Fen and Karaca *Advances in Difference Equations* 2014, 2014:263
http://www.advancesindifferenceequations.com/content/2014/1/263

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of positive solutions for boundary value problems of p-Laplacian difference equations

Fatma T Fen[1,2*] and Ilkay Y Karaca[2]

*Correspondence: fatmatokmak@gazi.edu.tr
[1]Department of Mathematics, Gazi University, Teknikokullar, Ankara, 06500, Turkey
[2]Department of Mathematics, Ege University, Bornova, Izmir, 35100, Turkey

Abstract

In this paper, by using the Avery-Peterson fixed point theorem, we investigate the existence of at least three positive solutions for a third order p-Laplacian difference equation. An example is given to illustrate our main results.

MSC: 34B10; 34B15; 34J10

Keywords: positive solutions; fixed point theorem; p-Laplacian difference equation; boundary value problem

1 Introduction

The aim of this paper is to study the existence of positive solutions for a third order p-Laplacian difference equation

$$\begin{cases} \Delta[\phi_p(\Delta^2 y(n))] + q(n)f(n,y(n),\Delta y(n)) = 0, & n \in [0,N], \\ ay(0) - b\Delta y(0) = 0, \qquad cy(N+3) + d\Delta y(N+2) = 0, \qquad \Delta^2 y(0) = 0, \end{cases} \tag{1.1}$$

where

- $N > 1$ an integer;
- $a, c > 0$, and $b, d \geq 0$ with $ad + ac(N+3) + bc > 0$;
- f and q are continuous and positive;
- ϕ_p is called p-Laplacian, $\phi_p(x) = |x|^{p-2}x$ with $p > 1$, its inverse function is denoted by $\phi_q(x)$ with $\phi_q(x) = |x|^{q-2}x$ with $1/p + 1/q = 1$;
- $\sum_{i=r}^{s} x(i) = 0$ if $r, s \in \mathbb{Z}$ and $s < r$, where $\mathbb{Z}$ is the integer set, denote $[r,s] = \{r, r+1, \ldots, s\}$ for $r, s \in \mathbb{Z}$ with $r \leq s$.

Difference equations, the discrete analog of differential equations, have been widely used in many fields such as computer science, economics, neural network, ecology, cybernetics, *etc.* [1]. In the past decade, the existence of positive solutions for the boundary value problems (BVPs) of the difference equations has been extensively studied; to mention a few references, see [1–13] and the references therein. Also there has been much interest shown in obtaining the existence of positive solutions for the third order p-Laplacian dynamic equations on time scales. To mention a few papers along these lines, see [14–18].

We now discuss briefly several of the appropriate papers on the topic.

Liu [10] studied the following second order p-Laplacian difference equation with multi-point boundary conditions:

$$\begin{cases} \Delta[\phi(\Delta x(n))] + f(n, x(n+1), \Delta x(n), \Delta x(n+1)) = 0, \quad n \in [0, N], \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\eta_i) = A, \\ x(N+2) - \beta_i x(\eta_i) = B. \end{cases}$$

The sufficient conditions to guarantee the existence of at least three positive solutions of the above multi-point boundary value problem were established by using a new fixed point theorem obtained in [19].

Liu [12] studied the following boundary value problem:

$$\begin{cases} \Delta[\phi(\Delta x(n))] + f(n, x(n+1), \Delta x(n), \Delta x(n+1)) = 0, \quad n \in [0, N], \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\eta_i) = 0, \\ x(N+2) - \beta_i x(\eta_i) = B. \end{cases}$$

By using the five functionals fixed point theorem [20], Liu obtained the existence criteria of at least three positive solutions.

Therefore, in this paper, we will consider the existence of at least three positive solutions for the third order p-Laplacian difference equation (1.1) by using the Avery-Peterson fixed point theorem [3].

Throughout this paper we assume that the following condition holds:

(C1) $f : [0, N] \times [0, +\infty) \times \mathbb{R} \to (0, +\infty)$ and $q : [0, N] \to (0, +\infty)$ are continuous.

This paper is organized as follows. In Section 2, we give some preliminary lemmas which are key tools for our proof. The main result is given in Section 3. Finally, in Section 4, we give an example to demonstrate our result.

2 Preliminaries

In this section we present some lemmas, which will be needed in the proof of the main result.

Let γ and θ be nonnegative continuous convex functionals on $\mathcal{P}$, α be a nonnegative continuous concave functional on $\mathcal{P}$ and ψ be a nonnegative continuous functional on $\mathcal{P}$. Then for positive real numbers t, v, w, and z, we define the following convex sets of $\mathcal{P}$:

$$\mathcal{P}(\gamma, z) = \{y \in \mathcal{P} : \gamma(y) < z\},$$

$$\mathcal{P}(\alpha, v; \gamma, z) = \{y \in \mathcal{P} : v \leq \alpha(y), \gamma(y) \leq z\},$$

$$\mathcal{P}(\alpha, v; \theta, w; \gamma, z) = \{y \in \mathcal{P} : v \leq \alpha(y), \theta(y) \leq w, \gamma(y) \leq z\},$$

and a closed set

$$R(\psi, t; \gamma, z) = \{y \in \mathcal{P} : t \leq \psi(y), \gamma(y) \leq z\}.$$

The following fixed point theorem is fundamental and important to the proof of our main result.

