Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Dynamics of deterministic and stochastic multi-group MSIRS epidemic models with varying total population size

Zhigang Wang[1], Xiaoming Fan[1*], Fuquan Jiang[2] and Qiang Li[3]

*Correspondence:
fanxm093@163.com
[1]School of Mathematical Sciences, Harbin Normal University, Harbin, 150500, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we extend the deterministic single-group MSIRS epidemic model to a multi-group model, and we also extend the deterministic multi-group framework to a stochastic one and formulate it as a stochastic differential equation. In the deterministic multi-group model, the basic reproduction number $\mathscr{R}_0$ is a threshold that completely determines the persistence or extinction of the disease. By using Lyapunov function techniques, we show that if $\mathscr{R}_0 > 1$, then the disease will prevail, the infective condition persists and the endemic state is asymptotically stable in a feasible region. If $\mathscr{R}_0 \leq 1$, then the infective condition disappears and the disease dies out. For the stochastic version, we perform a detailed analysis on the asymptotic behavior of the stochastic model, which also depends on the value of $\mathscr{R}_0$, when $\mathscr{R}_0 > 1$, we determine the asymptotic stability of the endemic equilibrium by measuring the difference between the solution and the endemic equilibrium of the deterministic model in time-averaged data. Numerical methods are used to illustrate the dynamic behavior of the model and to solve the systems.

Keywords: multi-group MSIRS model; stochastic perturbation; graph theory; Brownian motion

1 Introduction

Many models of the outbreak and spread of disease have been analyzed mathematically and applied to specific diseases, and these models have provided some useful and valid reference data for the characteristics of disease transmission. Based on the results of these theoretical analyses, one can predict the future course of an outbreak and evaluate the strategies used to control an epidemic. In 1927, Kermack and McKendrick investigated the classic SIR epidemic model and proved the existence of a threshold [1]; thereafter, the number of studies on epidemiological modeling has rapidly increased, and a tremendous variety of models have now been formulated, mathematically analyzed, and applied to various infectious diseases. These models have involved many aspects of infectious diseases, such as stage of infection, age structure, vertical transmission, spatial spread, loss of vaccine and disease-acquired immunity, vaccination, and quarantine [2–9]. Compartments with the labels S, E, I, and R (susceptible, exposed, infectious, and recovered) are often used for the epidemiological classes. For some infections, including measles, infants are not born into the susceptible compartment but are immune to the disease for the first few months of life due to protection from maternal antibodies. Thus it is reasonable to add an M class (maternally derived immunity) at the beginning of the model. In [4], Hethcote discuss the MSEIR model in detail. Threshold theorems involving the basic reproduction number R_0, the contact number σ, and the replacement number R are surveyed for the classic MSEIR epidemic [4]. Based on the MSEIR model in [4], Lou and Ma also proposed a class of single-group MSIRS models [5], which are formulated as follows:

$$\begin{cases} \frac{\mathrm{d}\overline{M}}{\mathrm{d}t} = b(N - \overline{S}) - (\delta + d)\overline{M}, \\ \frac{\mathrm{d}\overline{S}}{\mathrm{d}t} = b\overline{S} + \tau\overline{R} + \delta\overline{M} - \sum_{j=1}^{n} \frac{\beta\overline{S}\overline{I}}{N} - d\overline{S}, \\ \frac{\mathrm{d}\overline{I}}{\mathrm{d}t} = \frac{\beta\overline{S}\overline{I}}{N} - (d + \gamma)\overline{I}, \\ \frac{\mathrm{d}\overline{R}}{\mathrm{d}t} = \gamma\overline{I} - (d + \tau)\overline{R}, \\ \frac{\mathrm{d}N}{\mathrm{d}t} = (b - d)N, \quad k = 1, 2, \ldots, n. \end{cases} \tag{1.1}$$

In this MSIRS model, the flow of disease transmission is as follows. A mother has been infected, and some IgG antibodies are transferred across the placenta, so that her newborn infant has temporary passive immunity to an infection. The class M contains infants with passive immunity. After the maternal antibodies disappear from the body, the infant moves to the susceptible class S. Infants who do not have any passive immunity, because their mothers were never infected, also enter the class S of susceptible individuals; next the susceptible enters the I class while they are infectious and then move to the recovered class R upon temporary recovery. The MSIRS model for infections that do not confer permanent immunity (*i.e.*, an infection does not leave a long-lasting immunity, and thus individuals who have recovered return to being susceptible), the individual enters the susceptible class S (for a detailed introduction, see [4]).

Considering the different contact patterns, a distinct number of sexual partners, or different geography, among other variables, it is more appropriate to divide individual hosts into groups in the modeling of an epidemic disease. Therefore, it is reasonable to propose multi-group models to describe the transmission dynamics of diseases in heterogeneous host populations. In fact, there are already many scholars who focus their studies on the various forms of multi-group epidemic models [6–11]. Kuniya investigated the global stability of a multi-group SVIR epidemic model and considered the heterogeneity of the population and the effect of immunity induced by vaccination [10]. Muroya *et al.* also proved the global stability of an endemic equilibrium of a multi-group SIRS epidemic model using varying population sizes by extending the Lyapunov function techniques, which is one of the main mathematical challenges in analyzing multi-group models [11]. Recently, Li *et al.* more closely examined these multi-group models [12–14]. They first proposed a graph-theoretic approach to the method of global Lyapunov functions and used it to establish global stability of the interior equilibrium for more general models [12]. In the present paper, based on system (1.1), we divide the population of size $N(t)$ into n distinct groups, and then the n-group ($n \geq 2$) MSIRS epidemic models are formulated by

$$\begin{cases} \frac{\mathrm{d}\overline{M}_k}{\mathrm{d}t} = b_k(N_k - \overline{S}_k) - (\delta_k + d_k^M)\overline{M}_k, \\ \frac{\mathrm{d}\overline{S}_k}{\mathrm{d}t} = b_k\overline{S}_k + \tau_k\overline{R}_k + \delta_k\overline{M}_k - \sum_{j=1}^{n} \frac{\beta_{kj}\overline{S}_k(t)\overline{I}_j(t)}{N_j} - d_k^S\overline{S}_k, \\ \frac{\mathrm{d}\overline{I}_k}{\mathrm{d}t} = \sum_{j=1}^{n} \frac{\beta_{kj}\overline{S}_k(t)\overline{I}_j(t)}{N_j} - (d_k^I + \gamma_k)\overline{I}_k, \\ \frac{\mathrm{d}\overline{R}_k}{\mathrm{d}t} = \gamma_k\overline{I}_k(t) - (d_k^R + \tau_k)\overline{R}_k, \\ \frac{\mathrm{d}N_k}{\mathrm{d}t} = (b_k - d_k)N_k, \quad k = 1, 2, \ldots, n. \end{cases} \tag{1.2}$$

Because the natural births and deaths are not balanced, that is, $b_k \neq d_k$, the total population of the model is of an exponentially changing size. Thus, it is more difficult to analyze mathematically because the population size N_k is an additional variable that is governed by a differential equation. In accordance with Guo *et al.* [14], we investigate the asymptotic behavior of system (1.2). When studying epidemic systems, we are interested in two problems: one is when the disease will die out, and the other is when the disease will prevail and persist in a population. For a deterministic system, we solve the problems by determining the stability of the two equilibria under different conditions. However, note that because of environmental noises, the deterministic approach has some limitations in the mathematical modeling of the transmission of an infectious disease; as a result, several authors have begun to consider the effect of white noise in epidemic models, which involves a parameter perturbation and perturbations around the positive endemic equilibrium of the epidemic models [15–18]. Beretta *et al.* proved the stability of the epidemic model using stochastic time delays influenced by the probability under certain conditions [19]. Such stochastic perturbations were first proposed in [19, 20] and later were successfully used in many other papers for many different systems (see, *e.g.*, [21–25]). Yuan *et al.* [26] and Yu *et al.* [27] both investigated epidemic models with fluctuations around the positive equilibrium, and they proved the locally stochastically asymptotic stability of the positive equilibrium. Ji *et al.* also considered a multi-group SIR model with stochastic perturbation and deduced the globally asymptotic stability of the disease-free equilibrium when $R_0 \leq 1$, which means the disease will die out; the determined that when $R_0 > 1$, the disease will prevail, which is measured through the difference between the solution and the endemic equilibrium of the deterministic model using time-averaged data [28]. Imhof and Walcher [29] considered a stochastic chemostat model and they proved that the stochastic model led to extinction, even though the deterministic counterpart predicted persistence. In our previous work, we considered an SEIR epidemic model with constant immigration and random fluctuation around the endemic equilibrium, and we performed a detailed analysis on the asymptotic behavior of the stochastic model [30]; we also investigated a two-group epidemic model with distributed delays and random perturbation [31]. Because of the similarity between the transmission of human infectious diseases and the transmission of malicious objects in a computer network, we used the epidemic models to describe the transmission of malicious objects in the cyber world [32]. In the current paper, to examine the influence of white noise on system (1.2), we also consider a stochastic version of the MSIRS model by perturbing the deterministic system (1.2) using white noise and assuming that the perturbations are around the positive endemic equilibrium of the epidemic models. While some papers study the effect of stochastic perturbation on epidemic models, we are not aware of any literature addressing this issue in MSIRS epidemic models. This paper is an attempt to fill this gap.

This paper is organized as follows. We begin in Section 2 by providing the necessary background with respect to the deterministic multi-group MSIRS model and introduce some results of the graph theory used by Guo *et al.* in epidemic models. We establish the global dynamics of the disease-free state by using the basic reproduction number and present one of our main results (Theorem 3.2) in Section 3. We derive the asymptotic stability of a unique epidemic state in Section 4 (see Theorem 4.4). In Section 5, we derive the stochastic version from the deterministic model (1.2) and perform an analysis of the asymptotic behavior of the stochastic model by means of the method of Lyapunov functions and graph theory in Theorem 5.4. Numerical methods are used to simulate the dynamic behavior of the model. The effect of the rate of immunity loss is also analyzed in the deterministic models and the corresponding stochastic models in Section 6. Finally, we provide the conclusion of our article in Section 7.

2 Deterministic multi-group MSIRS models

To investigate the dynamical behavior, the first concern is whether the solution has a global existence. Moreover, for a population dynamics model, whether the value is nonnegative is also considered. Hence in this section we first show that the solution of system (1.2) is global and nonnegative.

The parameters in the model (1.2) are summarized in Table 1.

The disease transmission diagram is depicted in Figure 1.

We assume that $d_k^M = d_k^S = d_k^I = d_k^R = d_k > 0$, $b_k > 0$, $b_k - d_k = q_k > 0$, and that the rest of the parameters be nonnegative for all k. It is clear that the population size changes in an exponentially increasing manner. In particular, $\beta_{kj} = 0$ if there is no transmission of the disease between compartments S_k and I_j.

