Flaut *Advances in Difference Equations* 2014, 2014:279
http://www.advancesindifferenceequations.com/content/2014/1/279

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

A Clifford algebra associated to generalized Fibonacci quaternions

Cristina Flaut*

*Correspondence:
cflaut@univ-ovidius.ro;
cristina_flaut@yahoo.com
Faculty of Mathematics and Computer Science, Ovidius University, Bd. Mamaia 124, Constanta, 900527, Romania

Abstract

In this paper, using the construction of Clifford algebras, we associate to the set of generalized Fibonacci quaternions a quaternion algebra A (*i.e.*, a Clifford algebra of dimension four). Indeed, for the generalized quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$, denoting $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$, if $E(\beta_1, \beta_2) > 0$, therefore the algebra A is split. If $E(\beta_1, \beta_2) < 0$, then the algebra A is a division algebra. In this way, we provide a nice algorithm to obtain a division quaternion algebra starting from a quaternion non-division algebra and *vice versa*.

MSC: 11E88; 11B39

Keywords: Clifford algebras; generalized Fibonacci quaternions

1 Introduction

In 1878, WK Clifford discovered Clifford algebras. These algebras generalize the real numbers, complex numbers and quaternions (see [1]).

The theory of Clifford algebras is intimately connected with the theory of quadratic forms. In the following, we will consider K to be a field of characteristic not two. Let (V, q) be a K-vector space equipped with a nondegenerate quadratic form over the field K. A *Clifford algebra* for (V, q) is a K-algebra C with a linear map $i : V \to C$ satisfying the property

$$i(x)^2 = q(x) \cdot 1_C, \quad \forall x \in V,$$

such that for any K-algebra A and any K linear map $\gamma : V \to A$ with $\gamma^2(x) = q(x) \cdot 1_A$, $\forall x \in V$, there exists a unique K-algebra morphism $\gamma' : C \to A$ with $\gamma = \gamma' \circ i$.

Such an algebra can be constructed using the tensor algebra associated to a vector space V. Let $T(V) = K \oplus V \oplus (V \otimes V) \oplus \cdots$ be the tensor algebra associated to the vector space V, and let $\mathcal{J}$ be the two-sided ideal of $T(V)$ generated by all elements of the form $x \otimes x - q(x) \cdot 1$ for all $x \in V$. The associated Clifford algebra is the factor algebra $C(V, q) = T(V)/\mathcal{J}$ (see [2, 3]).

Theorem 1.1 (Poincaré-Birkhoff-Witt [2, p.44]) *If* $\{e_1, e_2, \ldots, e_n\}$ *is a basis of* V, *then the set* $\{1, e_{j_1} e_{j_2} \cdots e_{j_s}, 1 \leq s \leq n, 1 \leq j_1 < j_2 < \cdots < j_s \leq n\}$ *is a basis in* $C(V, q)$.

The most important Clifford algebras are those defined over real and complex vector spaces equipped with nondegenerate quadratic forms. Every nondegenerate quadratic form over a real vector space is equivalent to the following standard diagonal form:

$$q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2,$$

where $n = r + s$ is the dimension of the vector space. The pair of integers (r, s) is called *the signature* of the quadratic form. The real vector space with this quadratic form is usually denoted by $\mathbb{R}_{r,s}$ and the Clifford algebra on $\mathbb{R}_{r,s}$ is denoted by $\mathrm{Cl}_{r,s}(\mathbb{R})$. For other details about Clifford algebras, the reader is referred to [4–6] and [7].

Example 1.2

(i) For $p = q = 0$, we have $\mathrm{Cl}_{0,0}(K) \simeq K$.

(ii) For $p = 0$, $q = 1$, it results that $\mathrm{Cl}_{0,1}(K)$ is a two-dimensional algebra generated by a single vector e_1 such that $e_1^2 = -1$, and therefore $\mathrm{Cl}_{0,1}(K) \simeq K(e_1)$. For $K = \mathbb{R}$, it follows that $\mathrm{Cl}_{0,1}(\mathbb{R}) \simeq \mathbb{C}$.

