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Square-like functions generated by the Laplace-Bessel differential operator

Şeyda Keleş* and Simten Bayrakçı

*Correspondence:
seydaaltinkol@gmail.com
Department of Mathematics,
Faculty of Science, Akdeniz
University, Antalya, Turkey

Abstract

We introduce a wavelet-type transform associated with the Laplace-Bessel differential operator $\Delta_B = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k} + \frac{2\nu_k}{\partial x_k} \frac{\partial}{\partial x_k}$ and the relevant square-like functions. An analogue of the Calderón reproducing formula and the $L_{2,\nu}$ boundedness of the square-like functions are obtained.

MSC: 47G10; 42C40; 44A35

Keywords: square functions; generalized translation; wavelet transform; Calderón reproducing formula

1 Introduction

The classical square functions $f(x) \to S_\varphi(x) = (\int_0^\infty |(f * \varphi_t)(x)|^2 \frac{dt}{t})^{\frac{1}{2}}$, where $\varphi \in S, S \equiv S(\mathbb{R}^n)$ is the Schwartz test function space and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 0$, $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$, $t > 0$, play important role in harmonic analysis and its applications; see Stein [1]. There are a lot of diverse variants of square functions and their applications; see Daly and Phillips [2], Jones *et al.* [3], Pipher [4], Kim [5]. Square-like functions generated by a composite wavelet transform and its L_2 estimates are proved by Aliev and Bayrakci [6].

Note that the Laplace-Bessel differential operator Δ_B is known as an important operator in analysis and its applications. The relevant harmonic analysis, known as Fourier-Bessel harmonic analysis associated with the Bessel differential operator $B_t = \frac{d^2}{dt^2} + \frac{2\nu}{t}\frac{d}{dt}$, has been the research area for many mathematicians such as Levitan, Muckenhoupt, Stein, Kipriyanov, Klyuchantsev, Löfström, Peetre, Gadjiev, Aliev, Guliev, Triméche, Rubin and others (see [7–14]). Moreover, a lot of mathematicians studied a Calderón reproducing formula. For example, Amri and Rachdi [15], Guliyev and Ibrahimov [16], Kamoun and Mohamed [17], Pathak and Pandey [18], Mourou and Trimèche [19, 20] and others.

In this paper, firstly we introduce a wavelet-like transform associated with the Laplace-Bessel differential operator,

$$\Delta_B = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{2\nu_k}{\partial x_k}\frac{\partial}{\partial x_k}, \qquad \nu = (\nu_1, \nu_2, \dots, \nu_n), \nu > 0,$$

and then the relevant square-like function. The plan of the paper is as follows. Some necessary definitions and auxiliary facts are given in Section 2. In Section 3 we prove a Calderón-type reproducing formula and the $L_{2,\nu}$ boundedness of the square-like functions.

2 Preliminaries

$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, x_2 > 0, \dots, x_n > 0\}$ and let $S(\mathbb{R}^n_+)$ be the Schwartz space of infinitely differentiable and rapidly decreasing functions.

$L_{p,\nu} = L_{p,\nu}(\mathbb{R}^n_+)$ $(1 \le p < \infty$, $\nu = (\nu_1, \dots, \nu_n)$; $\nu_1 > 0, \dots, \nu_n > 0)$ space is defined as the class of measurable functions f on $\mathbb{R}^n_+$ for which

$$\|f\|_{p,\nu} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x^{2\nu}\,dx\right)^{\frac{1}{p}} < \infty, \quad x^{2\nu}\,dx = x_1^{2\nu_1} x_2^{2\nu_2} \cdots x_n^{2\nu_n}\,dx_1\,dx_2 \cdots dx_n.$$

In the case $p = \infty$, we identify $L_\infty \equiv L_{\infty,\nu}$ with C_0 the space of continuous functions vanishing at infinity, and set $\|f\|_\infty = \sup_{x\in\mathbb{R}^n_+} |f(x)|$.