The fractions of the population in the classes of $\overline{M}_k, \overline{S}_k, \overline{I}_k, \overline{R}_k$ are $M_k = \frac{\overline{M}_k}{N_k}$, $S_k = \frac{\overline{S}_k}{N_k}$, $I_k = \frac{\overline{I}_k}{N_k}$, $R_k = \frac{\overline{R}_k}{N_k}$, respectively. Note that the number of infectives $\overline{I}_k$ could go to infinity even though the fraction I_k goes to zero if the population size N_k grows faster than $\overline{I}_k$. To avoid

Table 1 Summary of notation

Notation	Explanation
$\overline{M}_k$	Passively immune infants in kth group
$\overline{S}_k$	Susceptibles in the kth group
$\overline{I}_k$	Infectives in the kth group
$\overline{R}_k$	Recovered people with immunity in the kth group
N_k	Total population size in the kth group
b_k	Fraction at which new-borns of group k have the passive immunity
β_{kj}	Rate of disease transmission between susceptible individuals in the kth group and infectious individuals in the jth group
$d_k^M, d_k^S, d_k^I, d_k^R$	Mortality rates of susceptible, infectious and recovered individuals in the kth group, respectively
τ_k	The rate of immunity loss in the kth group
δ_k	Rate of the transfer out of the passively immune class in the kth group
γ_k	Recovery rate of infectious individuals in the kth group

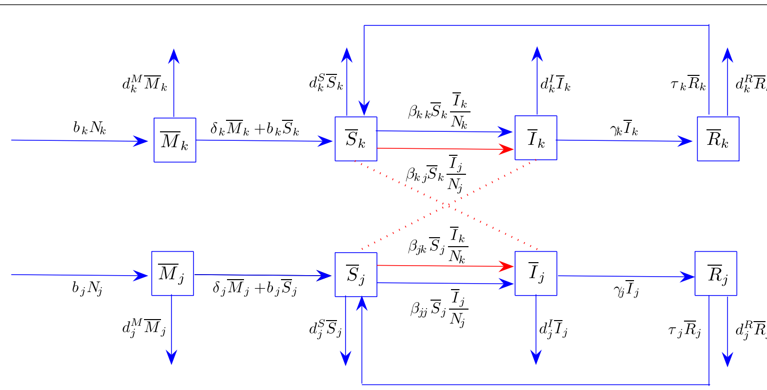

Figure 1 The transfer diagram for MSIRS model (1.2).

this ambiguity, we focus on the behavior of the fractions in the epidemiological classes. It is convenient to convert to differential equations for the fractions in the epidemiological classes with simplifications by using the differential equation for N_k; we then calculate

$$\frac{\mathrm{d}M_k}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\overline{M}_k}{\mathrm{d}t}}{N_k} - q_k M_k, \qquad \frac{\mathrm{d}S_k}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\overline{S}_k}{\mathrm{d}t}}{N_k} - q_k S_k,$$
$$\frac{\mathrm{d}I_k}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\overline{I}_k}{\mathrm{d}t}}{N_k} - q_k I_k, \qquad \frac{\mathrm{d}R_k}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\overline{R}_k}{\mathrm{d}t}}{N_k} - q_k R_k.$$

Furthermore, eliminating the differential equation for S_k by using $S_k = 1 - M_k - I_k - R_k$ and using $b_k - d_k = q_k > 0$, the ordinary differential equations for the MSIRS model becomes

$$\begin{cases} \frac{\mathrm{d}M_k}{\mathrm{d}t} = -\delta_k M_k + (d_k + q_k)I_k + (d_k + q_k)R_k, \\ \frac{\mathrm{d}I_k}{\mathrm{d}t} = \sum_{j=1}^{n} \beta_{kj}(1 - M_k - I_k - R_k)I_j - (d_k + \gamma_k + q_k)I_k, \\ \frac{\mathrm{d}R_k}{\mathrm{d}t} = \gamma_k I_k - (d_k + q_k + \tau_k)R_k, \quad k = 1, 2, \dots, n. \end{cases} \tag{2.1}$$

Equilibrium solutions $(M_1^*, I_1^*, R_1^*, \dots, M_n^*, I_n^*, R_n^*) \in R^{3n}$ of system (2.1) obey the following equation:

$$\begin{cases} 0 = -\delta_k M_k^* + (d_k + q_k)I_k^* + (d_k + q_k)R_k^*, \\ 0 = \sum_{j=1}^{n} \beta_{kj}(1 - M_k^* - I_k^* - R_k^*)I_j^* - (d_k + \gamma_k + q_k)I_k^*, \\ 0 = \gamma_k I_k^* - (d_k + q_k + \tau_k)R_k^*, \quad k = 1, 2, \dots, n. \end{cases} \tag{2.2}$$

Then it is easy to verify that the trivial solution of system (2.2) is given by $P_0 = (M_1^0, I_1^0, R_1^0, \dots, M_n^0, I_n^0, R_n^0) \in R_+^{3n}$, where $M_k^0 = I_k^0 = R_k^0 = 0$, $k \in \{1, \dots, n\}$. It is clear that $S_1^0 = \cdots = S_n^0 = 1$. In epidemiology it is called the disease-free equilibrium, at which the population remains in the absence of disease. Nontrivial solutions of system (2.2) with $I_k^* > 0$ for some $k \in \{1, \dots, n\}$ are called endemic equilibria, at which the disease persists. Our main task is to find some conditions that determine whether the disease dies out (*i.e.*, the fraction I_k goes to zero) or remains endemic (*i.e.*, the fraction I_k remains positive) for system (2.1).

We can check that the right sides of (2.1) are smooth, so that solution of system (2.1) with initial condition $(M_1(0), I_1(0), R_1(0), \dots, M_n(0), I_n(0), R_n(0)) \in R_+^{3n}$ has a unique solution and remain nonnegative. Therefore in what follows we consider system (2.1) in R_+^{3n}. A suitable bounded region in the nonnegative cone of R_+^{3n} for system (2.1) is

$$\Gamma = \{(M_1, I_1, R_1, \dots, M_n, I_n, R_n) \in R_+^{3n} | 0 \le M_k, I_k, R_k, M_k + I_k + R_k \le 1, 1 \le k \le n\}.$$

Because no solution paths leave through any boundary, it can be verified that region Γ is positively invariant for system (2.1) and the model is well posed. Our results in this paper will be stated for system (2.1) in Γ.

Let

$$\mathring{\Gamma} = \{(M_1, I_1, R_1, \dots, M_n, I_n, R_n) \in R_+^{3n} | 0 < M_k, I_k, R_k, M_k + I_k + R_k < 1, 1 \le k \le n\}.$$

It is clear that $\mathring{\Gamma}$ is the interior of Γ.

Set

$$\mathscr{F} = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{pmatrix}$$

and

$$\mathscr{V} = \operatorname{diag}(d_k + \gamma_k + q_k) = \begin{pmatrix} d_1 + \gamma_1 + q_1 & 0 & \cdots & 0 \\ 0 & d_2 + \gamma_2 + q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n + \gamma_n + q_n \end{pmatrix}.$$

Then the next generation matrix is

$$\mathscr{F}\mathscr{V}^{-1} = \begin{pmatrix} \frac{\beta_{11}}{d_1+\gamma_1+q_1} & \cdots & \frac{\beta_{1n}}{d_n+\gamma_n+q_n} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{n1}}{d_1+\gamma_1+q_1} & \cdots & \frac{\beta_{nn}}{d_n+\gamma_n+q_n} \end{pmatrix},$$

and hence the basic reproduction number $\mathscr{R}_0$ is

$$\mathscr{R}_0 = \rho\bigl(\mathscr{F}\mathscr{V}^{-1}\bigr) = \max\bigl\{|\lambda|; \lambda \in \sigma\bigl(\mathscr{F}\mathscr{V}^{-1}\bigr)\bigr\}, \tag{2.3}$$

where $\rho(\cdot)$ and $\sigma(\cdot)$ denote the spectral radius and the set of eigenvalues of a matrix, respectively. Since it can be verified that system (2.1) satisfies conditions (A1)-(A5) of Theorem 2 of [33], we have the following proposition.

Proposition 2.1 *For system* (2.1), *the disease-free equilibrium P_0 is locally asymptotically stable if $\mathscr{R}_0 < 1$ while it is unstable if $\mathscr{R}_0 > 1$.*

3 Asymptotic stability of the disease-free equilibrium

In the study of population systems, extinction and persistence are two of the most important issues. We will discuss the extinction of the deterministic MSIRS model (2.1) in this section, that is, we will find some conditions that determine when the disease dies out (*i.e.*, the fraction I_k goes to zero) for system (2.1). First, following [10, 14], we prepare a matrix whose spectral radius has a similar threshold property to that of $\mathscr{R}_0$. Let

$$\mathscr{V}^{-1}\mathscr{F} = \begin{pmatrix} \frac{\beta_{11}}{d_1+\gamma_1+q_1} & \cdots & \frac{\beta_{1n}}{d_1+\gamma_1+q_1} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{n1}}{d_n+\gamma_n+q_n} & \cdots & \frac{\beta_{nn}}{d_n+\gamma_n+q_n} \end{pmatrix}.$$

Then the following lemma immediately follows.

Lemma 3.1 $\rho(\mathscr{V}^{-1}\mathscr{F}) \le 1$ *if and only if* $\mathscr{R}_0 \le 1$.

Using Lemma 3.1, we obtain the following theorem, which is one of the main results of this paper. This proof is similar to that of [10, 14].

Theorem 3.2 *Assume $B = (\beta_{kj})$ is irreducible. If $\mathscr{R}_0 \le 1$, then the disease-free equilibrium P_0 of system* (2.2) *is globally asymptotically stable in Γ and there does not exist any endemic equilibrium P^*.*

Proof From Lemma 3.1 we have $\rho(\mathscr{V}^{-1}\mathscr{F}) \le 1$. Following [10, 14], we define the matrix-valued function

$$\mathcal{M}(\mathbf{M},\mathbf{I},\mathbf{R}) = \begin{pmatrix} \frac{\beta_{11}(1-M_1-I_1-R_1)}{d_1+\gamma_1+q_1} & \cdots & \frac{\beta_{1n}(1-M_1-I_1-R_1)}{d_1+\gamma_1+q_1} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{n1}(1-M_n-I_n-R_n)}{d_n+\gamma_n+q_n} & \cdots & \frac{\beta_{n1}(1-M_n-I_n-R_n)}{d_n+\gamma_n+q_n} \end{pmatrix}$$

on R_+^{3n}, where $M = (M_1,\dots,M_n)^T$, $I = (I_1,\dots,I_n)^T$ and $R = (R_1,\dots,R_n)^T$. Note that $\mathcal{M}(\mathbf{M}^0, \mathbf{I}^0,\mathbf{R}^0) = \mathscr{V}^{-1}\mathscr{F}$, where $\mathbf{M}^0 = (M_1^0,\dots,M_n^0)^T$, $\mathbf{I}^0 = (I_1^0,\dots,I_n^0)^T$, $\mathbf{R}^0 = (R_1^0,\dots,R_n^0)^T$, and first we claim that there does not exist any endemic equilibrium P^* in Γ. Suppose that $(\mathbf{M},\mathbf{I},\mathbf{R}) \neq (\mathbf{M}^0,\mathbf{I}^0,\mathbf{R}^0)$. Then we have $0 < \mathcal{M}(\mathbf{M},\mathbf{I},\mathbf{R}) < \mathscr{V}^{-1}\mathscr{F}$. Since nonnegative matrix $\mathcal{M}(\mathbf{M},\mathbf{I},\mathbf{R}) + \mathscr{V}^{-1}\mathscr{F}$ is irreducible, it follows from the Perron-Frobenius theorem (see [34]) that $\rho(\mathcal{M}(\mathbf{M},\mathbf{I},\mathbf{R})) < \rho(\mathscr{V}^{-1}\mathscr{F}) \le 1$. This implies that equation $\mathcal{M}(\mathbf{M},\mathbf{I},\mathbf{R})\mathbf{I} = \mathbf{I}$ has only the trivial solution $\mathbf{I} = 0$, where $\mathbf{I} = (I_1,\dots,I_n)^T$. Hence the claim is true.

Next we claim that the disease-free equilibrium P_0 is globally asymptotically stable in Γ. From the Perron-Frobenius theorem (see Theorem 2.1.4 of [34]) it follows that the nonnegative irreducible matrix $\mathscr{V}^{-1}\mathscr{F}$ has a strictly positive left eigenvector $\ell := (\ell_1,\dots,\ell_n) \ge 0$ associated with the eigenvalue $\rho(\mathscr{V}^{-1}\mathscr{F})$. Let us define a Lyapunov function

$$L(\mathbf{I}) = \sum_{i=1}^{n} \ell_i \frac{I_i}{d_i + \gamma_i + q_i} \tag{3.1}$$

on R_+^n, whose derivative along the trajectories of system (2.1) is

$$\begin{aligned} L'(\mathbf{I}) &= \sum_{i=1}^{n} \ell_i \left(\frac{(1 - M_i - I_i - R_i)\sum_{j=1}^{n} \beta_{ij} I_j}{d_i + \gamma_i + q_i} - I_i \right) \\ &= \ell \cdot \big(\mathcal{M}(\mathbf{M},\mathbf{I},\mathbf{R}) - \mathbf{E}_n\big)\mathbf{I} \\ &\le \ell \cdot \big(\mathscr{V}^{-1}\mathscr{F} - \mathbf{E}_n\big)\mathbf{I} \\ &= \ell \cdot \big(\rho\big(\mathscr{V}^{-1}\mathscr{F}\big) - 1\big)\mathbf{I}, \end{aligned} \tag{3.2}$$

where $\mathbf{E}_n$ and $\cdot$ denote the $n \times n$ identity matrix and the inner product of vectors, respectively. If $\rho(\mathscr{V}^{-1}\mathscr{F}) \le 1$, then $\ell \cdot (\rho(\mathscr{V}^{-1}\mathscr{F}) - 1)\mathbf{I} \le 0$. Suppose that $\rho(\mathscr{V}^{-1}\mathscr{F}) < 1$. Then $L'(\mathbf{I}) = 0$ if and only if $\mathbf{I} = 0$. Suppose that $\rho(\mathscr{V}^{-1}\mathscr{F}) = 1$. Then it follows from (3.2) that $L'(\mathbf{I}) = 0$ implies

$$\ell \cdot \mathcal{M}(\mathbf{M},\mathbf{I},\mathbf{R}) = \ell \cdot \mathbf{I}. \tag{3.3}$$

Hence, if $(\mathbf{M},\mathbf{I},\mathbf{R}) \neq (\mathbf{M}^0,\mathbf{I}^0,\mathbf{R}^0)$, then $\ell \cdot \mathcal{M}(\mathbf{M},\mathbf{I},\mathbf{R}) < \ell \cdot (\mathscr{V}^{-1}\mathscr{F}) = \rho(\mathscr{V}^{-1}\mathscr{F})\ell = \ell$ and thus $\mathbf{I} = 0$ is the only solution of (3.3). Summarizing the statements, we see that $L'(\mathbf{I}) = 0$ if and only if $\mathbf{I} = 0$ or $(\mathbf{M},\mathbf{I},\mathbf{R}) = (\mathbf{M}^0,\mathbf{I}^0,\mathbf{R}^0)$, which implies that the compact invariant subset of the set where $L'(\mathbf{I}) = 0$ is only the singleton $\{P_0\} \subset \Gamma$. Thus, from the LaSalle invariance principle [35], it follows that the disease-free equilibrium P_0 is globally asymptotically stable in Γ. □

4 Asymptotical stability of an endemic equilibrium in the deterministic MSIRS model

We discuss the persistence of the deterministic MSIRS model (2.1) in this section, our goal is to find some conditions that determine when the disease remains endemic (*i.e.*, the fraction I_k remains positive) for system (2.1). First, we introduce some specifics of the graph theory that are useful for the proofs of asymptotic stability of an endemic equilibrium in this section and the next section.