(iii) For $p = 0$, $q = 2$, the algebra $\mathrm{Cl}_{0,2}(K)$ is a four-dimensional algebra spanned by the set $\{1, e_1, e_2, e_1e_2\}$. Since $e_1^2 = e_2^2 = (e_1e_2)^2 = -1$ and $e_1e_2 = -e_2e_1$, we obtain that this algebra is isomorphic to the division quaternions algebra $\mathbb{H}$.

(iv) For $p = 1$, $q = 1$ or $p = 2$, $q = 0$, we obtain the algebra $\mathrm{Cl}_{1,1}(K) \simeq \mathrm{Cl}_{2,0}(K)$ which is isomorphic with a split (*i.e.*, nondivision) quaternion algebra [8].

2 Preliminaries

Let $\mathbb{H}(\beta_1, \beta_2)$ be a generalized real quaternion algebra, the algebra of the elements of the form $a = a_1 \cdot 1 + a_2e_2 + a_3e_3 + a_4e_4$, where $a_i \in \mathbb{R}$, $i \in \{1, 2, 3, 4\}$, and the elements of the basis $\{1, e_2, e_3, e_4\}$ satisfy the following multiplication table:

$\cdot$	1	e_2	e_3	e_4
1	1	e_2	e_3	e_4
e_2	e_2	$-\beta_1$	e_4	$-\beta_1 e_3$
e_3	e_3	$-e_4$	$-\beta_2$	$\beta_2 e_2$
e_4	e_4	$\beta_1 e_3$	$-\beta_2 e_2$	$-\beta_1\beta_2$

We denote by $\mathbf{n}(a)$ the norm of a real quaternion a. The norm of a generalized quaternion has the following expression $\mathbf{n}(a) = a_1^2 + \beta_1 a_2^2 + \beta_2 a_3^2 + \beta_1\beta_2 a_4^2$. For $\beta_1 = \beta_2 = 1$, we obtain the real division algebra $\mathbb{H}$, with the basis $\{1, i, j, k\}$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji$, $ik = -ki$, $jk = -kj$.

Proposition 2.1 ([3, Proposition 1.1]) *The quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$ is isomorphic to quaternion algebra $\mathbb{H}(x^2\beta_1, y^2\beta_2)$, where $x, y \in K^*$.*

The quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$ with $\beta_1, \beta_2 \in K^*$ is either a division algebra or is isomorphic to $\mathbb{H}(-1, -1) \simeq \mathcal{M}_2(K)$ [3].

For other details about the quaternions, the reader is referred, for example, to [3, 9, 10].

The Fibonacci numbers were introduced by *Leonardo of Pisa* (1170-1240) in his book *Liber abbaci*, book published in 1202 AD (see [11, pp.1, 3]). This name is attached to the following sequence of numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots,$$

with the nth term given by the formula

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2,$$

where $f_0 = 0, f_1 = 1$.

In [12], the author generalized Fibonacci numbers and gave many properties of them:

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where $h_0 = p$, $h_1 = q$, with p, q being arbitrary integers. In the same paper [12, relation (7)], the following relation between Fibonacci numbers and generalized Fibonacci numbers was obtained:

$$h_{n+1} = pf_n + qf_{n+1}. \tag{2.1}$$

For the generalized real quaternion algebra, the Fibonacci quaternions and generalized Fibonacci quaternions are defined in the same way:

$$F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4,$$

for the nth Fibonacci quaternions and

$$H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4 = pF_n + qF_{n+1}, \tag{2.2}$$

for the nth generalized Fibonacci quaternions.

In the following, we will denote the nth generalized Fibonacci number and the nth generalized Fibonacci quaternion element by $h_n^{p,q}$, respectively $H_n^{p,q}$. In this way, we emphasize the starting integers p and q.

