - f(s, v(s), {}^C D_{0+}^\sigma v(s))) ds \right| \\ &\leq 2l \|u - v\| B_2. \end{aligned} \tag{3.8}$$

Taking (3.7) and (3.8) into account, we acquire $\|Tu - Tv\| \leq \theta \|u - v\|$; then it is a contraction. As a consequence of the Banach fixed point theorem, we deduce that T has a fixed point which is the unique solution of the BVP (1.1). The proof is complete. \square

Next, we will use the Schauder fixed point theorem to prove our result.

For the sake of convenience, we set

$$L_0 = \max_{t \in [0,1]} t^\delta |f(t, u(t), {}^C D_{0+}^\sigma u(t))|, \quad r = \max\{L_0 B_1, L_0 B_2\}.$$

Theorem 3.2 Assume that $2 < q < 3$, $0 < \delta < 1$, $f : (0, 1] \times R \times R \rightarrow R$ is continuous, and $\lim_{t \rightarrow 0^+} f(t, \cdot, \cdot) = \infty$, $t^\delta f(t, \cdot, \cdot)$ is continuous on $[0, 1] \times R \times R$. Then the BVP (1.1) has a solution.

Proof Let $U = \{u : u \in E : \|u\| \leq r\}$. First, we prove that $T : U \rightarrow U$.

In fact, for each $t \in [0, 1]$, we have

$$\begin{aligned} |Tu(t)| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{-\delta} |s^{-\delta} f(s, u(s), {}^C D_{0^+}^\sigma u(s))| ds \\ &\quad + \frac{\Gamma(3-\sigma)t^2}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left| \left(\frac{(1-s)^{q-\sigma-1} s^{-\delta}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2} s^{-\delta}}{\Gamma(q-1)} \right) \right| \\ &\quad \times |s^\delta f(s, u(s), {}^C D_{0^+}^\sigma u(s))| ds \\ &\leq L_0 B_1, \\ |{}^C D_{0^+}^\sigma Tu(t)| &\leq \frac{1}{\Gamma(q-\sigma)} \int_0^t (t-s)^{q-\sigma-1} s^{-\delta} |s^{-\delta} f(s, u(s), {}^C D_{0^+}^\sigma u(s))| ds \\ &\quad + \frac{\Gamma(3)t^{2-\sigma}}{2\Gamma(3-\sigma) - \Gamma(3)} \int_0^1 \left| \left(\frac{(1-s)^{q-\sigma-1} s^{-\delta}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2} s^{-\delta}}{\Gamma(q-1)} \right) \right| \\ &\quad \times |s^\delta f(s, u(s), {}^C D_{0^+}^\sigma u(s))| ds \\ &\leq L_0 B_2. \end{aligned}$$

Hence, we can conclude that

$$\|Tu\| = \max \left\{ \max_{t \in [0,1]} |Tu(t)|, \max_{t \in [0,1]} |{}^C D_{0^+}^\sigma Tu(t)| \right\} \leq r.$$

From Lemma 3.1 and Lemma 3.2, we know that $Tu(t) \in C[0, 1]$, ${}^C D_{0^+}^\sigma Tu(t) \in C[0, 1]$. Consequently, $T : U \rightarrow U$. From Lemma 3.3, we find that $T : U \rightarrow U$ is completely continuous. By Lemma 2.5, we deduce that the problem (1.1) has a solution. This completes the proof. \square

4 Examples

We illustrate our work with two examples.

Example 4.1 Consider the following fractional boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{\frac{14}{5}} u = t^{-\frac{1}{2}}(0.029u + 0.028 {}^C D_{0^+}^{\frac{4}{5}} u + \cos t), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u'(1) = {}^C D_{0^+}^{\frac{4}{5}} u(1). \end{cases} \tag{4.1}$$

We have

$$f(t, x, y) = 0.029x + 0.028y + \cos t, \quad 2 < q = \frac{14}{5} < 3, \quad \sigma = \frac{4}{5} < 1$$

and

$$t^{\frac{1}{2}} |f(t, x_1, z_1) - f(t, x_2, z_2)| < 0.029(|x_1 - x_2| + |z_1 - z_2|),$$

where $\delta = \frac{1}{2} < 1$, $l = 0.029$. Simple calculus gives

$$\begin{aligned}
 B_1 &= \frac{B(q, 1 - \delta)}{\Gamma(q)} + \frac{\Gamma(3 - \sigma)}{2\Gamma(3 - \sigma) - \Gamma(3)} \left(\frac{B(q - \sigma, 1 - \delta)}{\Gamma(q - \sigma)} + \frac{B(q - 1, 1 - \delta)}{\Gamma(q - 1)} \right) \\
 &= 16.0968, \\
 B_2 &= \frac{B(q - \sigma, 1 - \delta)}{\Gamma(q - \sigma)} + \frac{\Gamma(3)}{2\Gamma(3 - \sigma) - \Gamma(3)} \left(\frac{B(q - \sigma, 1 - \delta)}{\Gamma(q - \sigma)} + \frac{B(q - 1, 1 - \delta)}{\Gamma(q - 1)} \right) \\
 &= 16.7697.
 \end{aligned}$$

So, $\theta = \max\{2lB_1, 2lB_2\} = 0.9726 < 1$, then by Theorem 3.1, the problem (4.1) has a unique solution.

Remark 4.1 If $l > 0.03$, then $\theta > 1$. However, by Theorem 3.2, the problem (4.1) still has a solution.

Example 4.2 Let us consider the fractional boundary value problem

$$\begin{cases}
 {}^C D_{0^+}^{\frac{14}{5}} u = t^{-\frac{1}{2}}(e^{-t}u^2 + ({}^C D_{0^+}^{\frac{4}{5}} u)^3 + (1 - t)^2), & 0 < t < 1, \\
 u(0) = u'(0) = 0, & u'(1) = {}^C D_{0^+}^{\frac{4}{5}} u(1).
 \end{cases} \quad (4.2)$$

Let $\sigma = \frac{1}{2}$, then all conditions in Theorem 3.2 are satisfied. Then the problem (4.2) has a solution.

Competing interests

The author declares to have no competing interests.

Author's contributions

The author read and approved the final manuscript.

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