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On some complex differential and difference equations concerning sharing function

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Abstract

By using the theory of complex differential equations, the purpose of this paper is to investigate a conjecture of Brück concerning an entire function f and its differential polynomial $L(f) = a_k(z)f^{(k)} + \cdots + a_0(z)f$ sharing a function $\alpha(z)$ and a constant β. We also study the problem on entire function and its difference polynomials sharing a function.

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1 Introduction and main results

Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. We shall use the following standard notations of the value distribution theory:

$$T(r,f), \qquad m(r,f), \qquad N(r,f), \qquad \overline{N}(r,f), \qquad \ldots$$

(see Hayman [1], Yang [2] and Yi and Yang [3]). We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to +\infty$, possibly outside of a set with finite measure. A meromorphic function $a(z)$ is called a small function with respect to f if $T(r,a) = S(r,f)$. In addition, we will use the notation $\sigma(f)$ to denote the order of meromorphic function $f(z)$, and $\tau(f)$ to denote the type of an entire function $f(z)$ with $0 < \sigma(f) = \sigma < +\infty$, which are defined to be (see [1])

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \qquad \tau(f) = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\sigma}}.$$

We use $\sigma_2(f)$ to denote the hyper-order of $f(z)$, $\sigma_2(f)$ is defined to be (see [3])

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In 1976, Rubel and Yang [4] proved the following result.

Theorem 1.1 [4] *Let f be a nonconstant entire function. If f and f' share two finite distinct values CM, then $f \equiv f'$.*

In 1996, Brück [5] gave the following conjecture.

Conjecture 1.1 [5] *Let f be a nonconstant entire function. Suppose that $\sigma(f)$ is not a positive integer or infinite, if f and f' share one finite value a CM, then*

$$\frac{f'-a}{f-a}=c$$

for some nonzero constant c.

In 1998, Gundersen and Yang [6] proved that Brück's conjecture holds for entire functions of finite order and obtained the following result.

Theorem 1.2 [6, Theorem 1] *Let f be a nonconstant entire function of finite order. If f and f' share one finite value a CM, then $\frac{f'-a}{f-a}=c$ for some nonzero constant c.*

The shared value problems relative to a meromorphic function f and its derivative $f^{(k)}$ have been a more widely studied subtopic of the uniqueness theory of entire and meromorphic functions in the field of complex analysis (see [7–12]).

In 2009, Chang and Zhu [13] further investigated the problem related to Brück's conjecture and proved that Theorem 1.2 remains valid if the value a is replaced by a function.

Theorem 1.3 [13, Theorem 1] *Let f be an entire function of finite order and $a(z)$ be a function such that $\sigma(a)<\sigma(f)<\infty$. If f and f' share $a(z)$ CM, then $\frac{f'-a}{f-a}=c$ for some nonzero constant c.*

Thus, there are natural questions to ask:

(i) What would happen when $\sigma(a)<\sigma(f)<\infty$ is replaced by $0<\sigma(a)=\sigma(f)<\infty$ in Theorem 1.3?

(ii) For Theorems 1.1-1.3, what would happen when f' is replaced by differential polynomial

$$L(f)=a_k(z)f^{(k)}+a_{k-1}(z)f^{(k-1)}+\cdots+a_1(z)f'+a_0(z)f, \tag{1}$$

where $a_k(z)$ ($\not\equiv 0$), $\ldots, a_0(z)$ are polynomials?

The main purpose of this article is to study the above questions and obtain the following theorems.

Theorem 1.4 *Let $f(z)$ and $\alpha(z)$ be two nonconstant entire functions and satisfy $0<\sigma(\alpha)=\sigma(f)<\infty$ and $\tau(f)>\tau(\alpha)$, and $L(f)$ be stated as in (1) such that*

$$\sigma(f)>1+\max\left\{\frac{\deg_z a_j-\deg_z a_k}{k-j},0\right\}.$$

If $f(z)$ and $L(f(z))$ share $\alpha(z)$ CM, then

$$\frac{L(f(z))-\alpha(z)}{f(z)-\alpha(z)}=c$$

for some nonzero constant c.