Lemma 2.1 ([3]) *Let $\mathbb{B}$ be a real Banach space and $\mathcal{P} \subset \mathbb{B}$ be a cone in $\mathbb{B}$. Let γ, θ be nonnegative continuous convex functionals on $\mathcal{P}$, let α be a nonnegative continuous concave functional on $\mathcal{P}$, and let ψ be a nonnegative continuous functional on $\mathcal{P}$ satisfying $\psi(\lambda y) \le \lambda\psi(y)$ for all $0 \le \lambda \le 1$, such that for some positive numbers z and M,*

$$\alpha(y) \le \psi(y), \qquad \|y\| \le M\gamma(y), \quad \text{for all } y \in \overline{\mathcal{P}(\gamma, z)}.$$

Suppose that $T : \overline{\mathcal{P}(\gamma, z)} \to \overline{\mathcal{P}(\gamma, z)}$ is completely continuous and there exist positive numbers t, v, and w with $0 < t < v < w$ such that

(i) *$\{y \in \mathcal{P}(\alpha, v; \theta, w; \gamma, z) | \alpha(y) > v\} \neq \emptyset$ and $\alpha(Ty) > v$ for $y \in \mathcal{P}(\alpha, v; \theta, w; \gamma, z)$;*

(ii) *$\alpha(Ty) > v$ for $y \in \mathcal{P}(\alpha, v; \gamma, z)$ with $\theta(Ty) > w$;*

(iii) *$0 \notin R(\psi, t; \gamma, z)$ and $\psi(Ty) < t$ for $y \in R(\psi, t; \gamma, z)$ with $\psi(y) = t$.*

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{\mathcal{P}(\gamma, z)}$ such that

$$\gamma(y_i) \le z, \quad i = 1, 2, 3, \qquad v < \alpha(y_1), \qquad t < \psi(y_2) \quad \text{with } \alpha(y_2) < v \text{ and } \psi(y_3) < t.$$

Let $h(n)$ ($n \in [0, N]$) be a positive sequence. Consider the following BVP:

$$\begin{cases} \Delta[\phi_p(\Delta^2 y(n))] + h(n) = 0, \quad n \in [0, N], \\ ay(0) - b\Delta y(0) = 0, \qquad cy(N+3) + d\Delta y(N+2) = 0, \qquad \Delta^2 y(0) = 0. \end{cases} \tag{2.1}$$

Lemma 2.2 *If y is a solution of BVP* (2.1)*, then there exists unique $n_0 \in [0, N+1]$ such that $\Delta y(n_0) > 0$ and $\Delta y(n_0 + 1) \le 0$.*

Proof Suppose y satisfies (2.1). It follows that

$$y(n) = y(0) + n\Delta y(0) - \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right), \quad n \in [0, N+3]. \tag{2.2}$$

The BCs in (2.1) imply that

$$ay(0) = b\Delta y(0)$$

and

$$cy(0) + c(N+3)\Delta y(0) - c\sum_{r=0}^{N+2} \sum_{i=0}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) + d\Delta y(0) - d\sum_{i=0}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) = 0.$$

It follows that

$$y(0) = \frac{b}{a}\Delta y(0)$$

and

$$\Delta y(0) = \frac{a}{ad + ac(N+3) + bc}\left(c\sum_{r=0}^{N+2} \sum_{i=0}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) + d\sum_{i=0}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right)\right).$$

Similarly, we get

$$y(n) = y(N+3) - (N+3-n)\Delta y(N+2) - \sum_{r=n}^{N+2}\sum_{i=r}^{N+1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right),$$
$$n \in [0, N+3]. \tag{2.3}$$

The BCs in (2.1) imply that

$$y(N+3) = -\frac{d}{c}\Delta y(N+2)$$

and

$$\begin{aligned}&\Delta y(N+2)\\&\quad = -\frac{c}{ad+ac(N+3)+bc}\left(a\sum_{r=0}^{N+2}\sum_{i=r}^{N+1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right) + b\sum_{i=0}^{N+1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right)\right).\end{aligned}$$

Since $a, b \geq 0$, and $c, d > 0$ with $ad + ac(N+3) + bc > 0$ and $h(n)$ a positive sequence, one can easily see that $\Delta y(0) > 0$ and $\Delta y(N+2) \leq 0$. It follows from $\Delta y(0) > 0$, $\Delta y(N+2) \leq 0$, and the fact that $\Delta y(n)$ is decreasing on $[0, N+2]$ that there exists unique $n_0 \in [0, N+1]$ such that $\Delta y(n_0) > 0$ and $\Delta y(n_0+1) \leq 0$. The proof is complete. □

Lemma 2.3 *If y is a solution of BVP* (2.1), *then $y(0) \geq 0$, $y(N+3) \geq 0$, and $y(n) > 0$ for all $n \in [1, N+2]$.*

Proof We get from Lemma 2.2 the result that there exists unique $n_0 \in [0, N]$ such that $\Delta y(n_0) > 0$ and $\Delta y(n_0+1) \leq 0$. It follows from (2.1) that

$$\Delta y(n) = \begin{cases}\Delta y(n_0+1) - \sum_{i=n_0+1}^{n-1}\phi_q(\sum_{s=0}^{i-1}h(s)), & n \in [n_0+1, N+2],\\ \Delta y(n_0) + \sum_{i=n}^{n_0-1}\phi_q(\sum_{s=0}^{i-1}h(s)), & n \in [0, n_0].\end{cases}$$

Then

$$y(n) = \begin{cases}y(N+3) - (N+3-n)\Delta y(n_0+1) + \sum_{r=n}^{N+2}\sum_{i=n_0+1}^{r-1}\phi_q(\sum_{s=0}^{i-1}h(s)),\\ \quad n \in [n_0+1, N+3],\\ y(0) + n\Delta y(n_0) + \sum_{r=0}^{n-1}\sum_{i=r}^{n_0-1}\phi_q(\sum_{s=0}^{i-1}h(s)), \quad n \in [0, n_0+1],\end{cases}$$

with

$$\begin{aligned}y(n_0+1) &= y(0) + (n_0+1)\Delta y(n_0) + \sum_{r=0}^{n_0}\sum_{i=r}^{n_0-1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right)\\ &= y(N+3) - (N+2-n_0)\Delta y(n_0+1)\sum_{r=n_0+1}^{N+2}\sum_{i=n_0+1}^{r-1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right).\end{aligned}$$

