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Existence and stability of the solutions for systems of nonlinear fractional differential equations with deviating arguments

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Abstract

In this paper, we give sufficient conditions for the existence and uniqueness of the solution for a class of nonlinear fractional differential systems, with variable delays. Our analysis relies on the Banach fixed point theorem. Furthermore, we prove the uniform stability of the solution. Some examples are given to illustrate our results.

Keywords: Caputo fractional derivatives; nonlinear fractional differential equations; deviating arguments; fixed point theorem; stability

1 Introduction

In recent years, many research works have been interested in fractional differential equations. This is due, first, to their widespread applications in diverse fields of engineering and natural sciences, and secondly to the intensive development of the theory of fractional calculus (see [1–12]). Furthermore, fractional differential equations with delays have proven more realistic in the description of natural phenomena than those without delays. Therefore, the study of these equations has drawn much attention (see *e.g.*, [13–17]).

El-Sayed and Gaafar [16] established sufficient conditions for the existence and uniqueness of a solution to some nonlinear Riemann-Liouville fractional differential systems with constant delays. Also, they proved the stability of the solution. Recently, this study has been extended to another class of nonlinear fractional differential equations with delay in [17].

In the current paper, motivated and inspired by the works of [16, 17], we treat the same questions, but with time-dependent delays. Thus, we consider a system of nonlinear fractional differential equations with variable delays of the form

$$^{c}D^{\alpha}x_i(t)=\sum_{j=1}^{n}f_{ij}\bigl(t,x_i(t),x_j\bigl(t-\tau_j(t)\bigr)\bigr),\quad i=1,2,\dots,n,t>0, \tag{1.1}$$

$$\mathbf{x}(t)=\Phi(t),\quad t\in[-\tau,0], \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha\in(0,1)$, $\mathbf{x}(t)=(x_1(t),\dots,x_n(t))'$, where $'$ denotes the transpose of the vector, and $f_{ij}:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions for $i,j=1,2,\dots,n$, τ_j are continuous real-valued functions defined on $\mathbb{R}^+$, such that $\tau=\max\{\sup_{t\in\mathbb{R}^+}\tau_j(t):j=1,2,\dots,n\}>0$, and $\Phi(t)=(\phi_1(t),\dots,\phi_n(t))'$ is a given vector function defined on $[-\tau,0]$ with values in $\mathbb{R}^n$.

Our purpose is to establish sufficient conditions for the existence and uniqueness of a solution to the problem (1.1)-(1.2), by applying the Banach contraction principle. Furthermore, we prove the uniform stability of the solution.

While most existing research focuses on constant delays, the considered equations (1.1) contain variable delays. Moreover, it is important to note that our results are valid even in the case where the equations are of mixed type. Namely, equations of mixed type are those that have both retarded and advanced arguments. This makes a net difference with the previous works (see Remark 3.3). Thus, the present work generalizes the results obtained in [16, 17].

This paper is organized as follows. In Section 2, we introduce some basic definitions and notations, which are used in the sequel of the paper. In Section 3 and Section 4, we present our main results. Finally, in Section 5, two examples are given, as applications to illustrate our results.

2 Definitions and notations

Let us start by giving the definition of Riemann-Liouville fractional integral, and Caputo fractional derivatives. Further details of related basic properties used in the text can be found in [3, 5, 8].

Definition 2.1 Let $\alpha \in \mathbb{R}$. The Riemann-Liouville fractional integral operator I^α is defined on $L^1[0,T]$ by

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \quad t \in [0,T],$$

where $\Gamma(\cdot)$ is the gamma function.

For $\alpha = 0$, we set $I^0 := Id$, the identity operator. The operator I^α has the semigroup property, that is, for $\alpha, \beta \in \mathbb{R}^+$ and $f \in L^1[0,T]$, the identity

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$$

holds almost everywhere on $[0,T]$. Moreover, if $f \in C[0,T]$ or $\alpha + \beta \geq 1$, then the identity holds everywhere on $[0,T]$.

If $n \in \mathbb{N}^*$, and $D^n f$ (or $f^{(n)}$) means the nth derivative of a function f, then we have the following definition.

Definition 2.2 Let $n = [\alpha]$, and assume $D^n f \in L^1[0,T]$. The Caputo fractional derivative of order a real number $\alpha \geq 0$ is defined by

$${}^cD^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, \quad t \geq 0.$$

3 Existence and uniqueness

In this section we prove the existence and uniqueness of the solution of the problem (1.1)-(1.2).