Theorem 1.5 *Let $f(z)$ be a nonconstant transcendental entire function with $\sigma_2(f) < \infty$, let $\sigma_2(f)$ be not an integer, and let $L(f)$ be stated as in* (1). *If f and $L(f)$ share a nonzero constant a CM and $\delta(0,f) > 0$, then*

$$\frac{L(f(z)) - a}{f(z) - a} = c$$

for some nonzero constant c.

Recently, some papers have studied Brück's conjecture related to difference of entire function (including [14, 15]). In 2009, Heittokangas *et al.* [14] got the following result.

Theorem 1.6 [14, Theorem 1] *Let f be a nonconstant meromorphic function of finite order $\sigma(f) < 2$, and let η be a nonzero complex number. If $f(z+\eta)$ and $f(z)$ share a finite complex value a CM, then $f(z+\eta) - a = c(f(z) - a)$ for all $z \in \mathbb{C}$, where c is some nonzero complex number.*

In this paper, we further investigate Brück's conjecture related to entire function and its difference polynomial and obtain the following result.

Theorem 1.7 *Let $f(z)$ be a nonconstant entire function of finite order $0 < \sigma(f) < \infty$, $L_1(f)$ be difference polynomial of f of the form*

$$L_1\bigl(f(z)\bigr) = f(z+\eta_k) + f(z+\eta_{k-1}) + \cdots + f(z+\eta_1),$$

where $\eta_k, \eta_{k-1}, \ldots, \eta_1$ are nonzero complex numbers. If $L_1(f(z)) = cf(z)$ and ξ $(\neq 0)$ is a Borel exceptional value of $f(z)$, then $L_1(f) = kf(z)$.

2 Some lemmas

To prove our theorems, we will require some lemmas as follows.

Lemma 2.1 [16] *Let $f(z)$ be a transcendental entire function, $\nu(r,f)$ be the central index of $f(z)$. Then there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure, we choose z satisfying $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r,f)$, we get*

$$\frac{f^{(j)}(z)}{f(z)} = \left\{\frac{\nu(r,f)}{z}\right\}^j \bigl(1 + o(1)\bigr) \quad \textit{for } j \in N.$$

Lemma 2.2 [17] *Let $f(z)$ be an entire function of finite order $\sigma(f) = \sigma < \infty$, and let $\nu(r,f)$ be the central index of f. Then, for any ε (> 0), we have*

$$\limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log r} = \sigma.$$

Lemma 2.3 [18] *Let f be a transcendental entire function, and let $E \subset [1, +\infty)$ be a set having finite logarithmic measure. Then there exists $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi)$, $\lim_{n\to\infty} \theta_n = \theta_0 \in [0, 2\pi)$, $r_n \notin E$ and if $0 < \sigma(f) < \infty$, then, for any given $\varepsilon > 0$ and sufficiently large r_n,*

$$r_n^{\sigma(f)-\varepsilon} < \nu(r_n, f) < r_n^{\sigma(f)+\varepsilon}.$$

Lemma 2.4 [16] *Let* $P(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$ *with* $b_n \neq 0$ *be a polynomial. Then, for every* $\varepsilon > 0$, *there exists* $r_0 > 0$ *such that for all* $r = |z| > r_0$ *the inequalities*

$$(1-\varepsilon)|b_n|r^n \le \left|P(z)\right| \le (1+\varepsilon)|b_n|r^n$$

hold.

Lemma 2.5 *Let* $f(z)$ *and* $A(z)$ *be two entire functions with* $0 < \sigma(f) = \sigma(A) = \sigma < \infty$, $0 < \tau(A) < \tau(f) < \infty$, *then there exists a set* $E \subset [1, +\infty)$ *that has infinite logarithmic measure such that for all* $r \in E$ *and a positive number* $\kappa > 0$, *we have*

$$\frac{M(r,A)}{M(r,f)} < \exp\{-\kappa r^\sigma\}.$$

Proof By definition, there exists an increasing sequence $\{r_m\} \to \infty$ satisfying $(1+\frac{1}{m})r_m < r_{m+1}$ and

$$\lim_{m\to\infty} \frac{\log M(r_m,f)}{r_m^\sigma} = \tau(f). \tag{2}$$

For any given β $(\tau(A) < \beta < \tau(f))$, there exists some positive integer m_0 such that for all $m \ge m_0$ and for any given ε $(0 < \varepsilon < \tau(f) - \beta)$, we have

$$\log M(r_m,f) > \bigl(\tau(f) - \varepsilon\bigr) r_m^\sigma. \tag{3}$$

Thus, there exists some positive integer m_1 such that for all $m \ge m_1$, we have

$$\left(\frac{m}{m+1}\right)^\sigma > \frac{\beta}{\tau(f) - \varepsilon}. \tag{4}$$