It follows from $h(n)$ being positive, $\Delta y(n_0) > 0$, and $\Delta y(n_0+1) \leq 0$ that

$$y(n) > \begin{cases}y(N+3), & n \in [n_0+1, N+2],\\ y(0), & n \in [1, n_0+1].\end{cases}$$

So $y(n) \geq \min\{y(0), y(N+3)\}$ for all $n \in [0, N+3]$. From BCs in BVP (2.1), we get

$$ay(0) = b\Delta y(0) \geq 0,$$
$$cy(N+3) = -d\Delta y(N+2) \geq 0.$$

Then

$$\min\{y(0), y(N+3)\} \geq 0.$$

Hence $y(n) > 0$ for all $n \in [1, N+2]$. The proof is complete. □

Lemma 2.4 *If* y *is a solution of BVP* (2.1), *then*

$$y(n) \geq \sigma_n \max_{n\in[0,N+3]} y(n) \quad \textit{for all } n \in [0, N+3], \tag{2.4}$$

where $\sigma_n = \min\{\frac{n}{N+3}, \frac{N+3-n}{N+3}\}$.

Proof It follows from Lemma 2.2 and Lemma 2.3 that $y(n) \geq 0$ for $n \in [0, N+3]$. Suppose that $y(n_0) = \max\{y(n) : n \in [0, N+3]\}$. Since $\Delta y(0) > 0$ and $\Delta y(N+1) \leq 0$, we get $n_0 \in [1, N+1]$. For $n \in [1, n_0]$, it is easy to see that

$$\begin{aligned}\frac{y(n_0) - y(0)}{n_0} n + y(0) - y(n) &= \frac{n \sum_{s=0}^{n_0-1} \Delta y(s) - n_0 \sum_{s=0}^{n-1} \Delta y(s)}{n_0} \\ &= \frac{-(n_0 - n) \sum_{s=0}^{n-1} \Delta y(s) + n \sum_{s=n}^{n_0-1} \Delta y(s)}{n_0}.\end{aligned}$$

Since $\Delta^2 y(n) = -\phi_q(\sum_{s=0}^{n-1} h(s)) < 0$ for all $n \in [0, N+1]$, we get $\Delta y(s) \leq \Delta y(j)$ for all $s \geq j$. Then $-(n_0 - n) \sum_{s=0}^{n-1} \Delta y(s) + n \sum_{s=n}^{n_0-1} \Delta y(s) \leq 0$. It follows that $\frac{y(n_0)-y(0)}{n_0} n + y(0) - y(n) \leq 0$. Then

$$y(n) \geq \frac{n}{n_0} y(n_0) + \left(1 - \frac{n}{n_0}\right) y(0) \geq \frac{n}{N+3} \max_{n\in[0,N+3]} y(n) \quad \text{for all } n \in [1, n_0].$$

Similarly, if $n \in [n_0, N+2]$, we get

$$y(n) \geq \frac{N+3-n}{N+3} \max_{n\in[0,N+3]} y(n) \quad \text{for all } n \in [n_0, N+2].$$

Then

$$y(n) \geq \min\left\{\frac{n}{N+3}, \frac{N+3-n}{N+3}\right\} \max_{n\in[0,N+3]} y(n) \quad \text{for all } n \in [0, N+3].$$ □

Lemma 2.5 *If* y *is a solution of BVP* (2.1), *then*

$$y(n) = \frac{b + na}{a} A_h - \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right), \tag{2.5}$$

where A_h satisfies the equation

$$A_h = \frac{a}{ad + ac(N+3) + bc}\left(c\sum_{r=0}^{N+2}\sum_{i=0}^{r-1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right) + d\sum_{i=0}^{N+1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right)\right).$$

Proof The proof follows from Lemma 2.2 and is omitted. □

Lemma 2.6 *If y is a solution of BVP* (2.1), *then there exists an $n_0 \in [0, N]$ such that*

$$\begin{aligned}\max_{n\in[0,N+3]} y(n) &= y(n_0+1)\\ &\geq \max\left\{\sum_{r=0}^{n_0}\sum_{i=r}^{n_0-1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right), \sum_{r=n_0+1}^{N+2}\sum_{i=n_0+1}^{r-1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right)\right\}.\end{aligned}$$

Proof It follows from Lemma 2.2 that there is $n_0 \in [0, N+1]$ such that $\Delta y(n_0) > 0$ and $\Delta y(n_0+1) \leq 0$, $\Delta y(n) > 0$ for all $n \in [0, n_0]$ and $\Delta y(n) \leq 0$ for all $n \in [n_0+1, N+2]$. Then

$$\max_{n\in[0,N+3]} y(n) = y(n_0+1),$$

there exists $\xi \in (n_0, n_0+1]$ such that

$$\frac{\Delta y(n_0+1) - \Delta y(n_0)}{n_0+1-n_0} = \frac{0 - \Delta y(n_0)}{\xi - n_0}.$$

Then

$$\Delta y(n_0+1) = -\frac{n_0+1-\xi}{\xi - n_0}\Delta y(n_0). \tag{2.6}$$

It is easy to see from (2.1) that

$$0 < \Delta y(n_0) = A_h - \sum_{i=0}^{n_0-1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right), \tag{2.7}$$

$$0 \geq \Delta y(n_0+1) = A_h - \sum_{i=0}^{n_0}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right). \tag{2.8}$$