It is known that the expression for the nth term of a Fibonacci element is

$$f_n = \frac{1}{\sqrt{5}}\left[\alpha^n - \beta^n\right] = \frac{\alpha^n}{\sqrt{5}}\left[1 - \frac{\beta^n}{\alpha^n}\right], \tag{2.3}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

From the above, we obtain the following limit:

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{n}(F_n) &= \lim_{n\to\infty} \left(f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1\beta_2 f_{n+3}^2\right) \\
&= \lim_{n\to\infty} \left(\frac{\alpha^{2n}}{5} + \beta_1 \frac{\alpha^{2n+2}}{5} + \beta_2 \frac{\alpha^{2n+4}}{5} + \beta_1\beta_2 \frac{\alpha^{2n+6}}{5}\right) \\
&= \operatorname{sgn} E(\beta_1, \beta_2) \cdot \infty,
\end{aligned}$$

where $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$, since $\alpha^2 = \alpha + 1$ (see [13]).

If $E(\beta_1, \beta_2) > 0$, there exists a number $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have $\mathbf{n}(F_n) > 0$. In the same way, if $E(\beta_1, \beta_2) < 0$, there exists a number $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have $\mathbf{n}(F_n) < 0$. Therefore, for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E(\beta_1, \beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there is a natural number $n_0 = \max\{n_1, n_2\}$ such that $\mathbf{n}(F_n) \neq 0$. Hence F_n is an invertible element for all $n \geq n_0$. Using the same arguments, we can compute the following limit:

$$\lim_{n\to\infty}\bigl(\mathbf{n}\bigl(H_n^{p,q}\bigr)\bigr) = \lim_{n\to\infty}\bigl(h_n^2 + \beta_1 h_{n+1}^2 + \beta_2 h_{n+2}^2 + \beta_1\beta_2 h_{n+3}^2\bigr) = \operatorname{sgn} E'(\beta_1,\beta_2)\cdot\infty,$$

where $E'(\beta_1,\beta_2) = \frac{1}{5}(p+\alpha q)^2 E(\beta_1,\beta_2)$, if $E'(\beta_1,\beta_2) \neq 0$ (see [13]).

Therefore, for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1,\beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1,\beta_2)$ there exists a natural number n_0' such that $\mathbf{n}(H_n^{p,q}) \neq 0$, hence $H_n^{p,q}$ is an invertible element for all $n \geq n_0'$.

Theorem 2.2 ([13, Theorem 2.6]) *For all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1,\beta_2) \neq 0$, there exists a natural number n' such that for all $n \geq n'$, Fibonacci elements F_n and generalized Fibonacci elements $H_n^{p,q}$ are invertible elements in the algebra $\mathbb{H}(\beta_1,\beta_2)$.*

Theorem 2.3 ([13, Theorem 2.1]) *The set $\mathcal{H}_n = \{H_n^{p,q}/p, q \in \mathbb{Z}, n \geq m, m \in \mathbb{N}\} \cup \{0\}$ is a $\mathbb{Z}$-module.*

3 Main results

Remark 3.1 We remark that the $\mathbb{Z}$-module from Theorem 2.3 is a free $\mathbb{Z}$-module of rank two. Indeed, $\varphi : \mathbb{Z}\times\mathbb{Z} \to \mathcal{H}_n$, $\varphi((p,q)) = H_n^{p,q}$ is a $\mathbb{Z}$-module isomorphism and $\{\varphi(1,0) = F_n, \varphi(0,1) = F_{n+1}\}$ is a basis in $\mathcal{H}_n$.