From (2)-(4), for all $m \ge m_2 = \max\{m_0, m_1\}$ and for any $r \in [r_m, (1+\frac{1}{m})r_m]$, we have

$$\begin{aligned} M(r,f) &\ge M(r_m,f) > \exp\bigl\{\bigl(\tau(f)-\varepsilon\bigr)r_m^\sigma\bigr\} \\ &\ge \exp\left\{\bigl(\tau(f)-\varepsilon\bigr)\left(\frac{m}{m+1}r\right)^\sigma\right\} > \exp\bigl\{\beta r^\sigma\bigr\}. \end{aligned} \tag{5}$$

Set $E = \bigcup_{m=m_2}^\infty [r_m, (1+\frac{1}{m})r_m]$, then

$$m_l E = \sum_{m=m_2}^\infty \int_{r_m}^{(1+\frac{1}{m})r_m} \frac{dt}{t} = \sum_{m=m_2}^\infty \log\left(1 + \frac{1}{m}\right) = \infty.$$

From the definition of type of entire function, for any sufficiently small $\varepsilon > 0$, we have

$$M(r,A) < \exp\bigl\{\bigl(\tau(A)+\varepsilon\bigr)r^\sigma\bigr\}. \tag{6}$$

By (5) and (6), set $\kappa = \beta - \tau(A) - \varepsilon$, for all $r \in E$, we have

$$\frac{M(r,A)}{M(r,f)} < \exp\bigl\{-\bigl(\beta - \tau(A) - \varepsilon\bigr)r^\sigma\bigr\} = e^{-\kappa r^\sigma}.$$

Thus, this completes the proof of this lemma. □

Lemma 2.6 [19, Theorem 2.1] *Let $f(z)$ be a meromorphic function of finite order σ, and let η be a fixed nonzero complex number, then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.7 [19, Corollary 2.5] *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f)$, $\sigma < +\infty$, and let η be a fixed nonzero complex number, then, for each $\varepsilon > 0$, we have*

$$T\left(r, f(z+\eta)\right) = T(r,f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).$$

Lemma 2.8 [1, 20] *Let $g : (0, +\infty) \to R$, $h : (0, +\infty) \to R$ be monotone increasing functions such that $g(r) \le h(r)$ outside of an exceptional set E with finite linear measure, or $g(r) \le h(r)$, $r \notin H \cup (0, 1]$, where $H \subset (1, \infty)$ is a set of finite logarithmic measure. Then, for any $\alpha > 1$, there exists r_0 such that $g(r) \le h(\alpha r)$ for all $r \ge r_0$.*

3 The proof of Theorem 1.4

Since $f(z)$ is an entire function, and $f(z)$ and $L(f(z))$ share $\alpha(z)$ *CM*, then there is an entire function $\gamma(z)$ such that

$$\frac{L(f(z)) - \alpha(z)}{f(z) - \alpha(z)} = e^{\gamma(z)}. \tag{7}$$

Next, we will claim that $\gamma(z)$ is a constant.

Suppose that $\gamma(z)$ is transcendental. It follows that $\sigma(e^{\gamma(z)}) = \infty$. However, since $0 < \sigma(f) = \sigma(\alpha) < \infty$, it follows from the left-hand side of (7) that $\sigma(\frac{L(f(z))-\alpha(z)}{f(z)-\alpha(z)}) < \infty$, a contradiction. Thus, $\gamma(z)$ is not transcendental.