The Fourier-Bessel transform and its inverse are defined by

$$f^\wedge(x) = F_\nu(f)(x) = \int_{\mathbb{R}^n_+} f(y)\left(\prod_{k=1}^n j_{\nu_k - \frac{1}{2}}(x_k y_k)\right) y^{2\nu}\,dy, \tag{2.1}$$

$$F_\nu^{-1}(f)(x) = c_\nu(n)(F_\nu f)(x), \quad c_\nu(n) = \left[2^{2n}\prod_{k=1}^n \Gamma^2\left(\nu_k + \frac{1}{2}\right)\right]^{-1}, \tag{2.2}$$

where $j_{\nu-\frac{1}{2}}$ is the normalized Bessel function, which is also the eigenfunction of the Bessel operator $B_t = \frac{d^2}{dt^2} + \frac{2\nu}{t}\frac{d}{dt}$; $j_{\nu-\frac{1}{2}}(0) = 1$ and $j'_{\nu-\frac{1}{2}}(0) = 0$ (see [10]).

Denote by T^y $(y \in \mathbb{R}^n_+)$ the generalized translation operator acting according to the law:

$$T^y f(x) = \pi^{-n/2}\prod_{k=1}^n \Gamma\left(\nu_k + \frac{1}{2}\right)\Gamma^{-1}(\nu_k)\int_0^\pi \cdots \int_0^\pi f\left(\sqrt{x_1^2 - 2x_1y_1\cos\alpha_1 + y_1^2}, \dots,\right.$$
$$\left.\sqrt{x_n^2 - 2x_ny_n\cos\alpha_n + y_n^2}\right)\prod_{k=1}^n \sin^{2\nu_k - 1}\alpha_k\,d\alpha_1 \cdots d\alpha_n.$$

T^y is closely connected with the Bessel operator B_t (see [10]). It is known that (see [11])

$$\left\|T^y f\right\|_{p,\nu} \le \|f\|_{p,\nu} \quad \forall y \in \mathbb{R}^n_+, 1 \le p \le \infty, \tag{2.3}$$

$$\left\|T^y f - f\right\|_{p,\nu} \to 0, \quad |y| \to 0, 1 \le p \le \infty. \tag{2.4}$$

The generalized convolution 'B-convolution' associated with the generalized translation operator is $(f * g)(x) = \int_{\mathbb{R}^+_n} f(y)(T^y g(x)) y^{2\nu}\,dy$ for which

$$(f * g)^\wedge = f^\wedge g^\wedge. \tag{2.5}$$

We consider the B-maximal operator (see [8, 21])

$$M_B f(x) = \sup_{r>0}\left|E_+(0,r)\right|_{2\nu}^{-1}\int_{E_+(0,r)} T^y\left|f(x)\right| y^{2\nu}\,dy,$$

where $E_+(0,r) = \{y \in \mathbb{R}^n_+ : |y| < r\}$ and $|E_+(0,r)|_{2\nu} = \int_{E_+(0,r)} x^{2\nu}\,dx = Cr^{n+2\nu}$. Moreover, the following inequalities are satisfied (see for details [22]).

(a) If $f \in L_{1,\nu}(\mathbb{R}_+^n)$, then for every $\alpha > 0$,

$$\left|\left\{x : M_B f(x) > \alpha\right\}\right|_{2\nu} \leq \frac{c}{\alpha} \int_{\mathbb{R}_+^n} \left|f(x)\right| x^{2\nu}\, dx,$$

where $c > 0$ is independent of f.

(b) If $f \in L_{p,\nu}(\mathbb{R}_+^n)$, $1 < p \leq \infty$, then $M_B f \in L_{p,\nu}(R_+^n)$ and

$$\|M_B f\|_{p,\nu} \leq C_p \|f\|_{p,\nu},$$

where c_p is independent of f.