Lemma 3.1 *The vector function* $\mathbf{x}(t) := (x_1(t), \dots, x_n(t))$ *is a solution of the problem* (1.1)-(1.2) *if and only if*

$$x_i(t) = \begin{cases} \phi_i(0) + \sum_{j=1}^{n} I^\alpha f_{ij}(t, x_i(t), x_j(t - \tau_j(t))), & t > 0, \\ \phi_i(t), & t \in [-\tau, 0], i = 1, 2, \dots, n. \end{cases} \tag{3.1}$$

Proof For $t > 0$ and $i = 1, 2, \dots, n$, (1.1) can be written as

$$I^{1-\alpha} Dx_i(t) = \sum_{j=1}^{n} f_{ij}\bigl(t, x_i(t), x_j\bigl(t - \tau_j(t)\bigr)\bigr).$$

Applying the operator I^α on both sides of the last equality, we obtain

$$IDx_i(t) = \sum_{j=1}^{n} I^\alpha f_{ij}\bigl(t, x_i(t), x_j\bigl(t - \tau_j(t)\bigr)\bigr),$$

$$x_i(t) - x_i(0) = \sum_{j=1}^{n} I^\alpha f_{ij}\bigl(t, x_i(t), x_j\bigl(t - \tau_j(t)\bigr)\bigr).$$

Then

$$x_i(t) = \phi_i(0) + \sum_{j=1}^{n} I^\alpha f_{ij}\bigl(t, x_i(t), x_j\bigl(t - \tau_j(t)\bigr)\bigr).$$

□

Let us denote by E the class of all continuous column vector-valued functions $C(\mathbb{R}^+, \mathbb{R}^n)$ equipped with the norm given by

$$\|\mathbf{x}\|_N = \sum_{i=1}^{n} \sup_{t \in \mathbb{R}^+} \bigl\{e^{-Nt}\bigl|x_i(t)\bigr|\bigr\}, \quad \mathbf{x} \in E,$$

where $N \in \mathbb{R}^+$ will be chosen later. We define the integral operator $F : E \to E$ by

$$Fx_i(t) = \begin{cases} \phi_i(0) + \sum_{j=1}^{n} I^\alpha f_{ij}(t, x_i(t), x_j(t - \tau_j(t))), & t > 0, \\ \phi_i(t), & t \in [-\tau, 0]. \end{cases}$$

Theorem 3.2 *Assume that the following hypotheses are satisfied*:

(H_1) *Let* $f_{ij} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function, satisfying the Lipschitz condition*

$$\bigl|f_{ij}(t, x_i, y_j) - f_{ij}(t, u_i, v_j)\bigr| \le k_i |x_i - u_i| + h_j |y_j - v_j|,$$

where $k_i, h_j > 0$, $i, j = \overline{1, n}$.

(H_2) *For* $j = 1, 2, \dots, n$, $\tau_j \in C(\mathbb{R}^+, \mathbb{R})$ *and*

$$\tau_j(t) > -\tau, \quad t > 0.$$

(H_3) *For* $j = 1, 2, \dots, n$, $\exists t_j > 0$ *such that*

$$\begin{cases} \tau_j(t) \ge t, & \forall t \in [0, t_j], \\ \tau_j(t) < t, & \forall t \in]t_j, +\infty[. \end{cases}$$

(H_4)

$$n\tau^{\alpha}\left[\sum_{i=1}^{n} k_i + \sum_{j=1}^{n} h_j e\right] < 1.$$

Then the problem (1.1)-(1.2) *has a unique solution.*

Proof Let $\mathbf{x}, \mathbf{y} \in E$, then for $i = 1, 2, \dots, n$ and $t > 0$ we have