Here $A_h = \Delta y(0)$. So (2.6)-(2.8) imply that

$$A_h - \sum_{i=0}^{n_0}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right) = -\frac{n_0+1-\xi}{\xi-n_0}A_h + \frac{n_0+1-\xi}{\xi-n_0}\sum_{i=0}^{n_0-1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right).$$

Then

$$A_h = (n_0+1-\xi)\sum_{i=0}^{n_0-1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right) + (\xi - n_0)\sum_{i=0}^{n_0}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right).$$

We get

$$\sum_{i=0}^{n_0-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) \le A_h \le \sum_{i=0}^{n_0} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right). \tag{2.9}$$

Lemma 2.3 implies that $B_h = y(0) \ge 0$. Furthermore, one has from (2.6)

$$\begin{aligned} y(n_0+1) &= B_h + (n_0+1)A_h - \sum_{r=0}^{n_0}\sum_{i=0}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) \\ &\ge \sum_{r=0}^{n_0}\sum_{i=0}^{n_0-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) - \sum_{r=0}^{n_0}\sum_{i=0}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) \\ &= \sum_{r=0}^{n_0}\sum_{i=r}^{n_0-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right). \end{aligned}$$

On the other hand, by a discussion similar to Lemma 2.2 and Lemma 2.3, we have $\overline{A}_h = \Delta y(N+2)$, $\overline{B}_h = y(N+3)$ with

$$\Delta y(n) = \overline{A}_h + \sum_{i=n}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right)$$

and

$$y(n) = \overline{B}_h - (N+3-n)\overline{A}_h - \sum_{r=n}^{N+2}\sum_{i=r}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right).$$

It follows that

$$\begin{aligned} \Delta y(n_0) &= \overline{A}_h + \sum_{i=n_0}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) > 0, \\ \Delta y(n_0+1) &= \overline{A}_h + \sum_{i=n_0+1}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) \le 0. \end{aligned}$$

So

$$\overline{A}_h + \sum_{i=n_0+1}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) = -\frac{n_0+1-\xi}{\xi-n_0}\overline{A}_h - \frac{n_0+1-\xi}{\xi-n_0}\sum_{i=n_0}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right).$$

Then

$$\overline{A}_h = -(\xi-n_0)\sum_{i=n_0+1}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) - (n_0+1-\xi)\sum_{i=n_0}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right).$$

We get

$$-\sum_{i=n_0}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) \ge \overline{A}_h \ge -\sum_{i=n_0+1}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right).$$

One has from Lemma 2.3 $\overline{B}_h = y(N+3) \geq 0$. Therefore

$$\begin{aligned} y(n_0+1) &= \overline{B}_h - (N+2-n_0)\overline{A}_h - \sum_{r=n_0+1}^{N+2} \sum_{i=n_0+1}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) \\ &\geq (N+2-n_0) \sum_{i=n_0+1}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) - \sum_{r=n_0+1}^{N+2} \sum_{i=r}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right) \\ &= \sum_{r=n_0+1}^{N+2} \sum_{i=n_0+1}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right). \end{aligned}$$

Hence

$$\begin{aligned} \max_{n\in[0,N+3]} y(n) &= y(n_0+1) \\ &\geq \max\left\{\sum_{r=0}^{n_0} \sum_{i=r}^{n_0-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right), \sum_{r=n_0+1}^{N+2} \sum_{i=n_0+1}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} h(s)\right)\right\}. \end{aligned}$$ □

Let $h(n) = q(n)f(n, y(n), \Delta y(n))$. Then A_y satisfies the following equation:

$$\begin{aligned} A_y = \frac{a}{ad + ac(N+3) + bc}\Bigg(& c\sum_{r=0}^{N+2}\sum_{i=0}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} q(s)f\big(s, y(s), \Delta y(s)\big)\right) \\ &+ d\sum_{i=0}^{N+1} \phi_q\left(\sum_{s=0}^{i-1} q(s)f\big(s, y(s), \Delta y(s)\big)\right)\Bigg). \end{aligned}$$

Let $\mathbb{B} = \mathbb{R}^{N+4}$. We call $x \leq y$ for $x, y \in \mathbb{B}$ if $x(n) \leq y(n)$ for all $n \in [0, N+3]$.

Define the norm

$$\|y\| = \max\left\{\max_{n\in[0,N+3]} y(n), \max_{n\in[0,N+2]} \left|\Delta y(n)\right|\right\}.$$

It is easy to see that $\mathbb{B}$ is a semi-ordered real Banach space.

Choose

$$\mathcal{P} = \left\{ y \in \mathbb{B} : \begin{array}{c} y(n) \geq \sigma_n \max_{n\in[0,N+3]} y(n) \text{ for all } n \in [0, N+3], \\ \Delta^2 y(n) \leq 0 \text{ for } n \in [0, N+1], \\ ay(0) - b\Delta y(0) = 0 \end{array} \right\}, \tag{2.10}$$

where $\sigma_n = \min\{\frac{n}{N+3}, \frac{N+3-n}{N+3}\}$. Then $\mathcal{P}$ is a cone in $\mathbb{B}$.