The matrix $B = (\beta_{kj})$ denotes the contact matrix. Associated to B, one can construct a directed graph $\mathfrak{L} = G(B)$ whose vertex k represents the kth group, $k = 1,\dots,n$. A directed edge exists from vertex k to vertex j if and only if $\beta_{kj} > 0$. Throughout the paper, we assume that B is irreducible, which is equivalent to $G(B)$ being strongly connected. Biologically, this is the same as assuming that any two groups k and j have a direct or indirect route of transmission. More specifically, individuals in I_j can infect ones in S_k directly or indirectly.

We define

$$\overline{\beta}_{kj} = \beta_{kj}S_k^*I_j^* = \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr)I_j^*, \quad \overline{\beta}_{kj} > 0, 1 \le k, j \le n, \tag{4.1}$$

where $(M_1^*, I_1^*, R_1^*, \dots, M_n^*, I_n^*, R_n^*)$ is the endemic equilibrium solution of system (2.1). Now consider the linear system

$$\overline{\mathfrak{B}}\varsigma = 0, \tag{4.2}$$

where

$$\overline{\mathfrak{B}} = \begin{bmatrix} \sum_{i\neq 1}\overline{\beta}_{1i} & -\overline{\beta}_{21} & \cdots & -\overline{\beta}_{n1} \\ -\overline{\beta}_{12} & \sum_{i\neq 2}\overline{\beta}_{2i} & \cdots & -\overline{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\beta}_{1n} & \overline{\beta}_{2n} & \cdots & \sum_{i\neq n}\overline{\beta}_{ni} \end{bmatrix},$$

and let $\mathfrak{L} = G(B)$ denote the directed graph associated with matrix B (and $(\overline{\beta}_{kj})_{n\times n}$), and C_{jk} denote the cofactor of the (j,k) entry of $\overline{\mathfrak{B}}$.

We have the following fundamental lemma [12].

Lemma 4.1 (Kirchhoff's Matrix-Tree theorem) *Assume $(\overline{\beta}_{kj})_{n\times n}$ is irreducible and $n \ge 2$. Then the following results hold*:

(1) *The solution space of system* (4.2) *has dimension* 1, *with a basis* $(\varsigma_1, \varsigma_2, \dots, \varsigma_n) = (C_{11}, C_{22}, \dots, C_{nn})$.

(2) *For* $1 \le k \le n$,

$$C_{kk} = \sum_{T\in\mathbb{T}_k} W(T) = \sum_{T\in\mathbb{T}_k}\prod_{(r,m)\in E(T)}\overline{\beta}_{rm} > 0,$$

where $\mathbb{T}_k$ is the set of all directed spanning subtrees of $\mathfrak{L}$ that are rooted at vertex k, $W(T)$ is the weight of a directed tree T, and $E(T)$ denotes the set of directed arcs in a directed tree T.

Let $\mathscr{R}_0$ be defined in (2.3). If $\mathscr{R}_0 > 1$, it follows from Proposition 2.1 that the disease-free equilibrium P_0 is unstable. From a uniform persistence result of [8, 9, 33, 36, 37], we can deduce that the instability of P_0 implies the uniform persistence of system (2.1) in $\mathring{\Gamma}$, the following proposition for system (2.1) is known in the literature, and its proof is standard (see [8, 9, 33, 36, 37]).

Proposition 4.2 *Assume $B = (\beta_{kj})$ is irreducible. Then the following statement applies. If $\mathscr{R}_0 > 1$, then P_0 is unstable and system* (2.1) *is uniformly persistent in* Γ.

The uniform persistence of (2.1), together with uniform boundedness of solutions in $\mathring{\Gamma}$, implies the existence of an endemic equilibrium of system (2.1) in $\mathring{\Gamma}$. Summarizing the statements, we have the following corollary.

Corollary 4.3 *Assume $B = (\beta_{kj})$ is irreducible. If $\mathscr{R}_0 > 1$, then system* (2.1) *has at least one endemic equilibrium.*

Moreover, based on the result of the existence of endemic equilibrium, we can prove the following theorem, which is one of the main results of this paper.

Theorem 4.4 *Assume that $B = (\beta_{kj})$ is irreducible. If $\mathscr{R}_0 > 1$, then system* (2.1) *has a unique endemic equilibrium P^* which is asymptotically stable.*

Proof When $n = 1$, system (2.1) becomes

$$\begin{cases} \mathrm{d}M/\mathrm{d}t = -\delta M + (d+q)I + (d+q)R, \\ \mathrm{d}I/\mathrm{d}t = \beta(1-M-I-R)I - \beta(M+I+R)I - (d+\gamma+q)I, \\ \mathrm{d}R/\mathrm{d}t = \gamma_k I - (d+q+\tau)R. \end{cases} \tag{4.3}$$

For this single-group model, Lou and Ma proved that the endemic equilibrium P^* is globally asymptotically stable, which of course is asymptotically stable. Here, we will consider the case of $n \geq 2$. First, using the change the variables of $\widetilde{M}_k = M_k - M_k^*$, $\widetilde{I}_k = I_k - I_k^*$, $\widetilde{R}_k = R_k - R_k^*$ and system (2.1) can be written as

$$\begin{cases} \frac{\mathrm{d}\widetilde{M}_k}{\mathrm{d}t} = -\delta_k \widetilde{M}_k + (d_k+q_k)\widetilde{I}_k + (d_k+q_k)\widetilde{R}_k, \\ \frac{\mathrm{d}\widetilde{I}_k}{\mathrm{d}t} = \sum_{j=1}^n \beta_{kj}(1-M_k^*-I_k^*-R_k^*)\widetilde{I}_j \\ \qquad - \sum_{j=1}^n \beta_{kj}(\widetilde{M}_k+\widetilde{I}_k+\widetilde{R}_k)(I_j^*+\widetilde{I}_j) - (d_k+\gamma_k+q_k)\widetilde{I}_k, \\ \frac{\mathrm{d}\widetilde{R}_k}{\mathrm{d}t} = \gamma_k\widetilde{I}_k - (d_k+q_k+\tau_k)\widetilde{R}_k, \quad k = 1,2,\ldots,n. \end{cases} \tag{4.4}$$

Note that $\overline{\mathfrak{B}}$ is the Laplacian matrix of the matrix $(\overline{\beta}_{kj})_{n\times n}$. Because $(\beta_{kj})_{n\times n}$ is irreducible, the matrices $(\overline{\beta}_{kj})_{n\times n}$ and $\overline{\mathfrak{B}}$ are also irreducible. The column sums of the Laplacian matrix $\overline{\mathfrak{B}}$ are zero. Therefore, it follows from Lemma 4.1 that the solution space of the linear system $\overline{\mathfrak{B}}\varsigma = 0$ has dimension 1, with a basis

$$\varsigma := (\varsigma_1, \varsigma_2, \ldots, \varsigma_n)^T = (C_{11}, C_{22}, \ldots, C_{nn})^T,$$

where C_{kk} denotes the cofactor of the kth diagonal entry of $\overline{\mathfrak{B}}$. For such $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_n)$, we define the Lyapunov function

$$\begin{aligned} W(\mathbf{M}, \mathbf{I}, \mathbf{R}) :=& \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \biggl[\frac{(\tau_k + \delta_k)\widetilde{M}_k^2}{2(d_k + q_k)(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \\ &+ \frac{(\tau_k + \delta_k)\widetilde{R}_k^2}{2\gamma_k(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \\ &+ \frac{(\widetilde{M}_k + \widetilde{R}_k)^2}{2(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \biggr] + \sum_{k=1}^{n} \frac{\varsigma_k}{2I_k^*} \widetilde{I}_k^2, \end{aligned} \tag{4.5}$$

$$\begin{aligned} W'(\mathbf{M}, \mathbf{I}, \mathbf{R}) :=& \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \biggl[\frac{(\tau_k + \delta_k)\widetilde{M}_k \dot{\widetilde{M}}_k}{(d_k + q_k)(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \\ &+ \frac{(\tau_k + \delta_k)\widetilde{R}_k \dot{\widetilde{R}}_k}{\gamma_k(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \\ &+ \frac{(\widetilde{M}_k + \widetilde{R}_k)(\dot{\widetilde{M}}_k + \dot{\widetilde{R}}_k)}{(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \biggr] + \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \widetilde{I}_k \dot{\widetilde{I}}_k \\ =& W_1 + W_2, \end{aligned}$$

where

$$\begin{aligned} W_1 =& \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \biggl[\frac{(\tau_k + \delta_k)\widetilde{M}_k \dot{\widetilde{M}}_k}{(d_k + q_k)(\tau_k + \delta_k + d_k + q_k + \gamma_k)} + \frac{(\tau_k + \delta_k)\widetilde{R}_k \dot{\widetilde{R}}_k}{\gamma_k(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \\ &+ \frac{(\widetilde{M}_k + \widetilde{R}_k)(\dot{\widetilde{M}}_k + \dot{\widetilde{R}}_k)}{(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \biggr], \\ W_2 =& \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \widetilde{I}_k \dot{\widetilde{I}}_k. \end{aligned}$$

We calculate

$$\begin{aligned} W_1 =& \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \biggl[-\frac{(\tau_k + \delta_k)\delta_k \widetilde{M}_k^2}{(d_k + q_k)(\tau_k + \delta_k + d_k + q_k + \gamma_k)} + \frac{(\tau_k + \delta_k)\widetilde{M}_k \widetilde{I}_k}{\tau_k + \delta_k + d_k + q_k + \gamma_k} \\ &+ \frac{(\tau_k + \delta_k)\widetilde{M}_k \widetilde{R}_k}{\tau_k + \delta_k + d_k + q_k + \gamma_k} + \frac{(\tau_k + \delta_k)\widetilde{R}_k \widetilde{I}_k}{\tau_k + \delta_k + d_k + q_k + \gamma_k} - \frac{(d_k + q_k + \gamma_k)\widetilde{R}_k^2}{\gamma_k(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \\ &- \frac{\delta_k \widetilde{M}_k^2}{\tau_k + \delta_k + d_k + q_k + \gamma_k} - \frac{\tau_k \widetilde{R}_k^2}{\tau_k + \delta_k + d_k + q_k + \gamma_k} + \frac{(d_k + q_k + \gamma_k)(\widetilde{M}_k + \widetilde{R}_k)\widetilde{I}_k}{\tau_k + \delta_k + d_k + q_k + \gamma_k} \\ &- \frac{(\tau_k + \delta_k)\widetilde{M}_k \widetilde{R}_k}{\tau_k + \delta_k + d_k + q_k + \gamma_k} \biggr] \\ =& \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \biggl[-\frac{(\tau_k + \delta_k + d_k + q_k)\delta_k \widetilde{M}_k^2}{(d_k + q_k)(\tau_k + \delta_k + d_k + q_k + \gamma_k)} + \widetilde{M}_k \widetilde{I}_k + \widetilde{R}_k \widetilde{I}_k \\ &- \frac{(d_k + q_k + \gamma_k + \gamma_k \tau_k)\widetilde{R}_k^2}{\gamma_k(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \biggr], \end{aligned}$$