Furthermore, if $f \in L_{p,\nu}(\mathbb{R}_+^n)$, $1 \leq p \leq \infty$, then

$$\lim_{r \to 0} \left|E_+(0,r)\right|_{2\nu}^{-1} \int_{E_+(0,r)} T^y f(x) y^{2\nu}\, dy = f(x).$$

Now, we will need the generalized Gauss-Weierstrass kernel defined as

$$g_\nu(x,t) = F_\nu^{-1}\left(e^{-t|\cdot|^2}\right)(x) = \sqrt{c_\nu(n)}\, t^{\frac{-(n+2|\nu|)}{2}} e^{\frac{-x^2}{4t}}, \quad x \in \mathbb{R}_+^n, t > 0 \tag{2.6}$$

$c_\nu(n)$ being defined by (2.2) and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n$.

The kernel $g_\nu(x,t)$ possesses the following properties:

$$\text{(a)} \quad F_\nu\left(g_\nu(\cdot,t)\right)(x) = e^{-t|x|^2} \quad (t > 0); \tag{2.7}$$

$$\text{(b)} \quad \int_{\mathbb{R}_+^n} g_\nu(y,t)\, dy = 1 \quad (t > 0). \tag{2.8}$$

Given a function $f : \mathbb{R}_n^+ \to \mathbb{C}$, the generalized Gauss-Weierstrass semigroup, $G_t f(x)$ is defined as

$$G_t f(x) = \int_{\mathbb{R}_+^n} g_\nu(y,t)\left(T^y f(x)\right) y^{2\nu}\, dy, \quad t > 0. \tag{2.9}$$

This semigroup is well known and arises in the context of stable random processes in probability, in pseudo-differential parabolic equations and in integral geometry; see Koldobsky, Landkof, Fedorjuk, Aliev, Rubin, Sezer and Uyhan (see [23–26]).

The following lemma contains some properties of the semigroup $\{G_t f\}_{t \geq 0}$. (Compare with the analogous properties of the classical Gauss-Weierstrass integral [1, 27, 28].)

Lemma 2.1 *If $f \in L_{p,\nu}$, $1 \leq p \leq \infty$ ($L_\infty \equiv C_0$), then*

$$\text{(a)} \quad \|G_t f\|_{p,\nu} \leq c\|f\|_{p,\nu}, \tag{2.10}$$

$$\text{(b)} \quad \lim_{t \to 0} G_t f(x) = f(x). \tag{2.11}$$

The limit is understood in $L_{p,\nu}$ norm and pointwise almost all $x \in \mathbb{R}_+^n$. If $f \in C_0$, then the limit is uniform on $\mathbb{R}_+^n$.

$$\text{(c)} \quad \sup_{t>0}\left|G_t f(x)\right| \leq c M_B f(x), \tag{2.12}$$

where $M_B f$ is the well-known Hardy-Littlewood maximal function.

Moreover, let $h(z)$ be an absolutely continuous function on $[0,\infty)$ and

$$\alpha = \int_0^\infty \frac{h(z)}{z}\,dz < \infty. \tag{2.13}$$

If we denote $w(z) = h'(z)$, we have from (2.13)

$$h(0) = 0 \quad \text{and} \quad h(\infty) = 0 \tag{2.14}$$

(see for details [29]).

Now, we define the following wavelet-like transform:

$$V_t f(x) = \frac{1}{\alpha}\int_0^\infty G_{tz} f(x) w(z)\,dz, \tag{2.15}$$

where $w(z)$ is known as 'wavelet function', $\int_0^\infty w(z)\,dz = 0$, and the function $G_{tz}f(x)$ is the generalized Gauss-Weierstrass semigroup.