Remark 3.2 By extension of scalars, we obtain that $\mathbb{R}\otimes_{\mathbb{Z}}\mathcal{H}_n$ is an $\mathbb{R}$-vector space of dimension two. A basis is $\{\overline{e}_1 = 1\otimes F_n, \overline{e}_2 = 1\otimes F_{n+1}\}$. We have that $\mathbb{R}\otimes_{\mathbb{Z}}\mathcal{H}_n$ is an isomorphic with the $\mathbb{R}$-vector space $\mathcal{H}_n^{\mathbb{R}} = \{H_n^{p,q}/p, q \in \mathbb{R}\} \cup \{0\}$. A basis in $\mathcal{H}_n^{\mathbb{R}}$ is $\{F_n, F_{n+1}\}$.

Let $T(\mathcal{H}_n^{\mathbb{R}})$ be the tensor algebra associated to the $\mathbb{R}$-vector space $\mathcal{H}_n^{\mathbb{R}}$, and let $C(\mathcal{H}_n^{\mathbb{R}})$ be the Clifford algebra associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$. From Theorem 1.1, it results that this algebra has dimension four.

Case 1: $\mathbb{H}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is a division algebra

Remark 3.3 Since in this case $E(\beta_1,\beta_2) > 0$ for all $n \geq n'$ (as in Theorem 2.2), then $\mathcal{H}_n^{\mathbb{R}}$ is an Euclidean vector space. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1F_n + x_2F_{n+1}$, $w = y_1F_n + y_2F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as follows:

$$\langle z, w\rangle = x_1y_1\mathbf{n}(F_n) + x_2y_2\mathbf{n}(F_{n+1}).$$

We remark that all properties of the inner product are fulfilled. Indeed, since for all $n \geq n'$ we have $\mathbf{n}(F_n) > 0$ and $\mathbf{n}(F_{n+1}) > 0$, it results that $\langle z, z\rangle = x_1^2\mathbf{n}(F_n) + x_2^2\mathbf{n}(F_{n+1}) = 0$ if and only if $x_1 = x_2 = 0$, therefore $z = 0$.

On $\mathcal{H}_n^{\mathbb{R}}$ with the basis $\{F_n, F_{n+1}\}$, we define the following quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}} : \mathcal{H}_n^{\mathbb{R}} \to \mathbb{R}$:

$$q_{\mathcal{H}_n^{\mathbb{R}}}(x_1F_n + x_2F_{n+1}) = \mathbf{n}(F_n)x_1^2 + \mathbf{n}(F_{n+1})x_2^2.$$

Let $Q_{\mathcal{H}_n^{\mathbb{R}}}$ be a bilinear form associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$,

$$\begin{aligned} Q_{\mathcal{H}_n^{\mathbb{R}}}(x,y) &= \frac{1}{2}\bigl(q_{\mathcal{H}_n^{\mathbb{R}}}(x+y) - q_{\mathcal{H}_n^{\mathbb{R}}}(x) - q_{\mathcal{H}_n^{\mathbb{R}}}(y)\bigr) \\ &= \mathbf{n}(F_n)x_1y_1 + \mathbf{n}(F_{n+1})x_2y_2. \end{aligned}$$

The matrix associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is

$$A = \begin{pmatrix} \mathbf{n}(F_n) & 0 \\ 0 & \mathbf{n}(F_{n+1}) \end{pmatrix}.$$

We remark that $\det A = \mathbf{n}(F_n)\mathbf{n}(F_{n+1}) > 0$ for all $n \geq n'$. Since $E(\beta_1, \beta_2) > 0$, therefore $\mathbf{n}(F_n) > 0$ for $n > n'$. We obtain that the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to a split quaternion algebra.

From the above results and using Proposition 2.1, we obtain the following theorem.