$$\begin{aligned}
W_2 = {} & \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*}\widetilde{I}_k \Biggl(\sum_{j=1}^{n} \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr)\widetilde{I}_j \\
& - \sum_{j=1}^{n} \beta_{kj}(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\bigl(I_j^* + \widetilde{I}_j\bigr) - (d_k + \gamma_k + q_k)\widetilde{I}_k \Biggr) \\
= {} & \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*}\biggl[- \sum_{j=1}^{n} \beta_{kj} I_j^* \widetilde{I}_k^2 + \sum_{j=1}^{n} \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr)\widetilde{I}_k\widetilde{I}_j \\
& - \sum_{j=1}^{n} \beta_{kj} \frac{(1 - M_k^* - I_k^* - R_k^*)I_j^*}{I_k^*}\widetilde{I}_k^2 \\
& - \sum_{j=1}^{n} \beta_{kj} I_j^* (\widetilde{M}_k + \widetilde{R}_k)\widetilde{I}_k - \sum_{j=1}^{n} \beta_{kj}\widetilde{I}_k(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\widetilde{I}_j \\
= {} & \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*}\Biggl[\Biggl(\sum_{j=1}^{n} \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr) I_k^* I_j^* \frac{\widetilde{I}_k}{I_k^*}\frac{\widetilde{I}_j}{I_j^*} \\
& - \sum_{j=1}^{n} \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr) I_k^* I_j^* \biggl(\frac{\widetilde{I}_k}{I_k^*}\biggr)^2 \Biggr) \\
& - \sum_{j=1}^{n} \beta_{kj} I_j^* \widetilde{I}_k^2 - \sum_{j=1}^{n} \beta_{kj} I_j^* (\widetilde{M}_k + \widetilde{R}_k)\widetilde{I}_k - \sum_{j=1}^{n} \beta_{kj}\widetilde{I}_k(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\widetilde{I}_j \Biggr] \\
= {} & \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*}\Biggl[\Biggl(\sum_{j=1}^{n} \overline{\beta}_{kj} I_k^* \frac{\boldsymbol{v}_k}{I_k^*}\frac{\widetilde{I}_j}{I_j^*} - \sum_{j=1}^{n} \overline{\beta}_{kj} I_k^* \biggl(\frac{\widetilde{I}_k}{I_k^*}\biggr)^2 \Biggr) - \sum_{j=1}^{n} \beta_{kj} I_j^* \widetilde{I}_k^2 \\
& - \sum_{j=1}^{n} \beta_{kj} I_j^* (\widetilde{M}_k + \widetilde{R}_k)\widetilde{I}_k - \sum_{k=1}^{n} \beta_{kj}\widetilde{I}_k(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\widetilde{I}_j \Biggr].
\end{aligned}$$

It follows from the arithmetic-geometric mean inequality that

$$\frac{I_k}{I_k^*}\frac{\widetilde{I}_j}{I_j^*} \le \frac{1}{2}\biggl(\frac{\widetilde{I}_k}{I_k^*}\biggr)^2 + \frac{1}{2}\biggl(\frac{\widetilde{I}_j}{I_j^*}\biggr)^2,$$

therefore

$$\begin{aligned}
W_2 \le {} & \sum_{k=1}^{n} \varsigma_k \Biggl[\sum_{j=1}^{n} \overline{\beta}_{kj} \Biggl(\frac{1}{2}\biggl(\frac{\widetilde{I}_k}{I_k^*}\biggr)^2 + \frac{1}{2}\biggl(\frac{\widetilde{I}_j}{I_j^*}\biggr)^2 - \biggl(\frac{\widetilde{I}_k}{I_k^*}\biggr)^2 \Biggr) \Biggr] \\
& - \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \Biggl[\sum_{j=1}^{n} \beta_{kj} I_j^* \widetilde{I}_k^2 + \sum_{j=1}^{n} \beta_{kj} I_j^* (\widetilde{M}_k + \widetilde{R}_k)\widetilde{I}_k + \sum_{k=1}^{n} \beta_{kj}\widetilde{I}_k(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\widetilde{I}_j \Biggr] \\
= {} & \frac{1}{2}\sum_{k=1}^{n} \varsigma_k \Biggl[\sum_{j=1}^{n} \overline{\beta}_{kj} \Biggl(\biggl(\frac{\widetilde{I}_j}{I_j^*}\biggr)^2 - \biggl(\frac{\widetilde{I}_k}{I_k^*}\biggr)^2 \Biggr) \Biggr] \\
& - \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \Biggl[\sum_{j=1}^{n} \beta_{kj} I_j^* \widetilde{I}_k^2 + \sum_{j=1}^{n} \beta_{kj} I_j^* (\widetilde{M}_k + \widetilde{R}_k)\widetilde{I}_k \\
& + \sum_{k=1}^{n} \beta_{kj}\widetilde{I}_k(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\widetilde{I}_j \Biggr].
\end{aligned} \tag{4.6}$$

Note that from (4.1), (4.2), and Lemma 4.1, we have

$$\sum_{j=1}^{n} \bar{\beta}_{jk}\varsigma_j = \sum_{k=1}^{n} \bar{\beta}_{ki}\varsigma_k = \sum_{j=1}^{n} \bar{\beta}_{kj}\varsigma_k. \tag{4.7}$$

It follows that

$$\begin{aligned}\sum_{k=1}^{n} \varsigma_k \sum_{j=1}^{n} \bar{\beta}_{kj}\left(\frac{\widetilde{I}_j}{I_j^*}\right)^2 &= \sum_{j=1}^{n} \varsigma_j \sum_{k=1}^{n} \bar{\beta}_{jk}\left(\frac{\widetilde{I}_k}{I_k^*}\right)^2 = \sum_{k=1}^{n}\left(\frac{\widetilde{I}_k}{I_k^*}\right)^2 \sum_{j=1}^{n} \bar{\beta}_{jk}\varsigma_j \\ &= \sum_{k=1}^{n}\left(\frac{\widetilde{I}_k}{I_k^*}\right)^2 \sum_{j=1}^{n} \bar{\beta}_{kj}\varsigma_k = \sum_{k=1}^{n} \varsigma_k \sum_{j=1}^{n} \bar{\beta}_{kj}\left(\frac{\widetilde{I}_k}{I_k^*}\right)^2.\end{aligned} \tag{4.8}$$

Substituting this into (4.6) yields

$$W_2 \le -\sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*}\left[\sum_{j=1}^{n} \beta_{kj} I_j^* \widetilde{I}_k^2 + \sum_{j=1}^{n} \beta_{kj} I_j^* (\widetilde{M}_k + \widetilde{R}_k)\widetilde{I}_k + \sum_{k=1}^{n} \beta_{kj}\widetilde{I}_k(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\widetilde{I}_j\right], \tag{4.9}$$

thus,

$$\begin{aligned} W'(\mathbf{M},\mathbf{I},\mathbf{R}) &= W_1 + W_2 \\ &= -\sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \left[\frac{(\tau_k + \delta_k + d_k + q_k)\delta_k \widetilde{M}_k^2}{(d_k + q_k)(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \right. \\ &\quad \left. + \frac{(d_k + q_k + \gamma_k + \gamma_k\tau_k)\widetilde{R}_k^2}{\gamma_k(\tau_k + \delta_k + d_k + q_k + \gamma_k)} + \widetilde{I}_k^2\right] \\ &\quad - \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{k=1}^{n} \beta_{kj}\widetilde{I}_k(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\widetilde{I}_j \\ &\le W_0 + \left(-\sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{k=1}^{n} \beta_{kj}\widetilde{I}_k(\widetilde{M}_k + \widetilde{I}_k + \widetilde{R}_k)\widetilde{I}_j\right), \end{aligned}$$

where

$$\begin{aligned} W_0 := &-\sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \left[\frac{(\tau_k + \delta_k + d_k + q_k)\delta_k \widetilde{M}_k^2}{(d_k + q_k)(\tau_k + \delta_k + d_k + q_k + \gamma_k)} \right. \\ &\left. + \frac{(d_k + q_k + \gamma_k + \gamma_k\tau_k)\widetilde{R}_k^2}{\gamma_k(\tau_k + \delta_k + d_k + q_k + \gamma_k)} + \widetilde{I}_k^2\right]. \end{aligned}$$

We denote $Y_k = (\widetilde{M}_k, \widetilde{I}_k, \widetilde{R}_k)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$; then

$$|Y_k| = \sqrt{\widetilde{M}_k^2(t) + \widetilde{I}_k^2 + \widetilde{R}_k^2}, \qquad |\mathbf{Y}| = \left(\sum_{k=1}^{n} |Y_k|^2\right)^{1/2}.$$

It is clear that $(-\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\sum_{k=1}^{n}\beta_{kj}\widetilde{I}_k(\widetilde{M}_k+\widetilde{I}_k+\widetilde{R}_k)\widetilde{I}_j)$ is an infinitesimal of higher order of $|\mathbf{Y}|^2$ for $|\mathbf{Y}|\to 0$. Let

$$\omega=\min_{k\in\{1,\ldots,n\}}\Biggl\{\frac{\varsigma_k}{I_k^*}\sum_{j=1}^{n}\beta_{kj}I_j^*\frac{(\tau_k+\delta_k+d_k+q_k)\delta_k}{(d_k+q_k)(\tau_k+\delta_k+d_k+q_k+\gamma_k)},$$
$$\frac{\varsigma_k}{I_k^*}\sum_{j=1}^{n}\beta_{kj}I_j^*\frac{(d_k+q_k+\gamma_k+\gamma_k\tau_k)}{\gamma_k(\tau_k+\delta_k+d_k+q_k+\gamma_k)},\frac{\varsigma_k}{I_k^*}\Biggr\}.$$

Thus

$$W'(\mathbf{M},\mathbf{I},\mathbf{R})\le-\omega\sum_{k=1}^{n}|Y_k|^2+o\bigl(|\mathbf{Y}|^2\bigr). \tag{4.10}$$

Hence $W'(\mathbf{M},\mathbf{I},\mathbf{R})$ is negative-definite in a sufficiently small neighborhood of $\mathbf{Y}=\mathbf{0}$ for $t\ge 0$. However, it is easy to see $W(\mathbf{M},\mathbf{I},\mathbf{R})$ is a positive-definite decrescent function, which implies the endemic equilibrium P^* is asymptotically stable. □

5 Stochastic stability of the endemic equilibrium of a multi-group stochastic MSIRS model

In this section, we consider the stochastic version of the deterministic MSIRS model. Under the assumption that $\mathscr{R}_0>1$ and $B=(\beta_{kj})_{n\times n}$ is irreducible, we know from Section 4 that there exists a unique positive endemic equilibrium P^* in $\mathring{\Gamma}$. Furthermore, we assume stochastic perturbations on the $M_k(t)$, $I_k(t)$, $R_k(t)$ are of white-noise type, which are directly proportional to deviations $M_k(t)$, $I_k(t)$, and $R_k(t)$ from the values of M_k^*, I_k^*, R_k^*, respectively. Thus, system (2.1) results in

$$\begin{cases}\frac{\mathrm{d}M_k}{\mathrm{d}t}=-\delta_kM_k+(d_k+q_k)I_k+(d_k+q_k)R_k+\sigma_{1k}(M_k-M_k^*)\frac{\mathrm{d}B_{1k}}{\mathrm{d}t},\\ \frac{\mathrm{d}I_k}{\mathrm{d}t}=\sum_{j=1}^{n}\beta_{kj}(1-M_k-I_k-R_k)I_j-(d_k+\gamma_k+q_k)I_k+\sigma_{2k}(I_k-I_k^*)\frac{\mathrm{d}B_{2k}}{\mathrm{d}t},\\ \frac{\mathrm{d}R_k}{\mathrm{d}t}=\gamma_kI_k-(d_k+q_k+\tau_k)R_k+\sigma_{3k}(R_k-R_k^*)\frac{\mathrm{d}B_{3k}}{\mathrm{d}t},\quad k=1,2,\ldots,n,\end{cases} \tag{5.1}$$

where $B_{1k}(t)$, $B_{2k}(t)$, and $B_{3k}(t)$ are independent standard Brownian motions and $\sigma_{ik}^2>0$ represent the intensities of $B_{ik}(t)$ $(i=1,2,3)$, respectively. Obviously, the stochastic system (5.1) has the same equilibrium points as system (2.1). Next, let us now proceed to discuss asymptotic stability of system (5.1). In this paper, unless otherwise specified, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge t_0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge t_0}$ satisfying the usual conditions (*i.e.* it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B_{ik}(t)$ be the Brownian motions defined on this probability space. If $\mathscr{R}_0>1$, then the stochastic system (5.1) can be centered at its endemic equilibrium $P^*=(M_1^*,I_1^*,R_1^*,\ldots,M_n^*,I_n^*,R_n^*)$, by the change of variables

$$\boldsymbol{u}_k=M_k-M_k^*,\qquad \boldsymbol{v}_k=I_k-I_k^*,\qquad \boldsymbol{w}_k=R_k-R_k^*,$$

we obtain the following system:

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{u}_k}{\mathrm{d}t} = -\delta_k \boldsymbol{u}_k + (d_k + q_k)\boldsymbol{v}_k + (d_k + q_k)\boldsymbol{w}_k + \sigma_{1k}\boldsymbol{u}_k \frac{\mathrm{d}B_{1k}}{\mathrm{d}t}, \\ \frac{\mathrm{d}\boldsymbol{v}_k}{\mathrm{d}t} = \sum_{j=1}^n \beta_{kj}(1 - M_k^* - I_k^* - R_k^*)\boldsymbol{v}_j \\ \qquad - \sum_{j=1}^n \beta_{kj}(\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)(I_j^* + \boldsymbol{v}_j) - (d_k + \gamma_k + q_k)\boldsymbol{v}_k + \sigma_{2k}\boldsymbol{v}_k \frac{\mathrm{d}B_{2k}}{\mathrm{d}t}, \\ \frac{\mathrm{d}\boldsymbol{w}_k}{\mathrm{d}t} = \gamma_k \boldsymbol{v}_k - (d_k + q_k + \tau_k)\boldsymbol{w}_k + \sigma_{3k}\boldsymbol{w}_k \frac{\mathrm{d}B_{3k}}{\mathrm{d}t}, \quad k = 1, 2, \ldots, n. \end{cases} \tag{5.2}$$

It is clear that the stability of equilibrium of system (5.1) is equivalent to the stability of zero solution of system (5.2). Considering the d-dimensional stochastic differential equation [38, 39]

$$d\boldsymbol{x}(t) = f\bigl(\boldsymbol{x}(t), t\bigr)\, dt + g\bigl(\boldsymbol{x}(t), t\bigr)\, dB(t), \quad t \ge t_0. \tag{5.3}$$

If the assumptions of the existence-and-uniqueness theorem are satisfied, then, for any given initial value $\boldsymbol{x}(t_0) = x_0 \in \mathbb{R}^d$, (5.3) has a unique global solution denoted by $\boldsymbol{x}(t; t_0, x_0)$. For the purpose of stability we assume in this section $f(\mathbf{0}, t) = 0$ and $g(\mathbf{0}, t) = 0$ for all $t \ge t_0$. So (5.3) admits a solution $\boldsymbol{x}(t) \equiv \mathbf{0}$, which is called the trivial solution or the equilibrium position.

Let κ denote the family of all continuous nondecreasing functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$. For $0 < h \le \infty$ and $S_h = \{\boldsymbol{x} \in \mathbb{R}^d, |\boldsymbol{x}| < h\}$, denote $C^{2,1}(S_h \times [t_0, \infty); \mathbb{R}_+)$ the family of all nonnegative functions $V(\boldsymbol{x}, t)$ on $S_h \times [t_0, \infty)$ which are continuously twice differentiable in $\boldsymbol{x}$ and once in t. Define the differential operator L associated with (5.3) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^d f_i(\boldsymbol{x}, t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d \bigl[g^T(\boldsymbol{x}, t) g(\boldsymbol{x}, t)\bigr]_{ij} \frac{\partial^2}{\partial x_i x_j}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, then

$$LV(\boldsymbol{x}, t) = V_t(\boldsymbol{x}, t) + V_t(\boldsymbol{x}, t) f(\boldsymbol{x}, t) + \frac{1}{2}\operatorname{trace}\bigl[g^T(\boldsymbol{x}, t) V_{xx}(\boldsymbol{x}, t) g(\boldsymbol{x}, t)\bigr].$$

Definition 5.1

(1) The trivial solution of (5.3) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$P\bigl\{\bigl|\boldsymbol{x}(t; t_0, x_0)\bigr| < r \text{ for all } t \ge t_0\bigr\} \ge 1 - \varepsilon$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

(2) The trivial solution is said to be stochastically asymptotically stable if it is stochastically stable and for every $\varepsilon \in (0, 1)$, there exists a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that

$$P\Bigl\{\lim_{t \to \infty} \boldsymbol{x}(t; t_0, x_0) = 0\Bigr\} \ge 1 - \varepsilon$$

whenever $|x_0| < \delta_0$.

(3) The trivial solution is said to be stochastically asymptotically stable in the large if it is stochastically stable and for all $x_0 \in \mathbb{R}^d$

$$P\Big\{\lim_{t\to\infty} \boldsymbol{x}(t;t_0,x_0) = 0\Big\} = 1.$$

Definition 5.2 A continuous nonnegative function $V(\boldsymbol{x},t)$ is said to be decrescent if for some $\mu \in \kappa$:

$$V(\boldsymbol{x},t) \le \mu\big(\big|\boldsymbol{x}(t)\big|\big) \quad \text{for all } (\boldsymbol{x},t) \in S_h \times [t_0,\infty).$$

Before presenting the main theorem we put forward a lemma from [39].

Lemma 5.3 [39] *If there exists a positive-definite decrescent function* $V(\boldsymbol{x},t) \in C^{2,1}(S^h \times [t_0,\infty);\mathbb{R}_+)$ *such that* $LV(\boldsymbol{x},t)$ *is negative-definite, then the trivial solution of* (5.3) *is stochastically asymptotically stable.*

From the above lemma, we can obtain the stochastically asymptotically stability of equilibrium as follows.

Theorem 5.4 *Assume that* $\mathfrak{B} = (\beta_{kj})$ *is irreducible and* $\mathscr{R}_0 > 1$. *Then, if the following condition is satisfied*:

$$\sigma_{1k}^2 < 2\delta_k, \qquad \sigma_{2k}^2 < 2\sum_{j=1}^n \beta_{kj} I_j^*, \qquad \sigma_{3k}^2 < \frac{2(d_k+q_k+\tau_k)(\tau_k+\delta_k)+2\tau_k\gamma_k}{\tau_k+\delta_k+\gamma_k}, \tag{5.4}$$

the endemic equilibrium P^* *is stochastically asymptotically stable.*

Proof It is easy to see that we only need to prove the zero solution of (5.1) is stochastically asymptotically stable. Let $\boldsymbol{x}_k(t) = (\boldsymbol{u}_k(t),\boldsymbol{v}_k(t),\boldsymbol{w}_k(t))^T \in R^3$, $k = 1,\ldots,n$ and $\boldsymbol{x}(t) = (\boldsymbol{x}_1(t),\ldots,\boldsymbol{x}_n(t))^T \in R^{3n}$. We define the Lyapunov function $V(\boldsymbol{x}(t))$ as follows:

$$V(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^n \big(a_k\boldsymbol{u}_k^2 + b_k\boldsymbol{v}_k^2 + c_k\boldsymbol{w}_k^2 + e_k(\boldsymbol{u}_k+\boldsymbol{w}_k)^2\big), \tag{5.5}$$

where $a_k > 0$, $b_k > 0$, $c_k > 0$ are real positive constants to be chosen later. Then it can be described as the quadratic form

$$V(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^n \boldsymbol{x}_k^T Q\boldsymbol{x}_k,$$

where

$$Q = \begin{pmatrix} a_k+e_k & 0 & e_k \\ 0 & b_k & 0 \\ e_k & 0 & c_k+e_k \end{pmatrix}$$

is a symmetric positive-definite matrix. So it is obviously that $V(\boldsymbol{x})$ is positive-definite decrescent. For the sake of simplicity, (5.5) may be divided into four functions: $V(\boldsymbol{x}) =$

$V_1(\boldsymbol{x}) + V_2(\boldsymbol{x}) + V_3(\boldsymbol{x}) + V_4(\boldsymbol{x})$, where

$$V_1(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n} a_k \boldsymbol{u}_k^2, \qquad V_2(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n} b_k \boldsymbol{v}_k^2,$$
$$V_3(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n} c_k \boldsymbol{w}_k^2, \qquad V_4(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n} e_k (\boldsymbol{u}_k + \boldsymbol{w}_k)^2.$$

Using Itô's formula, we compute

$$\begin{aligned} LV_1 &= \sum_{k=1}^{n} a_k \boldsymbol{u}_k \bigl(-\delta_k \boldsymbol{u}_k + (d_k + q_k)\boldsymbol{v}_k + (d_k + q_k)\boldsymbol{w}_k\bigr) + \frac{1}{2}\sum_{k=1}^{n} a_k \sigma_{1k}^2 \boldsymbol{u}_k^2 \\ &= -\sum_{k=1}^{n} a_k \left(\delta_k - \frac{1}{2}\sigma_{1k}^2\right)\boldsymbol{u}_k^2 + \sum_{k=1}^{n} a_k \bigl((d_k + q_k)\boldsymbol{u}_k \boldsymbol{v}_k + (d_k + q_k)\boldsymbol{u}_k \boldsymbol{w}_k\bigr). \end{aligned} \tag{5.6}$$

Similarly, from Itô's formula, we obtain

$$\begin{aligned} LV_2 = {} & \sum_{k=1}^{n} b_k \boldsymbol{v}_k \left(\sum_{j=1}^{n} \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr)\boldsymbol{v}_j - \sum_{j=1}^{n} \beta_{kj}(\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)\bigl(I_j^* + \boldsymbol{v}_j\bigr) \right. \\ & \left. - (d_k + \gamma_k + q_k)\boldsymbol{v}_k \right) + \frac{1}{2}\sum_{k=1}^{n} b_k \sigma_{2k}^2 \boldsymbol{v}_k^2 \\ = {} & -\sum_{k=1}^{n} b_k \left(\left(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2 \right) \boldsymbol{v}_k^2 + \sum_{j=1}^{n} \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr)\boldsymbol{v}_k \boldsymbol{v}_j \right. \\ & - \sum_{j=1}^{n} \beta_{kj} \frac{(1 - M_k^* - I_k^* - R_k^*) I_j^*}{I_k^*} \boldsymbol{v}_k^2 - \sum_{j=1}^{n} \beta_{kj} I_j^* (\boldsymbol{u}_k + \boldsymbol{w}_k)\boldsymbol{v}_k \\ & \left. - \sum_{j=1}^{n} \beta_{kj} \boldsymbol{v}_k (\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)\boldsymbol{v}_j \right) \\ = {} & \sum_{k=1}^{n} b_k \left(\sum_{j=1}^{n} \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr) I_k^* I_j^* \frac{\boldsymbol{v}_k}{I_k^*}\frac{\boldsymbol{v}_j}{I_j^*} \right. \\ & \left. - \sum_{j=1}^{n} \beta_{kj}\bigl(1 - M_k^* - I_k^* - R_k^*\bigr) I_k^* I_j^* \left(\frac{\boldsymbol{v}_k}{I_k^*}\right)^2 \right) \\ & - \sum_{k=1}^{n} b_k \left(\left(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2 \right) \boldsymbol{v}_k^2 - \sum_{j=1}^{n} \beta_{kj} I_j^* (\boldsymbol{u}_k + \boldsymbol{w}_k)\boldsymbol{v}_k \right. \\ & \left. + \sum_{j=1}^{n} \beta_{kj} \boldsymbol{v}_k (\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)\boldsymbol{v}_j \right) \\ = {} & \sum_{k=1}^{n} b_k \left(\sum_{j=1}^{n} \overline{\beta}_{kj} I_k^* \frac{\boldsymbol{v}_k}{I_k^*}\frac{\boldsymbol{v}_j}{I_j^*} - \sum_{j=1}^{n} \overline{\beta}_{kj} I_k^* \left(\frac{\boldsymbol{v}_k}{I_k^*}\right)^2 \right) - \sum_{k=1}^{n} b_k \left(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2 \right) \boldsymbol{v}_k^2 \\ & - \sum_{k=1}^{n} b_k \sum_{j=1}^{n} \beta_{kj} I_j^* (\boldsymbol{u}_k + \boldsymbol{w}_k)\boldsymbol{v}_k - \sum_{k=1}^{n} b_k \sum_{j=1}^{n} \beta_{kj} \boldsymbol{v}_k (\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)\boldsymbol{v}_j \end{aligned}$$

$$\begin{aligned}
&\le \sum_{k=1}^{n}\frac{1}{2}b_k\left(\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\left(\left(\frac{\mathbf{v}_k}{I_k^*}\right)^2+\left(\frac{\mathbf{v}_j}{I_j^*}\right)^2\right)-\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\left(\frac{\mathbf{v}_k}{I_k^*}\right)^2\right)\\
&\quad-\sum_{k=1}^{n}b_k\sum_{j=1}^{n}\beta_{kj}I_j^*(\mathbf{u}_k+\mathbf{w}_k)\mathbf{v}_k\\
&\quad-\sum_{k=1}^{n}b_k\sum_{j=1}^{n}\beta_{kj}\mathbf{v}_k(\mathbf{u}_k+\mathbf{v}_k+\mathbf{w}_k)\mathbf{v}_j-\sum_{k=1}^{n}b_k\left(\sum_{j=1}^{n}\beta_{kj}I_j^*-\frac{1}{2}\sigma_{2k}^2\right)\mathbf{v}_k^2\\
&=\frac{1}{2}\sum_{k=1}^{n}b_k\left(\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\left(\frac{\mathbf{v}_j}{I_j^*}\right)^2-\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\left(\frac{\mathbf{v}_k}{I_k^*}\right)^2\right)-\sum_{k=1}^{n}b_k\sum_{j=1}^{n}\beta_{kj}I_j^*(\mathbf{u}_k+\mathbf{w}_k)\mathbf{v}_k\\
&\quad-\sum_{k=1}^{n}b_k\sum_{j=1}^{n}\beta_{kj}\mathbf{v}_k(\mathbf{u}_k+\mathbf{v}_k+\mathbf{w}_k)\mathbf{v}_j-\sum_{k=1}^{n}b_k\left(\sum_{j=1}^{n}\beta_{kj}I_j^*-\frac{1}{2}\sigma_{2k}^2\right)\mathbf{v}_k^2.
\end{aligned}\tag{5.7}$$