Using wavelet-like transform (2.15), we define the following square-like functions:

$$(Sf)(x) = \left(\int_0^\infty \left|V_t f(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}. \tag{2.16}$$

3 Main theorems and proofs

Theorem 3.1

(a) *Let* $f \in L_{p,\nu}$, $1 \le p \le \infty$ $(L_\infty \equiv C_0)$, $\nu > 0$. *We have*

$$\|V_t f\|_{p,\nu} \le c_1 c_2 \|f\|_{p,\nu} \quad (\forall t > 0), \tag{3.1}$$

where $c_1 = 2^{2|\nu|-n}$, $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n$, $c_2 = \frac{1}{\alpha}\int_0^\infty |w(z)|\,dz < \infty$.

(b) *Let* $f \in L_{p,\nu}$, $1 < p \le \infty$ $(L_\infty \equiv C_0)$. *We have*

$$\int_0^\infty V_t f(x)\frac{dt}{t} \equiv \lim_{\substack{\epsilon\to 0\\ \rho\to\infty}} \int_\epsilon^\rho V_t f(x)\frac{dt}{t} = f(x), \tag{3.2}$$

where limit can be interpreted in the $L_{p,\nu}$ *norm and pointwise for almost all* $x \in \mathbb{R}_+^n$. *If* $f \in C_0$, *the convergence is uniform on* $\mathbb{R}_+^n$.

Theorem 3.2 *If* $f \in L_{2,\nu}$, *then*

$$\|Sf\|_{2,\nu} \le \frac{1}{2}\|f\|_{2,\nu}. \tag{3.3}$$

Proof of Theorem 3.1 (a) By using the Minkowski inequality, we have

$$\begin{aligned}\|V_t f\|_{p,\nu} &= \frac{1}{\alpha}\left(\int_{\mathbb{R}_+^n}\left|\int_0^\infty G_{tz} f(x) w(z)\,dz\right|^p x^{2\nu}\,dx\right)^{\frac{1}{p}} \\ &\le \frac{1}{\alpha}\int_0^\infty \left|w(z)\right| \|G_{tz} f\|_{p,\nu}\,dz,\end{aligned}$$

$$\begin{aligned}
\|G_{tz}f\|_{p,\nu} &= \left(\int_{\mathbb{R}_+^n}\left|\int_{\mathbb{R}_+^n} g_\nu(y,tz)T^y f(x)y^{2\nu}\,dy\right|^p x^{2\nu}\,dx\right)^{\frac{1}{p}} \\
&\le \int_{\mathbb{R}_+^n}\bigl|g_\nu(y,tz)\bigr|\left(\int_{\mathbb{R}_+^n}\bigl|T^y f(x)\bigr|^p x^{2\nu}\,dx\right)^{\frac{1}{p}} y^{2\nu}\,dy \\
&\le \|f\|_{p,\nu}\int_{\mathbb{R}_+^n}\bigl|g_\nu(y,tz)\bigr|y^{2\nu}\,dy = c_1\|f\|_{p,\nu}.
\end{aligned}$$

Taking into account the following equality for $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$, $p > 0$ (see [30, p.370])

$$\int_0^\infty x^{\nu-1}e^{-\mu x^p}\,dx = \frac{1}{p}\mu^{-\frac{\nu}{p}}\Gamma\left(\frac{\nu}{p}\right),$$

we have

$$\int_0^\infty x^{2\nu}e^{-x^2}\,dx = \frac{1}{2}\Gamma\left(\nu+\frac{1}{2}\right), \quad \nu > 0$$