Theorem 3.4 *If $\mathbb{H}(\beta_1, \beta_2)$ is a division algebra, there is a natural number n' such that for all $n \geq n'$, the Clifford algebra associated to the real vector space $\mathcal{H}_n^{\mathbb{R}}$ is isomorphic with the split quaternion algebra $\mathbb{H}(-1, -1)$.*

Case 2: $\mathbb{H}(\beta_1, \beta_2)$ is not a division algebra

Remark 3.5 (i) If $E(\beta_1, \beta_2) > 0$, then $\mathcal{H}_n^{\mathbb{R}}$ is an Euclidean vector space, for all $n \geq n'$, as in Theorem 2.2. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1 F_n + x_2 F_{n+1}$, $w = y_1 F_n + y_2 F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as follows:

$$\langle z, w \rangle = x_1 y_1 \mathbf{n}(F_n) + x_2 y_2 \mathbf{n}(F_{n+1}).$$

(ii) If $E(\beta_1, \beta_2) < 0$, then $\mathcal{H}_n^{\mathbb{R}}$ is also an Euclidean vector space, for all $n \geq n'$, as in Theorem 2.2. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1 F_n + x_2 F_{n+1}$, $w = y_1 F_n + y_2 F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as follows:

$$\langle z, w \rangle = -x_1 y_1 \mathbf{n}(F_n) - x_2 y_2 \mathbf{n}(F_{n+1}).$$

We have $\langle z, z \rangle = -x_1^2 \mathbf{n}(F_n) - x_2^2 \mathbf{n}(F_{n+1})$, and since for all $n \geq n'$ we have $\mathbf{n}(F_n) < 0$ and $\mathbf{n}(F_{n+1}) < 0$, it results that $\langle z, z \rangle = -x_1^2 \mathbf{n}(F_n) - x_2^2 \mathbf{n}(F_{n+1}) = 0$ if and only if $x_1 = x_2 = 0$, therefore $z = 0$.

On $\mathcal{H}_n^{\mathbb{R}}$ with the basis $\{F_n, F_{n+1}\}$,we define the following quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}} : \mathcal{H}_n^{\mathbb{R}} \to \mathbb{R}$:

$$q_{\mathcal{H}_n^{\mathbb{R}}}(x_1 F_n + x_2 F_{n+1}) = q_{\mathcal{H}_n^{\mathbb{R}}}(x_1 F_n + x_2 F_{n+1}) = \mathbf{n}(F_n) x_1^2 + \mathbf{n}(F_{n+1}) x_2^2.$$

Let $Q_{\mathcal{H}_n^{\mathbb{R}}}$ be a bilinear form associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$,

$$\begin{aligned} Q_{\mathcal{H}_n^{\mathbb{R}}}(x, y) &= \frac{1}{2}\left(q_{\mathcal{H}_n^{\mathbb{R}}}(x + y) - q_{\mathcal{H}_n^{\mathbb{R}}}(x) - q_{\mathcal{H}_n^{\mathbb{R}}}(y)\right) \\ &= \mathbf{n}(F_n) x_1 y_1 + \mathbf{n}(F_{n+1}) x_2 y_2. \end{aligned}$$

The matrix associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is

$$A = \begin{pmatrix} \mathbf{n}(F_n) & 0 \\ 0 & \mathbf{n}(F_{n+1}) \end{pmatrix}.$$

We remark that $\det A = \mathbf{n}(F_n)\mathbf{n}(F_{n+1}) > 0$ for all $n \geq n'$.

If $E(\beta_1, \beta_2) > 0$, therefore $\mathbf{n}(F_n) > 0$ for $n > n'$. We obtain that the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to a split quaternion algebra.

If $E(\beta_1, \beta_2) < 0$, therefore $\mathbf{n}(F_n) < 0$ for $n > n'$. Then the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $\mathrm{Cl}_{0,2}(K)$ which is isomorphic to the quaternion division algebra $\mathbb{H}$.

From the above results and using Proposition 2.1, we obtain the following theorem.