Let $b_k = \frac{C_{kk}}{I_k^*}$, so $\varsigma_k = b_k I_k^*$. It follows from $\overline{\mathfrak{B}}\varsigma = 0$ and $\overline{\beta}_{jk} = \beta_{jk}(1-M_j^*-I_j^*-R_j^*)I_k^*$ that

$$\sum_{j=1}^{n}\bar{\beta}_{jk}\varsigma_j = \sum_{k=1}^{n}\bar{\beta}_{kj}\varsigma_k,$$

which implies

$$\sum_{k=1}^{n}b_k\sum_{j=1}^{n}\bar{\beta}_{kj}I_k^*\left(\frac{\mathbf{v}_j}{I_j^*}\right)^2=\sum_{k=1}^{n}b_k\sum_{j=1}^{n}\bar{\beta}_{kj}I_k^*\left(\frac{\mathbf{v}_k}{I_k^*}\right)^2.$$

Hence inequality (5.7) becomes

$$\begin{aligned}
LV_2 &\le -\sum_{k=1}^{n}b_k\sum_{j=1}^{n}\beta_{kj}I_j^*(\mathbf{u}_k+\mathbf{w}_k)\mathbf{v}_k-\sum_{k=1}^{n}b_k\sum_{j=1}^{n}\beta_{kj}\mathbf{v}_k(\mathbf{u}_k+\mathbf{v}_k+\mathbf{w}_k)\mathbf{v}_j\\
&\quad-\sum_{k=1}^{n}b_k\left(\sum_{j=1}^{n}\beta_{kj}I_j^*-\frac{1}{2}\sigma_{2k}^2\right)\mathbf{v}_k^2,\\
LV_3 &= \sum_{k=1}^{n}c_k\mathbf{w}_k\big(\gamma_k\mathbf{v}_k-(d_k+q_k+\tau_k)\mathbf{w}_k\big)+\frac{1}{2}\sum_{k=1}^{n}c_k\sigma_{3k}^2\mathbf{w}_k^2\\
&= -\sum_{k=1}^{n}c_k\left(d_k+q_k+\tau_k-\frac{1}{2}\sigma_{3k}^2\right)\mathbf{w}_k^2+\sum_{k=1}^{n}c_k\gamma_k\mathbf{v}_k\mathbf{w}_k,\\
LV_4 &= \sum_{k=1}^{n}e_k(\mathbf{u}_k+\mathbf{w}_k)\big(-\delta_k\mathbf{u}_k+(d_k+q_k+\gamma_k)\mathbf{v}_k-\tau_k\mathbf{w}_k\big)+\frac{1}{2}\sum_{k=1}^{n}e_k\big(\sigma_{1k}^2\mathbf{u}_k^2+\sigma_{3k}^2\mathbf{w}_k^2\big)\\
&= -\sum_{k=1}^{n}e_k\delta_k\mathbf{u}_k^2-\sum_{k=1}^{n}e_k\tau_k\mathbf{w}_k^2-\sum_{k=1}^{n}e_k(\tau_k+\delta_k)\mathbf{u}_k\mathbf{w}_k\\
&\quad+\sum_{k=1}^{n}e_k(d_k+q_k+\gamma_k)(\mathbf{u}_k+\mathbf{w}_k)\mathbf{v}_k+\frac{1}{2}\sum_{k=1}^{n}e_k\big(\sigma_{1k}^2\mathbf{u}_k^2+\sigma_{3k}^2\mathbf{w}_k^2\big).
\end{aligned}$$

Then we compute

$$\begin{aligned}
LV &= LV_1 + LV_2 + LV_3 + LV_4 \\
&\le -\sum_{k=1}^{n}(a_k + e_k)\left(\delta_k - \frac{1}{2}\sigma_{1k}^2\right)\boldsymbol{u}_k^2 - \sum_{k=1}^{n} b_k \left(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2\right)\boldsymbol{v}_k^2 \\
&\quad - \sum_{k=1}^{n}\left[c_k(d_k + q_k + \tau_k) + e_k\tau_k - \frac{1}{2}(c_k + e_k)\sigma_{3k}^2\right]\boldsymbol{w}_k^2 \\
&\quad + \sum_{k=1}^{n} c_k\gamma_k \boldsymbol{v}_k\boldsymbol{w}_k + \sum_{k=1}^{n} a_k\big((d_k + q_k)\boldsymbol{u}_k\boldsymbol{v}_k + (d_k + q_k)\boldsymbol{u}_k\boldsymbol{w}_k\big) \\
&\quad - \sum_{k=1}^{n} b_k \sum_{j=1}^{n} \beta_{kj} I_j^* (\boldsymbol{u}_k + \boldsymbol{w}_k)\boldsymbol{v}_k \\
&\quad - \sum_{k=1}^{n} e_k(\tau_k + \delta_k)\boldsymbol{u}_k\boldsymbol{w}_k + \sum_{k=1}^{n} e_k(d_k + q_k + \gamma_k)(\boldsymbol{u}_k + \boldsymbol{w}_k)\boldsymbol{v}_k \\
&\quad - \sum_{k=1}^{n} b_k \sum_{j=1}^{n} \beta_{kj} \boldsymbol{v}_k(\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)\boldsymbol{v}_j \\
&= -\sum_{k=1}^{n}(a_k + e_k)\left(\delta_k - \frac{1}{2}\sigma_{1k}^2\right)\boldsymbol{u}_k^2 - \sum_{k=1}^{n} b_k \left(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2\right)\boldsymbol{v}_k^2 \\
&\quad - \sum_{k=1}^{n}\left[c_k(d_k + q_k + \tau_k) + e_k\tau_k - \frac{1}{2}(c_k + e_k)\sigma_{3k}^2\right]\boldsymbol{w}_k^2 \\
&\quad + \sum_{k=1}^{n}\left[c_k\gamma_k - b_k\sum_{j=1}^{n}\beta_{kj} I_j^* + e_k(d_k + q_k + \gamma_k)\right]\boldsymbol{v}_k\boldsymbol{w}_k \\
&\quad + \sum_{k=1}^{n}\left[a_k(d_k + q_k) - b_k\sum_{j=1}^{n}\beta_{kj} I_j^* + e_k(d_k + q_k + \gamma_k)\right]\boldsymbol{u}_k\boldsymbol{v}_k \\
&\quad + \sum_{k=1}^{n}\big[a_k(d_k + q_k) - e_k(\tau_k + \delta_k)\big]\boldsymbol{u}_k\boldsymbol{w}_k - \sum_{k=1}^{n} b_k \sum_{j=1}^{n} \beta_{kj} \boldsymbol{v}_k(\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)\boldsymbol{v}_j.
\end{aligned}$$

We can choose a_k, c_k, e_k such that

$$\begin{aligned}
&a_k(d_k + q_k) - e_k(\tau_k + \delta_k) = 0, \\
&a_k(d_k + q_k) - b_k\sum_{j=1}^{n}\beta_{kj} I_j^* + e_k(d_k + q_k + \gamma_k) = 0, \\
&c_k\gamma_k - b_k\sum_{j=1}^{n}\beta_{kj} I_j^* + e_k(d_k + q_k + \gamma_k) = 0,
\end{aligned}$$

i.e.,

$$a_k = \frac{\sum_{j=1}^{n}\beta_{kj} I_j^*(\tau_k + \delta_k)}{(d_k + q_k)(\tau_k + \delta_k + d_k + q_k + \gamma_k)} b_k,$$

$$
\begin{aligned}
e_k &= \frac{\sum_{j=1}^n \beta_{kj} I_j^*}{\tau_k + \delta_k + d_k + q_k + \gamma_k} b_k, \\
c_k &= \frac{\sum_{j=1}^n \beta_{kj} I_j^* (\tau_k + \delta_k)}{\gamma_k (\tau_k + \delta_k + d_k + q_k + \gamma_k)} b_k,
\end{aligned}
$$

then

$$
\begin{aligned}
LV \le & -\sum_{k=1}^n (a_k + e_k)\left(\delta_k - \frac{1}{2}\sigma_{1k}^2\right)\boldsymbol{u}_k^2 - \sum_{k=1}^n b_k \left(\sum_{j=1}^n \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2\right)\boldsymbol{v}_k^2 \\
& - \sum_{k=1}^n c_k \left(d_k + q_k + \tau_k + \frac{\tau_k \gamma_k}{\tau_k + \delta_k} - \frac{\tau_k + \delta_k + \gamma_k}{2(\tau_k + \delta_k)}\sigma_{3k}^2\right)\boldsymbol{w}_k^2 \\
& - \sum_{k=1}^n b_k \sum_{j=1}^n \beta_{kj} \boldsymbol{v}_k (\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)\boldsymbol{v}_j \\
= & \; V_0 - \sum_{k=1}^n b_k \sum_{j=1}^n \beta_{kj} \boldsymbol{v}_k (\boldsymbol{u}_k + \boldsymbol{v}_k + \boldsymbol{w}_k)\boldsymbol{v}_j,
\end{aligned} \tag{5.8}
$$

where

$$
\begin{aligned}
V_0 &= \sum_{k=1}^n \left(\mathcal{A}_k \boldsymbol{u}_k^2 + \mathcal{B}_k \boldsymbol{v}_k^2 + \mathcal{C}_k \boldsymbol{w}_k^2\right), \\
\mathcal{A}_k &= (a_k + e_k)\left(\delta_k - \frac{1}{2}\sigma_{1k}^2\right), \qquad \mathcal{B}_k = b_k \left(\sum_{j=1}^n \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2\right), \\
\mathcal{C}_k &= c_k \left(d_k + q_k + \tau_k + \frac{\tau_k \gamma_k}{\tau_k + \delta_k} - \frac{\tau_k + \delta_k + \gamma_k}{2(\tau_k + \delta_k)}\sigma_{3k}^2\right),
\end{aligned}
$$

and the proofs above show that if the condition (5.4) is satisfied, then $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k$ are positive constants. Let

$$
\lambda = \min_{k \in \{1,\dots,n\}} \{\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k\},
$$

then $\lambda > 0$. From (5.8), one sees that

$$
\begin{aligned}
LV &\le -\lambda \sum_{k=1}^n \left(\left|\boldsymbol{x}_k(t)\right|^2 + o\left(\left|\boldsymbol{x}_k(t)\right|^2\right)\right) \\
&= -\lambda \left|\boldsymbol{x}(t)\right|^2 + o\left(\left|\boldsymbol{x}(t)\right|^2\right),
\end{aligned} \tag{5.9}
$$

where $|\boldsymbol{x}_k(t)| = \sqrt{\boldsymbol{u}_k^2(t) + \boldsymbol{v}_k^2(t) + \boldsymbol{w}_k^2(t)}$, $|\boldsymbol{x}(t)| = (\sum_{k=1}^n |\boldsymbol{x}_k(t)|^2)^{1/2}$ and $o(|\boldsymbol{x}(t)|^2)$ is an infinitesimal of higher order of $|\boldsymbol{x}(t)|^2$ for $t \to \infty$. Hence $LV(\boldsymbol{x}, t)$ is negative-definite in a sufficiently small neighborhood of $\boldsymbol{x} = \boldsymbol{0}$ for $t \ge 0$. According to Lemma 5.3, we therefore conclude that the zero solution of (5.2) is stochastically asymptotically stable. The proof is complete. □