in one dimension. By using this equality, we get

$$\begin{aligned}
c_1 &= \int_{\mathbb{R}_+^n}\bigl|g_\nu(y,t)\bigr|y^{2\nu}\,dy \\
&= 2^{-n}\prod_{k=1}^n\Gamma^{-1}\left(\nu_k+\frac{1}{2}\right)t^{\frac{-(n+2|\nu|)}{2}}\int_{\mathbb{R}_+^n} e^{-\frac{|y|^2}{4t}}y^{2\nu}\,dy \quad \bigl(y = 2\sqrt{t}y, dy = 2^n t^{\frac{n}{2}}\,dy\bigr) \\
&= 2^{-n}\prod_{k=1}^n\Gamma^{-1}\left(\nu_k+\frac{1}{2}\right)t^{\frac{-(n+2|\nu|)}{2}}\int_{\mathbb{R}_+^n} e^{-|y|^2}2^{2|\nu|}t^{|\nu|}2^n t^{\frac{n}{2}}y^{2\nu}\,dy \\
&= 2^{2|\nu|}\prod_{k=1}^n\Gamma^{-1}\left(\nu_k+\frac{1}{2}\right)\int_{\mathbb{R}_+^n} e^{-|y|^2}y^{2\nu}\,dy \\
&= 2^{2|\nu|}\prod_{k=1}^n\Gamma^{-1}\left(\nu_k+\frac{1}{2}\right)\prod_{k=1}^n\Gamma\left(\nu_k+\frac{1}{2}\right)2^{-n} \\
&= 2^{2|\nu|-n}.
\end{aligned}$$

So we have $\|G_{tz}f\|_{p,\nu} \le 2^{2|\nu|-n}\|f\|_{p,\nu}$, and then inequality (3.1).

(b) Let $(A_{\epsilon,\rho}f)(x) = \int_\epsilon^\rho V_t f(x)\frac{dt}{t}$, $0 < \epsilon < \rho < \infty$. Applying Fubini's theorem, we get

$$\begin{aligned}
(A_{\epsilon,\rho}f)(x) &= \frac{1}{\alpha}\int_\epsilon^\rho\left(\int_0^\infty G_{tz}f(x)w(z)\,dz\right)\frac{dt}{t} \\
&= \frac{1}{\alpha}\int_0^\infty w(z)\left(\int_\epsilon^\rho G_{tz}f(x)\frac{dt}{t}\right)dz \\
&= \frac{1}{\alpha}\int_0^\infty w(z)\left(\int_{\epsilon z}^{\rho z} G_t f(x)\frac{dt}{t}\right)dz \\
&= \frac{1}{\alpha}\int_0^\infty\left(\int_{\frac{t}{\rho}}^{\frac{t}{\epsilon}} w(z)\,dz\right)G_t f(x)\frac{dt}{t}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\alpha}\int_0^\infty \frac{1}{t}\left[h\left(\frac{t}{\epsilon}\right) - h\left(\frac{t}{\rho}\right)\right] G_t f(x)\,dt \\
&= \frac{1}{\alpha}\int_0^\infty \frac{h(t)}{t} G_{\epsilon t} f(x)\,dt - \frac{1}{\alpha}\int_0^\infty \frac{h(t)}{t} G_{\rho t} f(x)\,dt \\
&= (A_\epsilon f)(x) - (A_\rho f)(x).
\end{aligned}$$

By Theorem 1.15 in [28, p.3], if $1 < p \leq \infty$ ($L_\infty \equiv C_0$), then

$$\lim_{\rho\to\infty} \|G_{\rho t} f\|_{p,\nu} = 0.$$

Therefore, by the Minkowski inequality and the Lebesgue dominated convergence theorem, taking into account Lemma 2.1, we have

$$\begin{aligned}
\|A_\rho f\|_{p,\nu} &= \frac{1}{\alpha}\left(\int_{\mathbb{R}_n^+}\left(\int_0^\infty \frac{h(t)}{t} G_{\rho t} f(x)\,dt\right)^p x^{2\nu}\,dx\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\alpha}\int_0^\infty \frac{h(t)}{t}\|G_{\rho t} f\|_{p,\nu}\,dt \\
&= \frac{1}{\alpha}\int_0^\infty \frac{h(\frac{t}{\rho})}{\frac{t}{\rho}}\|G_{\rho t} f\|_{p,\nu}\frac{1}{\rho}\,dt \to 0, \quad \rho\to\infty
\end{aligned}$$