Theorem 3.6 *If $\mathbb{H}(\beta_1, \beta_2)$ is not a division algebra, there is a natural number n' such that for all $n \geq n'$, if $E(\beta_1, \beta_2) > 0$, then the Clifford algebra associated to the real vector space $\mathcal{H}_n^{\mathbb{R}}$ is isomorphic with the split quaternion algebra $\mathbb{H}(-1, -1)$. If $E(\beta_1, \beta_2) < 0$, then the Clifford algebra associated to the real vector space $\mathcal{H}_n^{\mathbb{R}}$ is isomorphic to the division quaternion algebra $\mathbb{H}(1, 1)$.*

Example 3.7 (1) For $\beta_1 = 1$, $\beta_2 = -1$, we obtain the split quaternion algebra $\mathbb{H}(1, -1)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[-5 - 10\alpha] < 0$ and, for $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 + f_{n+1}^2 - f_{n+2}^2 - f_{n+3}^2 < 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 + f_{n+2}^2 - f_{n+3}^2 - f_{n+4}^2 < 0$ for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{0,2}(K)$ which is isomorphic to the quaternion division algebra $\mathbb{H}(1, 1)$.

(2) For $\beta_1 = -2$, $\beta_2 = -3$, we obtain the split quaternion algebra $\mathbb{H}(-2, -3)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[23 + 43\alpha] > 0$. For $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 - f_{n+1}^2 - f_{n+2}^2 + f_{n+3}^2 > 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 - f_{n+2}^2 - f_{n+3}^2 + f_{n+4}^2 > 0$ for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1, -1)$.

(3) For $\beta_1 = 2$, $\beta_2 = -3$, we obtain the split quaternion algebra $\mathbb{H}(2, -3)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[-33 - 44\alpha] < 0$. For $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 + 2f_{n+1}^2 - 3f_{n+2}^2 - 6f_{n+3}^2 < 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 + 2f_{n+2}^2 - 3f_{n+3}^2 - 6f_{n+4}^2 > 0$ for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{0,2}(K)$ which is isomorphic to the division quaternion algebra $\mathbb{H}(1, -1)$.

(4) For $\beta_1 = \beta_2 = -\frac{1}{2}$, we obtain the split quaternion algebra $\mathbb{H}(-\frac{1}{2}, -\frac{1}{2})$. Therefore $E(\beta_1, \beta_2) = \frac{3}{20} > 0$, and for $n' = 1$ we obtain $\mathbf{n}(F_n) > 0$ and $\mathbf{n}(F_{n+1}) > 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1, -1)$.

The algorithm

(1) Let $\mathbb{H}(\beta_1, \beta_2)$ be a quaternion algebra, $\alpha = \frac{1+\sqrt{5}}{2}$ and $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$.

(2) Let V be the $\mathbb{R}$-vector space $\mathcal{H}_n^{\mathbb{R}} = \{H_n^{p,q} / p, q \in \mathbb{R}\} \cup \{0\}$.

(3) If $E(\beta_1, \beta_2) > 0$, then the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1, -1)$.

(4) If $E(\beta_1, \beta_2) < 0$, then the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{0,2}(K)$ which is isomorphic to the division quaternion algebra $\mathbb{H}(1,1)$.

4 Conclusions

In this paper, we have extended the $\mathbb{Z}$-module of the generalized Fibonacci quaternions to a real vector space $\mathcal{H}_n^{\mathbb{R}}$. We have proved that the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to a split quaternion algebra or to a division algebra if $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$ is positive or negative. We also have given an algorithm which allows us to find a division quaternion algebra starting from a split quaternion algebra and *vice versa*.

Competing interests
The author declares that she has no competing interests.

Acknowledgements
I want thank the anonymous referees for their remarkable comments, suggestions and ideas which helped me to improve this paper. The author also thanks Professor Ravi P Agarwal and members of the Springer Open Team for their support.

Received: 9 July 2014 Accepted: 14 October 2014 Published: 31 Oct 2014

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10.1186/1687-1847-2014-279
Cite this article as: Flaut: A Clifford algebra associated to generalized Fibonacci quaternions. *Advances in Difference Equations* 2014, 2014:279