Suppose that $\gamma(z)$ is a nonconstant polynomial, let

$$\gamma(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,$$

where $b_m, \ldots, b_0$ are constants and $b_m \ne 0$, $m \ge 1$. Thus, it follows from (7) and Lemma 2.4 that

$$|b_m| r^m \left(1 + o(1)\right) = \left|\gamma(z)\right| = \left|\log \frac{\frac{L(f(z))}{f(z)} - \frac{\alpha(z)}{f(z)}}{1 - \frac{\alpha(z)}{f(z)}}\right|. \tag{8}$$

Since $L(f) = a_k f^k + a_{k-1} f^{(k-1)} + \cdots + a_0 f$, from Lemma 2.1, then there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that for some point $z = re^{i\theta}$ ($\theta \in [0, 2\pi)$), $r \notin E_1$ and $M(r,f) = |f(z)|$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left\{\frac{\nu(r,f)}{z}\right\}^j \left(1 + o(1)\right), \quad 1 \le j \le k.$$

Thus, it follows that

$$\begin{aligned} \frac{L(f(z))}{f(z)} &= a_k \left\{\frac{\nu(r,f)}{z}\right\}^k \left(1 + o(1)\right) + \cdots + a_1 \left\{\frac{\nu(r,f)}{z}\right\}\left(1 + o(1)\right) + a_0 \\ &= \frac{a_k}{z^k}\left(1 + o(1)\right)\left[\nu(r,f)^k + \sum_{j=1}^{k} \frac{a_{k-j}}{a_k} z^j \nu(r,f)^{k-j}\left(1 + o(1)\right)\right]. \end{aligned} \tag{9}$$

From Lemma 2.3, there exists $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n,f)$, $\theta_n \in [0,2\pi)$, $\lim_{n\to\infty}\theta_n = \theta_0 \in [0,2\pi)$, $r_n \notin E_1$, then, for any given ε satisfying

$$0 < \varepsilon < \min_{1\le j\le k} \frac{j\sigma(f) - j - d_{k-j}}{3k-j},$$

where $d_{k-j} = \deg_z a_{k-j} - \deg_z a_k$, and sufficiently large r_n, we have

$$r_n^{\sigma(f)-\varepsilon} < \nu(r_n,f) < r_n^{\sigma(f)+\varepsilon}. \tag{10}$$

Since $a_j(z)$, $j = 0,1,\dots,k$, are polynomials, let $a_j(z) = \sum_{t=0}^{s_j} l_{jt} z^t$, where $s_j = \deg_z a_j$, $j = 0,1,\dots,k$. Then, from Lemma 2.4 and (10), we have

$$\begin{aligned}\left|\frac{a_{k-j}}{a_k} z^j \nu(r,f)^{k-j}\bigl(1+o(1)\bigr)\right| &\le M\frac{|l_{k-j,s_{k-j}}|r_n^{s_{k-j}}}{|l_{k,s_k}|r_n^{s_k}} r_n^j r_n^{(\sigma(f)+\varepsilon)(k-j)} \\ &= M\frac{|l_{k-j,s_{k-j}}|}{|l_{k,s_k}|} r_n^{d_{k-j}+j+(\sigma(f)+\varepsilon)(k-j)} \\ &\le M\frac{|l_{k-j,s_{k-j}}|}{|l_{k,s_k}|} r_n^{k\sigma(f)-j\sigma(f)+d_{k-j}+j+(k-j)\varepsilon},\end{aligned} \tag{11}$$

where $d_{k-j} = s_{k-j} - s_k$ and M is a positive constant. Since $-j\sigma(f) + d_{k-j} + j + (k-j)\varepsilon < -2k\varepsilon < 0$, it follows from (11) that

$$\left|\frac{a_{k-j}}{a_k} z^j \nu(r,f)^{k-j}\bigl(1+o(1)\bigr)\right| < M\frac{|l_{k-j,s_{k-j}}|}{|l_{k,s_k}|} r_n^{k(\sigma(f)-2\varepsilon)}, \quad r_n \notin E_1. \tag{12}$$

Since $0 < \sigma(\alpha) = \sigma(f) < \infty$ and $\tau(\alpha) < \tau(f) < \infty$, from Lemma 2.5, there exists a set $E \subset [1,+\infty)$ that has infinite logarithmic measure such that for a sequence $\{r_n\}_1^\infty \in E_2 = E - E_1$, we have

$$\frac{M(r,\alpha)}{M(r,f)} < \exp\bigl\{-\kappa r_n^{\sigma(f)}\bigr\} \to 0 \quad \text{as } n \to \infty. \tag{13}$$

From (8), (9), (12), (13) and Lemma 2.2, we can get that

$$|b_m| r_n^m\bigl(1+o(1)\bigr) = \bigl|\gamma(z)\bigr| = O(\log r_n), \tag{14}$$

which is impossible. Thus, $\gamma(z)$ is not a polynomial.