6 Numerical simulation

Numerical methods are used to solve the systems (2.1) and (5.1) and to depict the behavior of the passively immune, infectious, and recovered with respect to time. We numerically simulate the solution of systems (2.1) and (5.1) when $n = 2$. In this case, we have

$$M_0 = \begin{bmatrix} \frac{\beta_{11}}{d_1^I+q_1+\gamma_1} & \frac{\beta_{12}}{d_1^I+q_1+\gamma_1} \\ \frac{\beta_{21}}{d_2^I+q_2+\gamma_2} & \frac{\beta_{22}}{d_2^I+q_2+\gamma_2} \end{bmatrix} := \begin{bmatrix} \beta_{11}K_1 & \beta_{12}K_1 \\ \beta_{21}K_2 & \beta_{22}K_2 \end{bmatrix}$$

and

$$\mathscr{R}_0 = \rho(M_0) = \frac{\beta_{11}K_1 + \beta_{22}K_2 + \sqrt{(\beta_{11}K_1 - \beta_{22}K_2)^2 + 4\beta_{12}\beta_{21}K_1K_2}}{2}.$$

The system parameters are given by

$$\beta_{11} = 0.25, \qquad \beta_{12} = 0.35, \qquad d_1 = 0.02, \qquad q_1 = 0.03,$$
$$\delta_1 = 0.15, \qquad \tau_1 = 0.05, \qquad \gamma_1 = 0.12,$$
$$\beta_{21} = 0.45, \qquad \beta_{22} = 0.65, \qquad d_2 = 0.03, \qquad q_2 = 0.05,$$
$$\delta_2 = 0.25, \qquad \tau_2 = 0.10, \qquad \gamma_2 = 0.25.$$

Hence, we obtain $M_1^* = 0.1763$, $I_1^* = 0.2405$, $R_1^* = 0.2886$, $M_2^* = 0.1717$, $I_2^* = 0.2247$, and $R_2^* = 0.3120$. It is easy to compute that

$$\mathscr{R}_0 = 3.5406 > 1.$$

Moreover, we choose $M_1(0) = 0.35$, $I_1(0) = 0.2$, $R_1(0) = 0.1$, $M_2(0) = 0.2$, $I_2(0) = 0.35$, $R_2(0) = 0.05$ as the initial values, and we keep all these initial values and parameter values unchanged in each example unless otherwise stated. In the absence of noise, we simulate the asymptotic stability of the endemic equilibrium of the deterministic system (2.1) in Figure 2.

By Lemma 4.1, we see that the endemic equilibrium P^* of the deterministic model (2.1) is asymptotically stable. The computer simulations shown in Figure 2 clearly support this result.

Next, we show the numerical simulation of the stochastic system (5.1). Given the discretization of system (5.1) for $t = 0, \triangle t, 2\triangle t, \dots, n\triangle t$, and $k = 1, 2$.

$$\begin{cases} M_{k,i+1} = M_{k,i} + (-\delta_k M_{k,i} + (d_k + q_k)I_{k,i} + (d_k + q_k)R_{k,i})\triangle t \\ \qquad\qquad + \sigma_{1k}(M_{k,i} - M_k^*)\sqrt{\triangle t}\varepsilon_{1k,i}, \\ I_{k,i+1} = I_{k,i} + (\beta_{k1}(1 - M_{k,i} - I_{k,i} - R_{k,i})I_{1,i} + \beta_{k2}(1 - M_{k,i} - I_{k,i} - R_{k,i})I_{2,i} \\ \qquad\qquad - (d_k + \gamma_k + q_k)I_{k,i})\triangle t + \sigma_{2k}(I_{k,i} - I_k^*)\sqrt{\triangle t}\varepsilon_{2k,i}, \\ R_{k,i+1} = R_{k,i} + (\gamma_k I_{k,i} + (d_k + q_k + \tau_k)R_{k,i})\triangle t + \sigma_{3k}(R_{k,i} - R_k^*)\sqrt{\triangle t}\varepsilon_{3k,i}, \end{cases} \tag{6.1}$$

where the time increment $\triangle t > 0$, and $\varepsilon_{1k,i}$, $\varepsilon_{2k,i}$, and $\varepsilon_{3k,i}$ are $N(0,1)$-distributed independent random variables, which can be generated numerically by pseudo-random number generators.

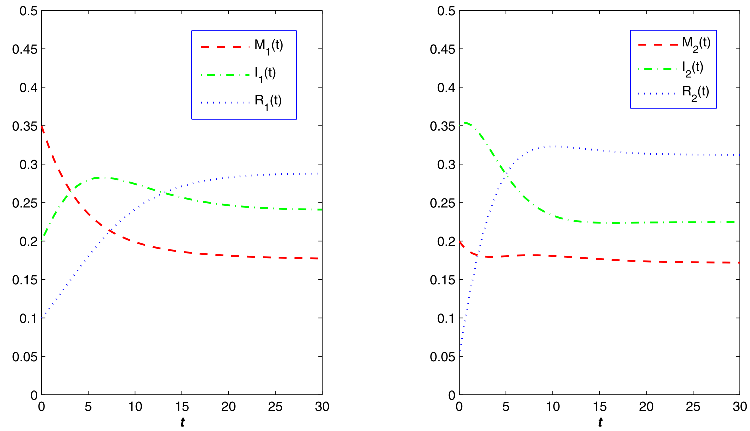

Figure 2 Deterministic trajectories of the MSIRS model (2.1) for the initial conditions $M_1(0) = 0.35$, $I_1(0) = 0.2$, $R_1(0) = 0.1$, $M_2(0) = 0.2$, $I_2(0) = 0.35$, and $R_2(0) = 0.05$.

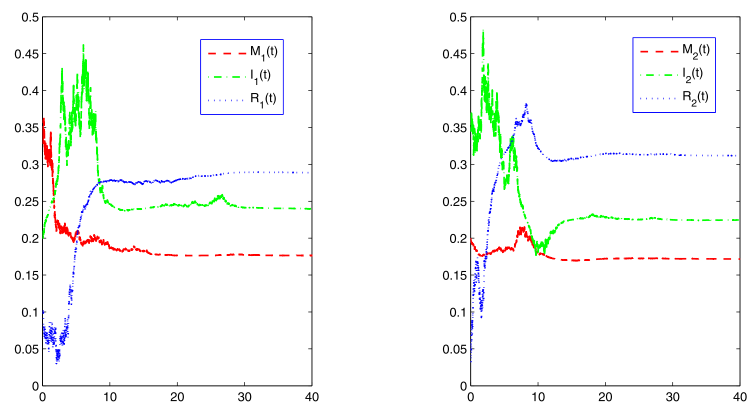

Figure 3 Stochastic trajectories of the MSIRS model (5.1) for $\sigma_{11} = 0.37$, $\sigma_{21} = 0.5$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.35$, $\sigma_{22} = 0.5$, $\sigma_{32} = 0.3$, and $\Delta t = 10^{-3}$.

To determine the effect of the noise intensity, we consider three series of different values. Figure 3 corresponds to $\sigma_{11} = 0.37$, $\sigma_{21} = 0.5$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.35$, $\sigma_{22} = 0.5$, and $\sigma_{32} = 0.3$, and it is easy to verify the noise intensity and system parameters obey condition (5.4). We can therefore conclude, by Theorem 5.4, that the endemic equilibrium P^* of stochastic model (5.1) is asymptotically stable. The computer simulations shown in Figure 3 agree well with the mathematical result. Figure 4 corresponds to $\sigma_{11} = 5.5$, $\sigma_{21} = 7.5$, $\sigma_{31} = 5.5$, $\sigma_{12} = 5.3$, $\sigma_{22} = 6.8$, and $\sigma_{32} = 5.7$, and the comparison of Figures 3 (left) and 4 (left) suggests that the fluctuations of at least one of the curves increase as the noise level increases. The same situation occurs for the comparison of Figures 3 (right) and 4 (right). Note that condition (5.4) is just a sufficient condition. When this condition is not satisfied, the stochastic system (5.1) may or may not be stable. For example, the intensities of the Brownian motions $\sigma_{11} = 5.5$, $\sigma_{21} = 7.5$, $\sigma_{31} = 5.5$, $\sigma_{12} = 5.3$, $\sigma_{22} = 6.8$, and $\sigma_{32} = 5.7$ do not obey the condition (5.4), but we can see from Figure 4 that the endemic equilibrium of the stochastic system (5.1) is still asymptotically stable. If we choose $\sigma_{11} = 43.5$, $\sigma_{21} = 46.5$, $\sigma_{31} = 51.3$, $\sigma_{12} = 65.3$, $\sigma_{22} = 52.2$, and $\sigma_{32} = 43.3$, then the solution of the stochastic sys-

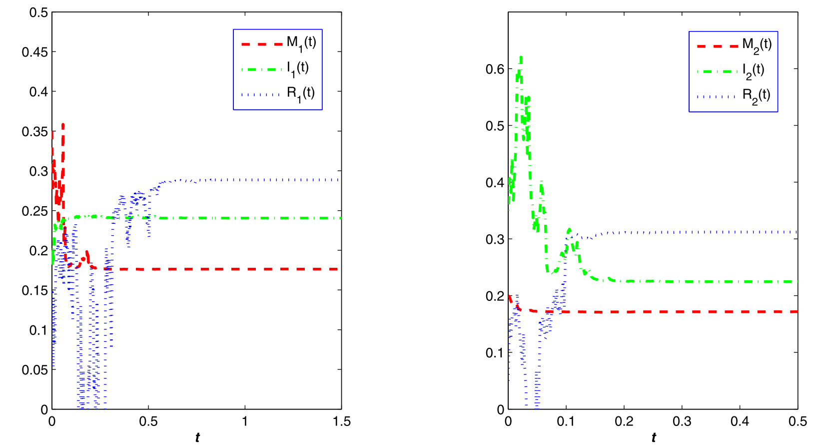

Figure 4 Stochastic trajectories of MSIRS model (5.1) for $\sigma_{11} = 5.5$, $\sigma_{21} = 7.5$, $\sigma_{31} = 5.5$, $\sigma_{12} = 5.3$, $\sigma_{22} = 6.8$, and $\sigma_{32} = 5.7$.

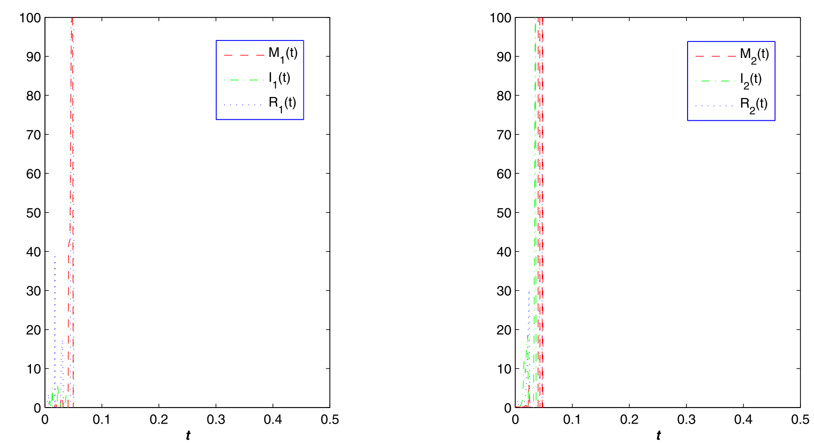

Figure 5 Stochastic trajectories of the MSIRS model (5.1) for $\sigma_{11} = 43.5$, $\sigma_{21} = 46.5$, $\sigma_{31} = 51.3$, $\sigma_{12} = 65.3$, $\sigma_{22} = 52.2$, and $\sigma_{32} = 43.3$.

tem (5.1) is not asymptotically stable but rather explodes to infinity in a finite time (see Figure 5).

To better understand the long time behavior of the deterministic and stochastic systems, we show the phase space portraits for the deterministic and MSIRS stochastic endemic model in Figures 6 and 7, respectively. For the stochastic case, Figure 7 corresponds to $\sigma_{11} = 0.37$, $\sigma_{21} = 0.5$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.35$, $\sigma_{22} = 0.5$, and $\sigma_{32} = 0.3$, in which the noise intensity and system parameters obey the condition (5.4). Therefore, by Theorem 5.4, the endemic equilibrium P^* of stochastic model (5.1) is asymptotically stable; we can see from Figure 7 that an oscillation appears under environmental driving forces, which actually affect the deterministic curves shown in Figure 6. These two trajectories in Figure 7 still maintains the same overall trend as those of Figure 6, and they reach the same equilibrium points as the deterministic version. The simulations agree with our results.