$$\begin{aligned}
&\left|Fx_i(t) - Fy_i(t)\right| \\
&\quad = \left|\sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{f_{ij}\bigl(s, x_i(s), x_j\bigl(s - \tau_j(s)\bigr)\bigr) - f_{ij}\bigl(s, y_i(s), y_j\bigl(s - \tau_j(s)\bigr)\bigr)\right\} ds\right| \\
&\quad \le \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f_{ij}\bigl(s, x_i(s), x_j\bigl(s - \tau_j(s)\bigr)\bigr) - f_{ij}\bigl(s, y_i(s), y_j\bigl(s - \tau_j(s)\bigr)\bigr)\right| ds \\
&\quad \le \sum_{j=1}^{n} k_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|x_i(s) - y_i(s)\right| ds \\
&\qquad + \sum_{j=1}^{n} h_j \int_0^{t_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|\phi_j\bigl(r_j(s)\bigr) - \phi_j\bigl(r_j(s)\bigr)\right| ds \\
&\qquad + \sum_{j=1}^{n} h_j \int_{t_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|x_j\bigl(r_j(s)\bigr) - y_j\bigl(r_j(s)\bigr)\right| ds \\
&\quad \le nk_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|x_i(s) - y_i(s)\right| ds \\
&\qquad + \sum_{j=1}^{n} h_j \int_{t_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|x_j\bigl(r_j(s)\bigr) - y_j\bigl(r_j(s)\bigr)\right| ds,
\end{aligned}$$

where $r_j(s) = s - \tau_j(s)$, thus

$$\begin{aligned}
&e^{-Nt}\left|Fx_i(t) - Fy_i(t)\right| \\
&\quad \le nk_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns} \left|x_i(s) - y_i(s)\right| ds \\
&\qquad + \sum_{j=1}^{n} h_j \int_{t_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-r_j(s))} e^{-Nr_j(s)} \left|x_j\bigl(r_j(s)\bigr) - y_j\bigl(r_j(s)\bigr)\right| ds \\
&\quad \le nk_i \sup_{\xi\in\mathbb{R}^+} \left\{e^{-N\xi}\left|x_i(\xi) - y_i(\xi)\right|\right\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)}\, ds \\
&\qquad + \sum_{j=1}^{n} h_j \sup_{\xi\in\mathbb{R}^+} \left\{e^{-Nr_j(\xi)}\left|x_j\bigl(r_j(\xi)\bigr) - y_j\bigl(r_j(\xi)\bigr)\right|\right\} \int_{t_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-r_j(s))}\, ds \\
&\quad \le nk_i \sup_{\xi\in\mathbb{R}^+} \left\{e^{-N\xi}\left|x_i(\xi) - y_i(\xi)\right|\right\} \frac{1}{N^{\alpha}} \int_0^{Nt} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}\, du \\
&\qquad + \sum_{j=1}^{n} h_j \sup_{\xi\in\mathbb{R}^+} \left\{e^{-N\xi}\left|x_j(\xi) - y_j(\xi)\right|\right\} \frac{1}{N^{\alpha}} \int_0^{N(t-t_j)} \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u} e^{-N\tau_j(t-\frac{u}{N})}\, du
\end{aligned}$$

$$\begin{aligned}
&\le nk_i \sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}|x_j(\xi)-y_j(\xi)|\}\frac{1}{N^\alpha}\int_0^{Nt}\frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}\,du \\
&\quad+\sum_{j=1}^{n} h_j \sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}|x_j(\xi)-y_j(\xi)|\}\frac{1}{N^\alpha}\int_0^{N(t-t_j)}\frac{u^{\alpha-1}}{\Gamma(\alpha)}e^{-u}e^{N\tau}\,du \\
&\le \frac{nk_i}{N^\alpha}\sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}|x_j(\xi)-y_j(\xi)|\}+\sum_{j=1}^{n}\frac{h_je^{N\tau}}{N^\alpha}\sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}|x_j(\xi)-y_j(\xi)|\} \\
&\le \frac{nk_i}{N^\alpha}\|\mathbf{x}-\mathbf{y}\|_N+\sum_{j=1}^{n}\frac{h_je^{N\tau}}{N^\alpha}\|\mathbf{x}-\mathbf{y}\|_N.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{n}\sup_{t\in\mathbb{R}^+}\{e^{-Nt}|Fx_i(t)-Fy_i(t)|\} &\le \left[\sum_{i=1}^{n}\frac{nk_i}{N^\alpha}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_je^{N\tau}}{N^\alpha}\right]\|\mathbf{x}-\mathbf{y}\|_N \\
&\le \frac{n}{N^\alpha}\left[\sum_{i=1}^{n}k_i+\sum_{j=1}^{n}h_je^{N\tau}\right]\|\mathbf{x}-\mathbf{y}\|_N.
\end{aligned}$$