and

$$\begin{aligned}
\|A_\epsilon f - f\|_{p,\nu} &= \left(\int_{\mathbb{R}_n^+}\left(\frac{1}{\alpha}\int_0^\infty \frac{h(t)}{t} G_{\epsilon t} f(x)\,dt - f(x)\right)^p x^{2\nu}\,dx\right)^{\frac{1}{p}} \\
&\overset{(2.13)}{=} \left(\int_{\mathbb{R}_n^+}\left(\frac{1}{\alpha}\int_0^\infty \frac{h(t)}{t} G_{\epsilon t} f(x)\,dt - \frac{1}{\alpha}\int_0^\infty \frac{h(t)}{t} f(x)\,dt\right)^p x^{2\nu}\,dx\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\alpha}\int_0^\infty \frac{h(t)}{t}\|G_{\epsilon t} f - f\|_{p,\nu}\,dt \to 0, \quad \epsilon\to 0.
\end{aligned}$$

Finally, for $1 < p \leq \infty$ ($L_\infty \equiv C_0$), we get

$$\|A_{\epsilon,\rho} f - f\|_{p,\nu} = \|A_\epsilon f - f\|_{p,\nu} + \|A_\rho f\|_{p,\nu} \to 0, \quad \epsilon\to 0, \rho\to\infty.$$

The a.e. convergence is based on the standard maximal function technique (see [31, p.60], [29] and [32]). □

Proof of Theorem 3.2 Firstly, let $f \in S(\mathbb{R}_+^n)$. By making use of the Fubini and Plancherel (for Fourier-Bessel transform) theorems, we get

$$\begin{aligned}
\|Sf\|_{2,\nu}^2 &= \int_{\mathbb{R}_+^n}\left(\int_0^\infty \left|V_t f(x)\right|^2 \frac{dt}{t}\right) x^{2\nu}\,dx \\
&= \int_0^\infty \left(\int_{\mathbb{R}_n^+}\left|V_t f(x)\right|^2 x^{2\nu}\,dx\right)\frac{dt}{t} \\
&= \int_0^\infty \left(\int_{\mathbb{R}_n^+}\left|(V_t f)^\wedge(x)\right|^2 x^{2\nu}\,dx\right)\frac{dt}{t}
\end{aligned}$$

and

$$\begin{aligned}(V_t f)^\wedge(x) &= F_\nu(V_t f)(x) = \frac{1}{\alpha}\int_{\mathbb{R}_n^+}\left(\int_0^\infty G_{tz}f(y)w(z)\,dz\right)\prod_{k=1}^n j_{\nu_k-\frac{1}{2}}(x_k y_k)y^{2\nu}\,dy \\ &= \frac{1}{\alpha}\int_0^\infty w(z)\left(\int_{\mathbb{R}_n^+} G_{tz}f(y)\prod_{k=1}^n j_{\nu_k-\frac{1}{2}}(x_k y_k)y^{2\nu}\,dy\right)dz \\ &= \frac{1}{\alpha}\int_0^\infty w(z)(G_{tz}f)^\wedge(x)\,dz \\ &\overset{(2.5)}{=} \frac{1}{\alpha}\int_0^\infty w(z)f^\wedge(x)e^{-tz|x|^2}\,dz.\end{aligned}$$

Now, by using Fubini's theorem, we have

$$\begin{aligned}\|Sf\|_{2,\nu}^2 &= \frac{1}{\alpha^2}\int_0^\infty\left[\int_{\mathbb{R}_n^+}\left(f^\wedge(x)\right)^2\left(\int_0^\infty w(z)e^{-tz|x|^2}\,dz\right)^2 x^{2\nu}\,dx\right]\frac{dt}{t} \\ &= \frac{1}{\alpha^2}\int_{\mathbb{R}_n^+}\left(f^\wedge(x)\right)^2\int_0^\infty\frac{dt}{t}\left(\int_0^\infty w(z)e^{-tz|x|^2}\,dz\right)^2 x^{2\nu}\,dx \\ &\quad \left(t = \tau|x|^{-2}, dt = |x|^{-2}\,d\tau\right) \\ &= \frac{1}{\alpha^2}\int_{\mathbb{R}_n^+}\left(f^\wedge(x)\right)^2\int_0^\infty\frac{d\tau}{\tau}\left(\int_0^\infty w(z)e^{-\tau z}\,dz\right)^2 x^{2\nu}\,dx \\ &= C^2\frac{1}{\alpha^2}\|f\|_{2,\nu}^2,\end{aligned}$$