Define the operator $T : \mathcal{P} \to \mathbb{B}$ by

$$(Ty)(n) = \frac{b + na}{a} A_y - \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \phi_q\left(\sum_{s=0}^{i-1} q(s)f\big(s, y(s), \Delta y(s)\big)\right),$$

for $y \in \mathcal{P}$, $n \in [0, N+3]$. Then

$$\begin{aligned}(Ty)(n) = {} & \frac{b+na}{ad+ac(N+3)+bc}\left(c\sum_{r=0}^{N+2}\sum_{i=0}^{r-1}\phi_q\left(\sum_{s=0}^{i-1}q(s)f\big(s,y(s),\Delta y(s)\big)\right)\right. \\ & \left. + d\sum_{i=0}^{N+1}\phi_q\left(\sum_{s=0}^{i-1}q(s)f\big(s,y(s),\Delta y(s)\big)\right)\right) \\ & - \sum_{r=0}^{n-1}\sum_{i=0}^{r-1}\phi_q\left(\sum_{s=0}^{i-1}q(s)f\big(s,y(s),\Delta y(s)\big)\right).\end{aligned}$$

Lemma 2.7 *Suppose that* (C1) *holds. Then*

(i) *Ty satisfies the following*:

$$\begin{cases}\Delta[\phi_p(\Delta^2(Ty)(n))] + q(n)f(n,y(n),\Delta y(n)) = 0, & 0 < n < N, \\ a(Ty)(0) - b\Delta(Ty)(0) = 0, & c(Ty)(N+3) + d\Delta(Ty)(N+2) = 0, \\ \Delta^2(Ty)(0) = 0.\end{cases} \tag{2.11}$$

(ii) *$Ty \in \mathcal{P}$ for each $y \in \mathcal{P}$.*

(iii) *y is a solution of BVP* (1.1) *if and only if y is a solution of the operator equation $y = Ty$.*

(iv) *$T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof

(i) By the definition of Ty, we get (2.11).

(ii) Note the definition of $\mathcal{P}$. Since (C1) holds, for $y \in \mathcal{P}$, (2.11), Lemma 2.2, Lemma 2.3 and Lemma 2.4 imply that $\Delta(Ty)(n)$ is decreasing on $[0, N+2]$ and $(Ty)(n) \geq \sigma_n \max_{n\in[0,N+3]}(Ty)(n)$ for all $n \in [0, N+3]$. Together with (2.11), it follows that $Ty \in \mathcal{P}$.

(iii) It is easy to see from (2.11) that y is a solution of BVP (1.1) if and only if y is a solution of the operator equation $y = Ty$.

(iv) It suffices to prove that T is continuous on $\mathcal{P}$ and T is relative compact.

We divide the proof into three steps:

Step 1. For each bounded subset $D \subset \mathcal{P}$, prove that A_y is bounded in $\mathbb{R}$ for $y \in \overline{D}$

Denote

$$L_1 = \max\left\{\max_{n\in[0,N+3]} y(n), \max_{n\in[0,N+1]}\left|\Delta y(n)\right| : x \in \overline{D}\right\}$$

and

$$L_2 = \max_{j\in[0,N]} f_{L_1}(j) = \max_{j\in[0,N]}\max_{u,|x|\leq L_1} q(j)f(j,u,x).$$

It follows from (2.9) in the proof of Lemma 2.6 that

$$0 \leq A_y \leq \sum_{i=0}^{N}\phi_q\left(\sum_{s=0}^{i-1}q(s)f\big(s,y(s),\Delta y(s)\big)\right) \leq (N+1)\phi_q(NL_2).$$

Hence A_y is bounded in $\mathbb{R}$.

Step 2. For each bounded subset $D \subset \mathcal{P}$, and each $y_0 \in D$, it is easy to prove that T is continuous at y_0.

Step 3. For each bounded subset $D \subset \mathcal{P}$, prove that T is relative compact on D.

In fact, for each bounded subset $\Omega \subseteq D$ and $y \in \Omega$. Suppose

$$\|y\| = \max\Big\{\max_{n\in[0,N+3]} y(n), \max_{n\in[0,N+2]} \big|\Delta y(n)\big|\Big\} < M_1,$$

and Step 1 implies that there exists a constant $M_2 > 0$ such that $A_y < M_2$. Then

$$\begin{aligned}
(Ty)(n) &= \frac{b+na}{a}A_y - \sum_{r=0}^{n-1}\sum_{i=0}^{r-1}\phi_q\left(\sum_{s=0}^{i-1} q(s)f\big(s,y(s),\Delta y(s)\big)\right) \\
&\le \frac{b+na}{a}M_2 + \sum_{r=0}^{N+2}\sum_{i=0}^{N+1}\phi_q\left(\sum_{s=0}^{N} q(s)f\big(s,y(s),\Delta y(s)\big)\right) \\
&\le \frac{b+(N+3)a}{a}M_2 + (N+3)(N+2)\phi_q\left(\sum_{s=0}^{N} f_{M_1}(s)\right) \\
&:= M_3,
\end{aligned}$$

where $f_{M_1}(s) = \max_{u\le M_1,|x|\le M_1} q(j)f(s,u,x)$. Similarly, one has

$$\begin{aligned}
\big|\Delta(Ty)(n)\big| &= \left|A_y - \sum_{i=0}^{n-1}\phi_q\left(\sum_{s=0}^{i-1} q(s)f\big(s,y(s),\Delta y(s)\big)\right)\right| \\
&\le M_2 + \sum_{i=0}^{N+1}\phi_q\left(\sum_{s=0}^{N} f_{M_1}(s)\right) \\
&= M_2 + (N+2)\phi_q\left(\sum_{s=0}^{N} f_{M_1}(s)\right) \\
&:= M_4.
\end{aligned}$$

It follows that $T\Omega$ is bounded. Since $\mathbb{B} = \mathbb{R}^{N+4}$, one knows that $T\Omega$ is relative compact. Steps 1, 2, and 3 imply that T is completely continuous. □

3 Main result

In this section, our objective is to establish the existence of at least three positive solutions for BVP (1.1) by using the Avery-Peterson fixed point theorem [3].