Therefore, $\gamma(z)$ is a constant, that is, there exists some nonzero constant c such that

$$\frac{L(f(z)) - \alpha(z)}{f(z) - \alpha(z)} = c.$$

Thus, this completes the proof of Theorem 1.4.

4 The proof of Theorem 1.5

Since $L(f)$ and f share the constant a *CM*, then there exists an entire function $\varphi(z)$ such that

$$\frac{L(f) - a}{f - a} = e^{\varphi}. \tag{15}$$

We will consider two cases as follows.

Case 1. If $a = 0$, it follows from (15) that

$$\frac{L(f(z))}{f(z)} = e^{\varphi(z)}. \tag{16}$$

Since $L(f(z)) = a_k(z)f^{(k)}(z) + \cdots + a_1(z)f'(z) + a_0(z)$ and $a_j(z), j = 0,1,\ldots,k$, are polynomials, it follows from (16) that

$$T\big(r,e^{\varphi}\big) = m\big(r,e^{\varphi}\big) = m\left(r,\frac{L(f)}{f}\right) \le \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + \sum_{i=0}^{k} m(r,a_i) = O\big(\log rT(r,f)\big),$$

outside of an exceptional set E_3 with finite linear measure. Thus, there exists a constant K such that

$$T\big(r,e^{\varphi}\big) \le K\log\big(rT(r,f)\big) \quad \text{for } r \notin E_3.$$

By Lemma 2.8, there exists an $r_0 > 0$, and for all $r \ge r_0$, we have

$$T\big(r,e^{\varphi}\big) \le K\log\big(\zeta rT(\zeta r,f)\big) \quad \text{for any } \zeta > 1. \tag{17}$$

Thus, we can deduce from (17) that $\sigma(e^{\varphi}) \le \sigma_2(f) < \infty$, that is, $\varphi(z)$ is a polynomial.

By using the same argument as in [21, Theorem 1.1], we can get that $\sigma_2(f) = \deg_z \varphi$, which is a contradiction to $\sigma_2(f)$ is not a positive integer. Thus, $\varphi(z)$ is only a constant, it follows from (15) that $L(f(z)) = cf(z)$, where c is a nonzero constant.

Case 2. If $a \not= 0$, from the derivation of (15) and eliminating e^{φ}, we can get

$$\varphi'(z) = \frac{L'(f(z))}{L(f(z)) - a} - \frac{f'(z)}{f(z) - a}. \tag{18}$$

If $\varphi'(z) \equiv 0$, that is, $\varphi(z) \equiv c$, c is a constant. Thus, we can prove the conclusion of Theorem 1.5 easily.

If $\varphi'(z) \not\equiv 0$, then it follows from (18) that

$$m\big(r,\varphi'\big) = S(r,f). \tag{19}$$

We can rewrite (18) in the following form:

$$\begin{aligned}\varphi' &= f\left[\frac{L(f)}{f}\frac{1}{L(f)}\frac{L'(f)}{L(f)-a} - \frac{1}{f}\frac{f'}{f-a}\right]\\ &= \frac{f}{a}\left[\frac{L(f)}{f}\frac{L'(f)}{L(f)-a} - \frac{L'(f)}{f} - \frac{f'}{f-a} + \frac{f'}{f}\right].\end{aligned} \tag{20}$$