Let us choose $N=\frac{1}{\tau}$. So, we have

$$\sum_{i=1}^{n}\sup_{t\in\mathbb{R}^+}\{e^{-Nt}|Fx_i(t)-Fy_i(t)|\}\le n\tau^\alpha\left[\sum_{i=1}^{n}k_i+\sum_{j=1}^{n}h_je\right]\|\mathbf{x}-\mathbf{y}\|_N.$$

From hypothesis (H_4) we have $n\tau^\alpha[\sum_{i=1}^n k_i+\sum_{j=1}^n h_je]<1$. So, $F:E\to E$ is a contraction. Hence, it has a unique fixed point $\mathbf{x}=F\mathbf{x}$ which is precisely the unique solution of our problem (1.1)-(1.2). □

Remark 3.3 Note that, if for some j the delay function $\tau_j(t)$ takes negative values, which is possible under the assumptions (H_2) and (H_3), then (1.1) are with advanced arguments. Thus, the sign of the delay functions being arbitrary, the equations of the considered system (1.1) may contain both types of deviation of argument *i.e.* both delay and advance. As far as we know, there are no published studies addressing these issues for such systems of equations. However, concerning boundary value problems of fractional order, some results on the existence of solutions are obtained in [18]. But in [18], the authors considered problems involving only an advanced argument, and they do not address the question about the uniqueness (and the stability) of the solution.

4 Stability

In this section we study the stability of the solution of the problem (1.1)-(1.2).

Definition 4.1 The solution of the problem (1.1)-(1.2) is stable if for any $\epsilon>0$, there exists $\delta>0$ such that for any two solutions $\mathbf{x}(t)=(x_1(t),\dots,x_n(t))'$ and $\tilde{\mathbf{x}}(t)=(\tilde{x}_1(t),\dots,\tilde{x}_n(t))'$ with the initial condition (1.2) and $\tilde{\mathbf{x}}(t)=\widetilde{\Phi}(t)$ for $t\in[-\tau,0]$, respectively, one has $\|\Phi-\widetilde{\Phi}\|\le\delta$, implies $\|\mathbf{x}-\tilde{\mathbf{x}}\|_N\le\epsilon$, where $\|\cdot\|$ denotes the supremum norm defined by $\|\Psi\|=\sum_{i=1}^n\max_{t\in[-\tau,0]}|\psi_i(t)|$, for all bounded vector function Ψ from $[-\tau,0]$ to $\mathbb{R}^n$.

Theorem 4.2 *Assume that hypotheses* (H_1)-(H_4) *in Theorem* 3.2 *are satisfied, then the solution of the problem* (1.1)-(1.2) *is uniformly stable*.

Proof Let $\mathbf{x}(t)$ and $\tilde{\mathbf{x}}(t)$ be the solutions of the system (1.1) under the conditions (1.2) and $\{\tilde{\mathbf{x}}(t) = \widetilde{\Phi}(t) \text{ for } t \in [-\tau, 0]\}$, respectively. Then for $t > 0$, from (3.1), we have

$$\begin{aligned} x_i(t) - \tilde{x}_i(t) &= \phi_i(0) - \tilde{\phi}_i(0) \\ &\quad + \sum_{j=1}^{n} I^{\alpha}\bigl\{f_{ij}\bigl(t, x_i(t), x_j\bigl(t - \tau_j(t)\bigr)\bigr) - f_{ij}\bigl(t, \tilde{x}_i(t), \tilde{x}_j\bigl(t - \tau_j(t)\bigr)\bigr)\bigr\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\bigl|x_i(t) - \tilde{x}_i(t)\bigr| \\ &\quad \leq \bigl|\phi_i(0) - \tilde{\phi}_i(0)\bigr| \\ &\qquad + \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f_{ij}\bigl(s, x_i(s), x_j\bigl(s - \tau_j(s)\bigr)\bigr) - f_{ij}\bigl(s, \tilde{x}_i(s), \tilde{x}_j\bigl(s - \tau_j(s)\bigr)\bigr)\bigr| \, ds \\ &\quad \leq \bigl|\phi_i(0) - \tilde{\phi}_i(0)\bigr| + \sum_{j=1}^{n} k_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|x_i(s) - \tilde{x}_i(s)\bigr| \, ds \\ &\qquad + \sum_{j=1}^{n} h_j \int_0^{t_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|\phi_j\bigl(s - \tau_j(s)\bigr) - \tilde{\phi}_j\bigl(s - \tau_j(s)\bigr)\bigr| \, ds \\ &\qquad + \sum_{j=1}^{n} h_j \int_{t_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|x_j\bigl(s - \tau_j(s)\bigr) - \tilde{x}_j\bigl(s - \tau_j(s)\bigr)\bigr| \, ds \\ &\quad \leq \max_{s \in [-\tau, 0]} \bigl|\phi_i(s) - \tilde{\phi}_i(s)\bigr| + nk_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|x_i(s) - \tilde{x}_i(s)\bigr| \, ds \\ &\qquad + \sum_{j=1}^{n} \max_{s \in [-\tau, 0]} \bigl|\phi_j(s) - \tilde{\phi}_j(s)\bigr| h_j \int_0^{t_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, ds \\ &\qquad + \sum_{j=1}^{n} h_j \int_{t_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|x_j\bigl(s - \tau_j(s)\bigr) - \tilde{x}_j\bigl(s - \tau_j(s)\bigr)\bigr| \, ds. \end{aligned}$$