Choose $[\frac{N+3}{2}] > k > 0$, where $[x]$ denotes the largest integer not greater than x, and denote $\sigma_k = \min\{\frac{k}{N+3}, \frac{N+3-k}{N+3}\}$.

Define the functionals on $\mathcal{P} : \mathcal{P} \to [0,+\infty)$ by

$$\gamma(y) = \max_{n\in[0,N+2]} \big|\Delta y(n)\big|,$$

$$\theta(y) = \psi(y) = \max_{n\in[0,N+3]} y(n),$$

$$\alpha(y) = \min_{n\in[k,N+3-k]} y(n).$$

For $y \in \mathcal{P}$ and $n \in [0, N+3]$ we have

$$\begin{aligned} y(n) &= y(n) - y(0) + y(0) \\ &\le \left| \sum_{i=0}^{n-1} \Delta y(i) \right| + y(0) \\ &= \left| \sum_{i=0}^{n-1} \Delta y(i) \right| + \frac{b}{a} \Delta y(0) \\ &\le \left| \sum_{i=0}^{N+2} \Delta y(i) \right| + \frac{b}{a} \Delta y(0) \\ &\le \left(N + 3 + \frac{b}{a} \right) \max_{n \in [0, N+2]} \left| \Delta y(n) \right|, \end{aligned}$$

i.e.,

$$\max_{n \in [0, N+3]} y(n) \le \left(N + 3 + \frac{b}{a} \right) \max_{n \in [0, N+2]} \left| \Delta y(n) \right|. \tag{3.1}$$

So,

$$\begin{aligned} \|y\| &= \max \left\{ \max_{n \in [0, N+3]} y(n), \max_{n \in [0, N+2]} \left| \Delta y(n) \right| \right\} \\ &\le \left(N + 3 + \frac{b}{a} \right) \max_{n \in [0, N+2]} \left| \Delta y(n) \right|. \end{aligned}$$

Hence, we obtain

$$\|y\| \le \left(N + 3 + \frac{b}{a} \right) \gamma(y), \quad y \in \mathcal{P}. \tag{3.2}$$

Let

$$\begin{aligned} \Omega &= 2 \sum_{i=0}^{N+2} \phi_q \left(\sum_{s=0}^{i-1} q(s) \right), \\ \Lambda &= \sigma_k \min \left\{ \sum_{r=k}^{[\frac{N+3}{2}]} \sum_{i=r}^{[\frac{N+3}{2}]-1} \phi_q \left(\sum_{s=k}^{i-1} q(s) \right), \sum_{r=[\frac{N+3}{2}]}^{N+3-k} \sum_{i=[\frac{N+3}{2}]}^{r-1} \phi_q \left(\sum_{s=k}^{i-1} q(s) \right) \right\}. \end{aligned}$$

Theorem 3.1 *Suppose that* (C1) *holds. If there are positive numbers*

$$t < \nu < \frac{\nu}{\sigma_k} < z \quad \text{with } \Omega \nu < \Lambda z,$$

such that the following conditions are satisfied:

(C2) $f(n, y(n), \Delta y(n)) \le \phi_p(\frac{z}{\Omega})$ *for all* $(n, y(n), \Delta y(n)) \in [0, N+3] \times [0, (N+3+\frac{b}{a})z] \times [-z, z]$;

(C3) $f(n, y(n), \Delta y(n)) > \phi_p(\frac{\nu}{\Lambda})$ *for all* $(n, y(n), \Delta y(n)) \in [k, N+3-k] \times [\nu, \frac{\nu}{\sigma_k}] \times [-z, z]$;

(C4) $f(n, y(n), \Delta y(n)) < \frac{a}{(N+3)a+b} \phi_p(\frac{t}{\Omega})$ *for all* $(n, y(n), \Delta y(n)) \in [0, N+3] \times [0, t] \times [-z, z]$,

then BVP (1.1) *has at least three positive solutions.*

Proof We choose positive numbers t, ν, $w = \frac{\nu}{\sigma_k}$, z with $t < \nu < \frac{\nu}{\sigma_k} < z$, $\phi_p(\frac{\nu}{\Lambda}) \le \min\{\phi_p(\frac{z}{\Omega}), \frac{a}{(N+3)a+b}\phi_p(\frac{t}{\Omega})\}$. Next we show that all the conditions of Lemma 2.1 are satisfied.

It is clear that for $y \in P$ and $\lambda \in [0,1]$, there are $\alpha(y) \le \psi(y)$, $\psi(\lambda y) = \lambda\psi(y)$. From (3.2), we have $\|y\| \le (N+3+\frac{b}{a})\gamma(y)$. Furthermore, $\psi(0) = 0 < t$ and therefore $0 \notin R(\psi, t; \gamma, z)$.

Now the proof is divided into four steps.