where

$$C = \left(\int_0^\infty\frac{d\tau}{\tau}\left(\int_0^\infty e^{-\tau z}w(z)\,dz\right)^2\right)^{1/2}.$$

Since $w(z) = h'(z)$, $h(z) \geq 0$, $h(\infty) = h(0) = 0$, it follows that

$$\begin{aligned}C &= \left(\int_0^\infty\frac{d\tau}{\tau}\left(\int_0^\infty e^{-\tau z}w(z)\,dz\right)^2\right)^{1/2} \\ &= \left(\int_0^\infty\left(\int_0^\infty\sqrt{\tau}e^{-\tau z}h(z)\,dz\right)^2 d\tau\right)^{1/2} \\ &\leq \int_0^\infty h(z)\left(\int_0^\infty \tau e^{-2\tau z}\,d\tau\right)^{1/2} dz \quad (2z\tau = t, 2zd\tau = dt) \\ &= \int_0^\infty h(z)\left(\int_0^\infty\frac{t}{2z}e^{-t}\frac{1}{2z}\,dt\right)^{1/2} dz \\ &= \int_0^\infty\frac{h(z)}{2z}\left(\int_0^\infty te^{-t}\,dt\right)^{1/2} dz = \frac{1}{2}\alpha.\end{aligned}$$

Finally, we get

$$\|Sf\|_{2,\nu} \leq \frac{1}{2}\|f\|_{2,\nu}.$$

For arbitrary $f \in L_{2,\nu}(\mathbb{R}^n_+)$, the result follows by density of the class $S(\mathbb{R}^n_+)$ in $L_{2,\nu}(\mathbb{R}^n_+)$. Namely, let (f_n) be a sequence of functions in $S(\mathbb{R}^n_+)$, which converge to f in $L_{2,\nu}(\mathbb{R}^n_+)$-norm. That is, $\lim_{n\to\infty} \|f_n(x) - f(x)\|_{2,\nu} = 0$, $\forall x \in \mathbb{R}^n_+$.

From the 'triangle inequality' $((\|u\|_{2,\nu} - \|v\|_{2,\nu})^2 \leq \|u - v\|^2_{2,\nu})$, we have

$$\begin{aligned}
\left|(Sf_n)(x) - (Sf_m)(x)\right|^2 &= \left[\left(\int_0^\infty \left|V_t f_n(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} - \left(\left(\int_0^\infty \left|V_t f_m(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right)\right]^2 \\
&\leq \int_0^\infty \left|V_t f_n(x) - V_t f_m(x)\right|^2 \frac{dt}{t} \\
&= \int_0^\infty \left|V_t (f_n - f_m)\right|^2 \frac{dt}{t} \\
&= \left(S(f_n - f_m)(x)\right)^2.
\end{aligned}$$

Hence

$$\|Sf_n - Sf_m\|_{2,\nu} \leq \left\|S(f_n - f_m)\right\|_{2,\nu} \leq \frac{1}{2}\|f_n - f_m\|_{2,\nu}.$$

This shows that the sequence (Sf_n) converges to Sf in $L_{2,\nu}(\mathbb{R}^n_+)$-norm. Thus

$$\|Sf\|_{2,\nu} \leq \frac{1}{2}\|f\|_{2,\nu}, \quad \forall f \in L_{2,\nu}\left(\mathbb{R}^n_+\right)$$

and the proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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