Hence,

$$\begin{aligned} &e^{-Nt}\bigl|x_i(t) - \tilde{x}_i(t)\bigr| \\ &\quad \leq e^{-Nt} \max_{s \in [-\tau, 0]} \bigl|\phi_i(s) - \tilde{\phi}_i(s)\bigr| + nk_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns} \bigl|x_i(s) - \tilde{x}_i(s)\bigr| \, ds \\ &\qquad + e^{-Nt} \sum_{j=1}^{n} \max_{s \in [-\tau, 0]} \bigl|\phi_j(s) - \tilde{\phi}_j(s)\bigr| \frac{h_j}{\Gamma(\alpha+1)} \bigl[t^{\alpha} - (t - t_j)^{\alpha}\bigr] \\ &\qquad + \sum_{j=1}^{n} h_j \int_{t_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t - r_j(s))} e^{-Nr_j(s)} \bigl|x_j\bigl(r_j(s)\bigr) - \tilde{x}_j\bigl(r_j(s)\bigr)\bigr| \, ds \\ &\quad \leq \max_{s \in [-\tau, 0]} \bigl|\phi_i(s) - \tilde{\phi}_i(s)\bigr| + \sum_{j=1}^{n} \max_{s \in [-\tau, 0]} \bigl|\phi_j(s) - \tilde{\phi}_j(s)\bigr| \frac{h_j t_j^{\alpha}}{\Gamma(\alpha+1)} \end{aligned}$$