The major difference between the MSIRS model and the MSIR model is that the MSIRS model does not confer permanent immunity to individuals in the model. Thus, let us an-

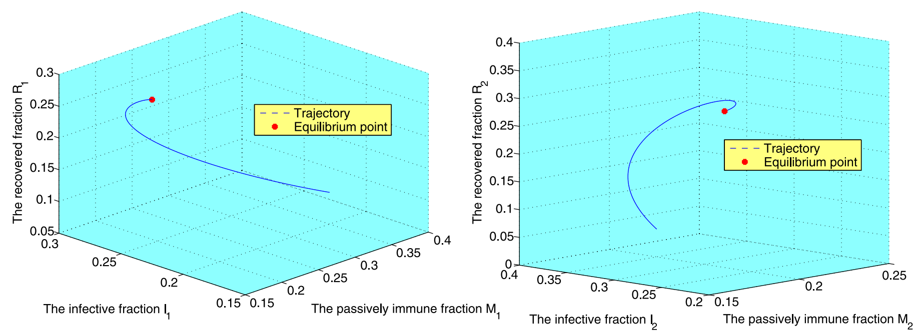

Figure 6 **Phase space portrait for the deterministic MSIRS endemic model.**

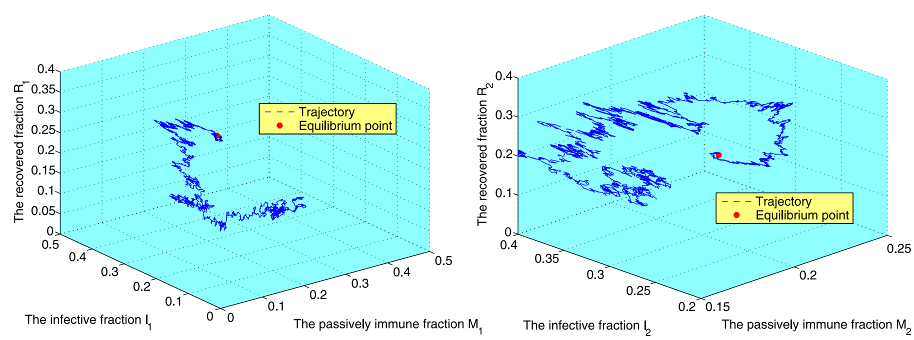

Figure 7 **Phase space portrait for the MSIRS endemic model with stochastic perturbations.**

Table 2 **Values of P^*, when fixing $\tau_2 = 0.1$ and changing τ_1**

	M_1^*	I_1^*	R_1^*	M_2^*	I_2^*	R_2^*
$\tau_1 = 0.05$	0.1763	0.2405	0.2886	0.1717	0.2247	0.3120
$\tau_1 = 0.1$	0.1704	0.2840	0.2272	0.1760	0.2303	0.3198
$\tau_1 = 0.15$	0.1669	0.3129	0.1877	0.1786	0.2336	0.3244
$\tau_1 = 0.2$	0.1646	0.3336	0.1601	0.1802	0.2358	0.3275

Table 3 **Values of P^*, when fixing $\tau_1 = 0.05$ and changing τ_2**

	M_1^*	I_1^*	R_1^*	M_2^*	I_2^*	R_2^*
$\tau_2 = 0.1$	0.1763	0.2405	0.2886	0.1717	0.2247	0.3120
$\tau_2 = 0.15$	0.1803	0.2459	0.2951	0.1689	0.2529	0.2748
$\tau_2 = 0.2$	0.1832	0.2498	0.2998	0.1668	0.2754	0.2459
$\tau_2 = 0.25$	0.1853	0.2527	0.3032	0.1653	0.2939	0.2226

alyze the impact of the rate τ_k of passive immune loss on the MSIRS model by assigning different values to it, as provided in Tables 2 and 3 by calculating the equilibrium of system (2.1), which is, of course, an equilibrium of system (5.1).

Analyzing the data in Tables 2 and 3, it shows that the higher the value τ_k of the rate of immunity loss is, the higher the value I_j^* ($j = 1, \ldots, n$) of the endemic equilibrium is. Thus, it will be of great importance for health management to take some effective measures to diminish the rate the immunity loss. For example, when the antibody concentration of a recovered person decreased, he can be required to undergo vaccination to achieve the protective antibody levels.

7 Conclusion

This paper presented a mathematical study describing the dynamical behavior of an MSIRS epidemic model. Our purpose was based on analyzing this behavior using both a deterministic model and a stochastic model. This result differs from the previous results obtained in [4, 5] for single-group MSIRS models. We proved that the deterministic model has a unique endemic equilibrium, which is asymptotically stable if the reproduction number $\mathscr{R}_0$ is greater than one; this means that the disease will persist at the endemic equilibrium level if it is initially present and the disease die out if $\mathscr{R}_0 \leq 1$. Furthermore, concerning the stochastic model, we obtained sufficient conditions for stochastic asymptotical stability of the endemic equilibrium P^* by using a suitable Lyapunov function and other stochastic analysis techniques. The investigation of this stochastic model revealed that the stochastic stability of P^* depends on the magnitude of the intensity of the noise as well as the parameters involved within the model system.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together and contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, Harbin Normal University, Harbin, 150500, P.R. China. [2]Department of Foundation, Harbin Finance University, Harbin, 150032, P.R. China. [3]College of Sciences, Qiqihar University, Qiqihar, 161006, P.R. China.

Acknowledgements
All authors of this paper were partially supported by Project of Science and Technology of Heilongjiang Province of China No. 12531187 and Natural Science Foundation of Heilongjiang Province of China No. A201410.

Received: 18 September 2014 Accepted: 15 October 2014 Published: 21 Oct 2014

References

1. Kermack, WO, McKendrick, AG: Contributions to the mathematical theory of epidemics, part 1. Proc. R. Soc. Lond. Ser. A **115**, 700-721 (1927)
2. Mulone, G, Straughan, B, Wang, W: Stability of epidemic models with evolution. Stud. Appl. Math. **118**, 117-132 (2007)
3. Wang, X, Wang, W, Zhang, G: Global analysis of predator-prey system with Hawk and Dove tactics. Stud. Appl. Math. **124**, 151-178 (2010)
4. Hethcote, HW: The mathematics of infectious diseases. SIAM Rev. **42**, 599-653 (2000)
5. Lou, J, Ma, Z: Stability of some epidemic models with passive immune. Acta Math. Sin. **23**, 357-368 (2003) (in Chinese)
6. Lajmanovich, A, Yorke, JA: A deterministic model for gonorrhea in a nonhomogeneous population. Math. Biosci. **28**, 221-236 (1976)
7. Beretta, E, Capasso, V: Global stability results for a multigroup SIR epidemic model. In: Hallam, TG, Gross, LJ, Levin, SA (eds.) Mathematical Ecology. World Scientific, Teaneck (1988)
8. Hethcote, HW: An immunization model for a heterogeneous population. Theor. Popul. Biol. **14**, 338-349 (1978)
9. Thieme, HR: Local stability in epidemic models for heterogeneous populations. In: Capasso, V, Grosso, E, Paveri-Fontana, SL (eds.) Mathematics in Biology and Medicine. Lecture Notes in Biomathematics, vol. 57, pp. 185-189. Springer, Berlin (1985)
10. Kuniya, T: Global stability of a multi-group SVIR epidemic model. Nonlinear Anal., Real World Appl. **14**, 1135-1143 (2013)
11. Muroya, Y, Enatsu, Y, Kuniya, T: Global stability for a multi-group SIRS epidemic model with varying population sizes. Nonlinear Anal., Real World Appl. **14**, 1693-1704 (2013)
12. Guo, H, Li, MY, Shuai, Z: A graph-theoretic approach to the method of global Lyapunov functions. Proc. Am. Math. Soc. **136**, 2793-2802 (2008)
13. Li, MY, Shuai, Z, Wang, C: Global stability of multi-group epidemic models with distributed delays. J. Math. Anal. Appl. **361**, 38-47 (2010)
14. Guo, H, Li, MY, Shuai, Z: Global stability of the endemic equilibrium of multigroup SIR epidemic models. Can. Appl. Math. Q. **14**, 259-284 (2006)
15. Dalal, N, Greenhalgh, D, Mao, X: A stochastic model of AIDS and condom use. J. Math. Anal. Appl. **325**, 36-53 (2007)
16. Ji, C, Jiang, D, Shi, N: Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation. J. Math. Anal. Appl. **359**, 482-498 (2009)
17. Ji, C, Jiang, D, Li, XY: Qualitative analysis of a stochastic ratio-dependent predator-prey system. J. Comput. Appl. Math. **235**, 1326-1341 (2011)
18. Tornatore, E, Buccellato, SM, Vetro, P: Stability of a stochastic SIR system. Physica A **354**, 111-126 (2005)

19. Beretta, E, Kolmanovskii, V, Shaikhet, L: Stability of epidemic model with time delays influenced by stochastic perturbations. Math. Comput. Simul. **45**, 269-277 (1998)
20. Shaikhet, L: Stability of predator-prey model with aftereffect by stochastic perturbation. Stab. Control: Theory Appl. **1**, 3-13 (1998)
21. Carletti, M: On the stability properties of a stochastic model for phage-bacteria interaction in open marine environment. Math. Biosci. **175**, 117-131 (2002)
22. Sarkar, RR, Banerjee, S: Cancer self remission and tumor stability-a stochastic approach. Math. Biosci. **196**, 65-81 (2005)
23. Shaikhet, L: Stability of a positive point of equilibrium of one nonlinear system with aftereffect and stochastic perturbations. Dyn. Syst. Appl. **17**, 235-253 (2008)
24. Shaikhet, L: Lyapunov Functionals and Stability of Stochastic Difference Equations. Springer, London (2011)
25. Shaikhet, L: Lyapunov Functionals and Stability of Stochastic Functional Differential Equations. Springer, Dordrecht (2013)
26. Yuan, C, Jiang, D, O'Regan, D, Agarwal, RP: Stochastically asymptotically stability of the multi-group SEIR and SIR models with random perturbation. Commun. Nonlinear Sci. Numer. Simul. **17**, 2501-2516 (2012)
27. Yu, J, Jiang, D, Shi, N: Global stability of two-group SIR model with random perturbation. J. Math. Anal. Appl. **360**, 235-244 (2009)
28. Ji, C, Jiang, D, Shi, N: Multigroup SIR epidemic model with stochastic perturbation. Physica A **390**, 1747-1762 (2011)
29. Imhof, L, Walcher, S: Exclusion and persistence in deterministic and stochastic chemostat models. J. Differ. Equ. **217**, 26-53 (2005)
30. Fan, X, Wang, Z: Stability analysis of an SEIR epidemic model with stochastic perturbation and numerical simulation. Int. J. Nonlinear Sci. Numer. Simul. **14**, 113-121 (2013)
31. Fan, X, Wang, Z, Xu, X: Global stability of two-group epidemic models, with distributed delays and random perturbation. Abstr. Appl. Anal. **2012**, Article ID 132095 (2012)
32. Wang, Z, Fan, X: Global stability of deterministic and stochastic multigroup SEIQR models in computer network. Appl. Math. Model. **37**, 8673-8686 (2013)
33. van den Driessche, P, Watmough, J: Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math. Biosci. **180**, 29-48 (2002)
34. Berman, A, Plemmons, RJ: Nonnegative Matrices in the Mathematical Sciences. Academic Press, New York (1979)
35. LaSalle, JP: The Stability of Dynamical Systems. Regional Conference Series in Applied Mathematics. SIAM, Philadelphia (1976)
36. Freedman, HI, Ruan, S, Tang, M: Uniform persistence and flows near a closed positively invariant set. J. Dyn. Differ. Equ. **6**, 583-600 (1994)
37. Li, MY, Graef, JR, Wang, L, Karsai, J: Global dynamics of a SEIR model with varying total population size. Math. Biosci. **160**, 191-213 (1999)
38. Gikhman, II, Skorokhod, AV: Stochastic Differential Equations. Springer, Berlin (1972)
39. Mao, X: Stochastic Differential Equations and Applications. Ellis Horwood, Chichester (1997)

10.1186/1687-1847-2014-270
Cite this article as: Wang et al.: **Dynamics of deterministic and stochastic multi-group MSIRS epidemic models with varying total population size.** *Advances in Difference Equations* 2014, 2014:270