Step 1. We will show that (C2) implies that

$$T : \overline{\mathcal{P}(\gamma, z)} \to \overline{\mathcal{P}(\gamma, z)}.$$

For $y \in \overline{\mathcal{P}(\gamma, z)}$, we have $\gamma(y) = \max_{n\in[0,N+2]} |\Delta y(n)| \le z$. From (3.2) we get

$$\big(n, y(n), \Delta y(n)\big) \in [0, N+3] \times \left[0, \left(N+3+\frac{b}{a}\right)z\right] \times [-z, z].$$

This implies that (C2) holds. Then one has from (C2) and (2.9) in the proof of Lemma 2.6

$$\begin{aligned}
\gamma(Ty) &= \max_{n\in[0,N+2]} \big|\Delta(Ty)(n)\big| \\
&= \max_{n\in[0,N+2]} \left| A_y - \sum_{i=0}^{n-1} \phi_q\left(\sum_{s=0}^{i-1} q(s) f\big(s, y(s), \Delta y(s)\big)\right)\right| \\
&\le \max_{n\in[0,N+2]} \left| \sum_{i=0}^{N+2} \phi_q\left(\sum_{s=0}^{i-1} q(s) f\big(s, y(s), \Delta y(s)\big)\right)\right. \\
&\quad \left. + \sum_{i=0}^{n-1} \phi_q\left(\sum_{s=0}^{i-1} q(s) f\big(s, y(s), \Delta y(s)\big)\right)\right| \\
&\le 2\sum_{i=0}^{N+2} \phi_q\left(\sum_{s=0}^{i-1} q(s) f\big(s, y(s), \Delta y(s)\big)\right) \\
&\le \frac{2z}{\Omega}\sum_{i=0}^{N+2} \phi_q\left(\sum_{s=0}^{i-1} q(s)\right) \\
&= z.
\end{aligned}$$

Therefore, $T : \overline{\mathcal{P}(\gamma, z)} \to \overline{\mathcal{P}(\gamma, z)}$. Hence, by Lemma 2.7, we know that $T : \overline{\mathcal{P}(\gamma, z)} \to \overline{\mathcal{P}(\gamma, z)}$ is completely continuous.

Step 2. We show that condition (i) in Lemma 2.1 holds.

Choose $y(n) = \frac{\nu}{\sigma_k} = w$ for all $n \in [0, N+3]$. It is easy to see that

$$\begin{aligned}
&\alpha(y) = \min_{n\in[k,N+3-k]} y(n) = \frac{\nu}{\sigma_k} > \nu, \qquad \theta(y) = \max_{n\in[0,N+3]} y(n) = \frac{\nu}{\sigma_k} = w, \\
&\gamma(y) = \max_{n\in[0,N+2]} \big|\Delta y(n)\big| = 0 < z,
\end{aligned}$$

since $\sigma_k = \min\{\frac{k}{N+3}, \frac{N+3-k}{N+3}\} < \frac{1}{2}$. Hence $\{y \in \mathcal{P}(\alpha, \nu; \theta, w; \gamma, z) | \alpha(y) > \nu\} \ne \emptyset$.

For $y \in \mathcal{P}(\alpha, \nu; \theta, w; \gamma, z)$, we have $\nu \le y(n) \le \frac{\nu}{\sigma_k}$ and $-z \le \Delta y(n) \le z$ for $n \in [k, N+3-k]$.

It follows from (C3) that

$$f\big(n, y(n), \Delta y(n)\big) > \phi_p\left(\frac{\nu}{\Lambda}\right), \quad \big(n, y(n), \Delta y(n)\big) \in [k, N+3-k] \times \left[\nu, \frac{\nu}{\sigma_k}\right] \times [-z, z].$$

Similarly to Lemma 2.6 there exists $n_0 \in [k, N+3-k]$ such that $\Delta y(n_0) > 0$ and $\Delta y(n_0+1) \leq 0$ and

$$\max_{n\in[0,N+3]}(Ty)(n) = (Ty)(n_0+1) \geq \max\bigl\{(Ty)(n_0), (Ty)(n_0+2)\bigr\},$$

we get from (2.4), (C1), (C3), and Lemma 2.6

$$\begin{aligned}
\alpha(Ty) &= \min_{n\in[k,N+3-k]}(Ty)(n) \\
&\geq \sigma_k \max_{n\in[0,N+3]}(Ty)(n) = \sigma_k (Ty)(n_0) \\
&\geq \sigma_k \max\left\{\sum_{r=0}^{n_0}\sum_{i=r}^{n_0-1}\phi_q\left(\sum_{s=0}^{i-1} q(s)f\bigl(s, y(s), \Delta y(s)\bigr)\right),\right. \\
&\quad \left.\sum_{r=n_0+1}^{N+2}\sum_{i=n_0+1}^{r-1}\phi_q\left(\sum_{s=0}^{i-1} q(s)f\bigl(s, y(s), \Delta y(s)\bigr)\right)\right\} \\
&\geq \sigma_k \max\left\{\sum_{r=0}^{[\frac{N+3}{2}]}\sum_{i=r}^{[\frac{N+3}{2}]-1}\phi_q\left(\sum_{s=0}^{i-1} q(s)f\bigl(s, y(s), \Delta y(s)\bigr)\right),\right. \\
&\quad \left.\sum_{r=[\frac{N+3}{2}]}^{N+2}\sum_{i=[\frac{N+3}{2}]}^{r-1}\phi_q\left(\sum_{s=0}^{i-1} q(s)f\bigl(s, y(s), \Delta y(s)\bigr)\right)\right\} \\
&\geq \sigma_k \min\left\{\sum_{r=k}^{[\frac{N+3}{2}]}\sum_{i=r}^{[\frac{N+3}{2}]-1}\phi_q\left(\sum_{s=k}^{i-1} q(s)f\bigl(s, y(s), \Delta y(s)\bigr)\right),\right. \\
&\quad \left.\sum_{r=[\frac{N+3}{2}]}^{N+2}\sum_{i=[\frac{N+3}{2}]}^{r-1}\phi_q\left(\sum_{s=k}^{i-1} q(s)f\bigl(s, y(s), \Delta y(s)\bigr)\right)\right\} \\
&\geq \frac{\nu\sigma_k}{\Lambda}\min\left\{\sum_{r=k}^{[\frac{N+3}{2}]}\sum_{i=r}^{[\frac{N+3}{2}]-1}\phi_q\left(\sum_{s=k}^{i-1} q(s)\right), \sum_{r=[\frac{N+3}{2}]}^{N+3-k}\sum_{i=[\frac{N+3}{2}]}^{r-1}\phi_q\left(\sum_{s=k}^{i-1} q(s)\right)\right\} \\
&= \nu.
\end{aligned}$$

We conclude that condition (i) of Lemma 2.1 holds.