$$
\begin{aligned}
&+ nk_i \sup_{\xi\in\mathbb{R}^+}\bigl\{e^{-N\xi}\bigl|x_i(\xi)-\tilde{x}_i(\xi)\bigr|\bigr\}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-N(t-s)}\,ds \\
&+ \sum_{j=1}^{n} h_j \sup_{\xi\in\mathbb{R}^+}\bigl\{e^{-Nr_j(\xi)}\bigl|x_j\bigl(r_j(\xi)\bigr)-\tilde{x}_j\bigl(r_j(\xi)\bigr)\bigr|\bigr\}\int_{t_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-N(t-r_j(s))}\,ds \\
\le{}& \max_{s\in[-\tau,0]}\bigl|\phi_i(s)-\tilde{\phi}_i(s)\bigr| + \sum_{j=1}^{n}\max_{s\in[-\tau,0]}\bigl|\phi_j(s)-\tilde{\phi}_j(s)\bigr|\frac{h_jt_j^\alpha}{\Gamma(\alpha+1)} \\
&+ nk_i \sup_{\xi\in\mathbb{R}^+}\bigl\{e^{-N\xi}\bigl|x_i(\xi)-\tilde{x}_i(\xi)\bigr|\bigr\}\frac{1}{N^\alpha}\int_0^{Nt}\frac{u^{\alpha-1}}{\Gamma(\alpha)}e^{-u}\,du \\
&+ \sum_{j=1}^{n} h_j \sup_{\xi\in\mathbb{R}^+}\bigl\{e^{-N\xi}\bigl|x_j(\xi)-\tilde{x}_j(\xi)\bigr|\bigr\}\frac{1}{N^\alpha}\int_0^{N(t-t_j)}\frac{u^{\alpha-1}}{\Gamma(\alpha)}e^{-u}e^{-N\tau_j(t-\frac{u}{N})}\,du \\
\le{}& \max_{s\in[-\tau,0]}\bigl|\phi_i(s)-\tilde{\phi}_i(s)\bigr| + \sum_{j=1}^{n}\|\Phi-\tilde{\Phi}\|\frac{h_jt_j^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{nk_i}{N^\alpha}\sup_{\xi\in\mathbb{R}^+}\bigl\{e^{-N\xi}\bigl|x_i(\xi)-\tilde{x}_i(\xi)\bigr|\bigr\} + \sum_{j=1}^{n}\frac{h_je^{N\tau}}{N^\alpha}\sup_{\xi\in\mathbb{R}^+}\bigl\{e^{-N\xi}\bigl|x_j(\xi)-\tilde{x}_j(\xi)\bigr|\bigr\} \\
\le{}& \max_{s\in[-\tau,0]}\bigl|\phi_i(s)-\tilde{\phi}_i(s)\bigr| + \sum_{j=1}^{n}\|\Phi-\tilde{\Phi}\|\frac{h_jt_j^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{nk_i}{N^\alpha}\|\mathbf{x}-\tilde{\mathbf{x}}\|_N + \sum_{j=1}^{n}\frac{h_je^{N\tau}}{N^\alpha}\|\mathbf{x}-\tilde{\mathbf{x}}\|_N.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{n}\sup_{t\in\mathbb{R}^+}\bigl\{e^{-Nt}\bigl|x_i(t)-\tilde{x}_i(t)\bigr|\bigr\} \\
&\quad\le \sum_{i=1}^{n}\max_{s\in[-\tau,0]}\bigl|\phi_i(s)-\tilde{\phi}_i(s)\bigr| + \sum_{i=1}^{n}\sum_{j=1}^{n}\|\Phi-\tilde{\Phi}\|\frac{h_jt_j^\alpha}{\Gamma(\alpha+1)} \\
&\qquad+ \sum_{i=1}^{n}\frac{nk_i}{N^\alpha}\|\mathbf{x}-\tilde{\mathbf{x}}\|_N + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{h_je^{N\tau}}{N^\alpha}\|\mathbf{x}-\tilde{\mathbf{x}}\|_N \\
&\quad\le \|\Phi-\widetilde{\Phi}\| + \frac{n\sum_{j=1}^n h_jt_j^\alpha}{\Gamma(\alpha+1)}\|\Phi-\tilde{\Phi}\| + \frac{n\sum_{i=1}^n k_i}{N^\alpha}\|\mathbf{x}-\tilde{\mathbf{x}}\|_N + \frac{n\sum_{j=1}^n h_je^{N\tau}}{N^\alpha}\|\mathbf{x}-\tilde{\mathbf{x}}\|_N \\
&\quad\le \biggl[1+\frac{n\sum_{j=1}^n h_jt_j^\alpha}{\Gamma(\alpha+1)}\biggr]\|\Phi-\widetilde{\Phi}\| + \frac{n[\sum_{i=1}^n k_i+\sum_{j=1}^n h_je^{N\tau}]}{N^\alpha}\|\mathbf{x}-\tilde{\mathbf{x}}\|_N.
\end{aligned}
$$

It follows that

$$
\biggl[1-\frac{n[\sum_{i=1}^n k_i+\sum_{j=1}^n h_je^{N\tau}]}{N^\alpha}\biggr]\|\mathbf{x}-\tilde{\mathbf{x}}\|_N \le \biggl[1+\frac{n\sum_{j=1}^n h_jt_j^\alpha}{\Gamma(\alpha+1)}\biggr]\|\Phi-\widetilde{\Phi}\|.
$$

We choose $N=\frac{1}{\tau}$, and we have

$$
\Biggl[1-n\tau^\alpha\Biggl[\sum_{i=1}^{n}k_i+\sum_{j=1}^{n}h_je\Biggr]\Biggr]\|\mathbf{x}-\tilde{\mathbf{x}}\|_N \le \biggl[1+\frac{n\sum_{j=1}^n h_jt_j^\alpha}{\Gamma(\alpha+1)}\biggr]\|\Phi-\widetilde{\Phi}\|.
$$