Since $\varphi' \not\equiv 0$ and f is transcendental, set

$$\Psi := \frac{L(f)}{f}\frac{L'(f)}{L(f)-a} - \frac{L'(f)}{f} - \frac{f'}{f-a} + \frac{f'}{f}, \tag{21}$$

then we have $m(r,\Psi) = S(r,f)$. Thus, it follows from (20) and (21) that

$$\frac{a}{f(z)} = \frac{\Psi(z)}{\varphi'(z)}. \tag{22}$$

Since $\varphi(z)$ is an entire function, from (18)-(22), then we have

$$\begin{aligned} m\left(r,\frac{1}{f}\right) &\le m(r,\Psi) + m\left(r,\frac{1}{\varphi'}\right) \le S(r,f) + T\left(r,\varphi'\right) \\ &= S(r,f) + m\left(r,\varphi'\right) = S(r,f). \end{aligned}$$

It follows that

$$\delta(0,f) = \liminf_{r\to\infty} \frac{m(r,\frac{1}{f})}{T(r,f)} = 0,$$

which is a contradiction to the assumption of Theorem 1.5.

Thus, from Case 1 and Case 2, we complete the proof of Theorem 1.5.

5 The proof of Theorem 1.7

Since $f(z)$ is an entire function of finite order $0 < \sigma(f) < \infty$ and ξ ($\neq 0$) is a Borel exceptional value of $f(z)$, then $f(z)$ can be written in the form

$$f(z) = \xi + p(z)e^{h(z)}, \tag{23}$$

where $h(z)$ is a polynomial of degree l and $p(z)$ is an entire function satisfying $\sigma(p(z)) < \sigma(f(z)) = \deg_z h(z) = l$. Thus, we have

$$f(z+\eta_j) = \xi + p(z+\eta_j)e^{h(z+\eta_j)}, \quad j = 1,2,\ldots,k. \tag{24}$$

From Lemma 2.7, we have $\sigma(p(z+\eta_j)) < \sigma(f(z+\eta_j)) = \sigma(f(z))$ and $\deg_z h(z+\eta_j) = \deg_z h(z) = l$ for $j = 1,2,\ldots,k$. Since $L_1(f(z)) = cf(z)$, it follows from (23) and (24) that

$$\sum_{j=1}^{k} p(z+\eta_j)e^{h(z+\eta_j)} = (c-k)d + cp(z)e^{h(z)}. \tag{25}$$

Set $h(z) = \mu_l z^l + \cdots$ and $\mu_l \neq 0$, then we can deduce from (25) that

$$\sum_{j=1}^{k} p(z+\eta_j)e^{\mu_{m-1}^{j} z^{m-1}+\cdots} = \frac{(c-k)d + cp(z)e^{h(z)}}{e^{\mu_l z^l}}. \tag{26}$$

Let $\Phi := \sum_{j=1}^{k} p(z+\eta_j)e^{\mu_{m-1}^{j} z^{m-1}+\cdots}$, it is easy to see that $\Phi \not\equiv 0$ and $\sigma(\Phi) < \sigma(f)$, that is, $T(r,\Phi) = o(T(r,f)) = o(T(r,e^{h(z)}))$.

Suppose that $c \neq k$. Since $\xi \neq 0$, it follows from (26) that

$$N\left(r,\frac{1}{e^{h(z)} - \frac{(c-k)\xi}{cp(z)}}\right) = N\left(r,\frac{1}{\Phi}\right) \le T(r,\Phi) = S\left(r,e^{h(z)}\right).$$

By the second fundamental theorem concerning small functions, for any $\varepsilon > 0$, we have

$$\begin{aligned} T\left(r, e^{h(z)}\right) &\le N\left(r, \frac{1}{e^{h(z)} - \frac{(c-k)\xi}{cp(z)}}\right) + \varepsilon T\left(r, e^{h(z)}\right) + S\left(r, e^{h(z)}\right) \\ &= \varepsilon T\left(r, e^{h(z)}\right) + S\left(r, e^{h(z)}\right). \end{aligned}$$

Since ε is arbitrary, we can get a contradiction from the above inequality. Thus, we can get that $c = k$.

Therefore, we prove that $L_1(f(z)) = kf(z)$, that is, the conclusion of Theorem 1.7 holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HW, L-ZY and H-YX completed the main part of this article, HW, H-YX corrected the main theorems. All authors read and approved the final manuscript.

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