Therefore, given any $\epsilon > 0$, there exists

$$\delta = \left[1 - n\tau^{\alpha}\left[\sum_{i=1}^{n} k_i + \sum_{j=1}^{n} h_j e\right]\right]\left[1 + \frac{n\sum_{j=1}^{n} h_j t_j^{\alpha}}{\Gamma(\alpha+1)}\right]^{-1}\epsilon > 0,$$

such that if $\|\Phi - \widetilde{\Phi}\| < \delta$, then $\|\mathbf{x} - \tilde{\mathbf{x}}\|_N \le \epsilon$, which shows that the solution of the problem (1.1)-(1.2) is uniformly stable. □

5 Applications

Example 5.1 Consider the problem

$$\begin{cases} {}^cD^{\alpha}x_1(t) = \frac{10^{-3}}{x_1^2(t)+1} + \frac{10^{-2}}{x_2^2(t-\tau_2(t))+1}, \\ {}^cD^{\alpha}x_2(t) = \frac{10^{-1}}{x_2^2(t)-x_2(t)+1} + \frac{10^{-2}}{x_3^2(t-\tau_3(t))-x_3(t-\tau_3(t))+1}, \qquad t > 0, \\ {}^cD^{\alpha}x_3(t) = \frac{10^{-3}}{x_3^2(t)+1} + \frac{10^{-2}}{x_1^2(t-\tau_1(t))+1}, \end{cases}$$

and

$$\mathbf{x}(t) = \Phi(t), \quad t \in [-\tau, 0],$$

where $\alpha = 0.8$, $\tau_j(t) = \frac{4}{3} - \frac{1}{j+t}$ for $j = 1, 2, 3$. $\tau = \max_{t\in\mathbb{R}^+} \tau_j(t) = \frac{4}{3}$.

It is easy to see that the conditions (H_1) and (H_2) of Theorem 3.2 hold. Also, $\exists t_j = \frac{4-3j+\sqrt{9j^2+24j-20}}{6}$ such that

$$\begin{cases} \tau_j(t) \ge t, & \forall t \in [0, t_j], \\ \tau_j(t) < t, & \forall t \in]t_j, +\infty[, \end{cases}$$

and

$$3\tau^{\alpha}\left[\sum_{i=1}^{3} k_i + \sum_{j=1}^{3} h_j e\right] = 0.9523063402 < 1,$$

where $k_1 = k_3 = 10^{-3}$, $k_2 = 10^{-1}\frac{8\sqrt{3}}{9}$, $h_1 = h_2 = 10^{-2}$ and $h_3 = 10^{-2}\frac{8\sqrt{3}}{9}$.

Hence, all hypotheses of Theorem 3.2 are fulfilled. Thus, the problem has a unique solution, and by Theorem 4.2 the solution is uniformly stable. Therefore, as a conclusion, the problem has a unique uniform stable solution.

Example 5.2 Consider the problem

$$\begin{cases} {}^cD^{\alpha}x_1(t) = 10^{-1}\{\sqrt{x_1^2(t)+1} + \sqrt{x_2^2(t-\tau_2(t))+1}\}, \\ {}^cD^{\alpha}x_2(t) = 10^{-2}\{x_2(t) + x_1(t-\tau_1(t))\}, \end{cases} \qquad t > 0,$$

and

$$\mathbf{x}(t) = \Phi(t), \quad t \in [-\tau, 0],$$

where $\alpha = 0.4$, $\tau_j(t) = \frac{5}{4} - \frac{1}{j+t}$ for $j = 1, 2$. $\tau = \max_{t\in\mathbb{R}^+} \tau_j(t) = \frac{5}{4}$.

It is easy to see that the conditions (H_1) and (H_2) of Theorem 3.2 hold. Also, $\exists t_j = \frac{-4j+\sqrt{16j^2+8}}{8}$ such that

$$\begin{cases} \tau_j(t) \geq t, & \forall t \in [0, t_j], \\ \tau_j(t) < t, & \forall t \in]t_j, +\infty[\end{cases}$$

and

$$2\tau^{\alpha}\left[\sum_{i=1}^{2} k_i + \sum_{j=1}^{2} h_j e\right] = 0.8943942329 < 1,$$

where $k_1 = h_2 = 10^{-1}$ and $k_2 = h_1 = 10^{-2}$.

Hence, all hypotheses of Theorem 3.2 are satisfied. Thus, the problem has a unique solution. Moreover, by Theorem 4.2 the solution is uniformly stable. Therefore, as a conclusion, the problem has a unique uniform stable solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LN has proposed the main idea of this paper and has directed this study. LN and AB have conceived and drafted the manuscript. All authors read and approved the final manuscript.

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