Step 3. We prove that condition (ii) of Lemma 2.1 holds. If $y \in \mathcal{P}(\alpha, \nu; \gamma, z)$ and $\theta(Ty) > w = \frac{\nu}{\sigma_k}$, then we have

$$\alpha(Ty) = \min_{n\in[k,N+3-k]}(Ty)(n) \geq \sigma_k \max_{n\in[0,N+3]}(Ty)(n) = \sigma_k\theta(Ty) > \nu.$$

Then condition (ii) of Lemma 2.1 is satisfied.

Step 4. Finally, we verify that (iii) of Lemma 2.1 also holds. Clearly, $0 \notin R(\psi, t; \gamma, z)$. Suppose that $y \in R(\psi, t; \gamma, z)$ with $\psi(y) = t$. Then by condition (C4) and (3.1), we obtain

$$\begin{aligned}
\psi(Ty) &= \max_{n\in[0,N+3]}(Ty)(n) \leq \left(N+3+\frac{b}{a}\right)\max_{n\in[0,N+2]}\bigl|\Delta(Ty)(n)\bigr| \\
&= \left(N+3+\frac{b}{a}\right)\max_{n\in[0,N+2]}\left|A_y - \sum_{i=0}^{n-1}\phi_q\left(\sum_{s=0}^{i-1} q(s)f\bigl(s, y(s), \Delta y(s)\bigr)\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(N+3+\frac{b}{a}\right) 2 \sum_{i=0}^{N+2} \phi_q\left(\sum_{s=0}^{i-1} q(s) f\bigl(s, y(s), \Delta y(s)\bigr)\right) \\
&< \left(N+3+\frac{b}{a}\right)\left(\frac{a}{(N+3)a+b}\right) \frac{2t}{\Omega} \sum_{i=0}^{N+2} \phi_q\left(\sum_{s=0}^{i-1} q(s)\right) \\
&= t.
\end{aligned}
$$

Thus, condition (iii) of Lemma 2.1 is satisfied.

From Steps 1-4 together with Lemma 2.1 we find that the operator T has at least three fixed points which are positive solutions y_1, y_2, and y_3 belonging to $\overline{\mathcal{P}(\gamma,z)}$ of (1.1) such that

$$
\gamma(y_i) \leq z, \quad i = 1,2,3, \qquad \nu < \alpha(y_1), \qquad t < \psi(y_2) \quad \text{with } \alpha(y_2) < \nu, \psi(y_3) < t. \qquad \square
$$

4 An example

Example 4.1 Consider the following BVP:

$$
\begin{cases} \Delta^3 y(n) + f(n, y(n), \Delta y(n)) = 0, \quad n \in [0, 99], \\ 2y(0) - \Delta y(0) = 0, \qquad 2y(102) + \Delta y(101) = 0, \qquad \Delta^2 y(0) = 0, \end{cases} \tag{4.1}
$$

where $f(n,y(n),\Delta y(n))$ is continuous and positive for all $(n,y(n),\Delta y(n)) \in [0,N] \times [0,+\infty) \times \mathbb{R}$. Corresponding to BVP (1.1), we have $N = 99$, $p = 2$, $q(n) = 1$, $n \in [0,N]$, $a = c = 2$, $b = d = 1$, $\phi_2(y) = y$.

It is easy to see that (C1) holds.

Choose the constant $k = 49$, then $\sigma_{49} = \min\{\frac{49}{102}, \frac{53}{102}\} = \frac{49}{102}$, $\Omega = 10{,}302$, $\Lambda = \frac{49}{51}$. Taking $t = 10$, $\nu = 50$, and $z = 600{,}000$, it is easy to check that

$$
10 = t < \nu = 50 < \frac{\nu}{\sigma_{49}} = \frac{5{,}100}{49} < 600{,}000, \qquad \Omega\nu = 515{,}100 < \Lambda z = 576{,}470.5882.
$$

If

$f(n,y(n),\Delta y(n)) \leq \frac{300{,}000}{5{,}151}$, for all $(n,y(n),\Delta y(n)) \in [0,102] \times [0,62{,}400{,}000] \times [-600{,}000, 600{,}000]$;
$f(n,y(n),\Delta y(n)) > \frac{2{,}550}{49}$ for all $(n,y(n),\Delta y(n)) \in [49,53] \times [50, \frac{5{,}100}{49}] \times [-600{,}000, 600{,}000]$;
$f(n,y(n),\Delta y(n)) < \frac{5}{535{,}704}$, for all $(n,y(n),\Delta y(n)) \in [0,102] \times [0,10] \times [-600{,}000, 600{,}000]$,

then Theorem 3.1 implies that BVP (4.1) has at least three positive solutions such that

$$
\begin{gathered}
\max_{n\in[0,101]} \bigl|\Delta y_i(n)\bigr| \leq 600{,}000, \quad i = 1,2,3, \qquad 50 < \min_{n\in[49,53]} y_1(n), \\
10 < \max_{n\in[0,102]} y_2(n) \quad \text{with } \min_{n\in[49,53]} y_2(n) < 50 \text{ and } \max_{n\in[0,102]} y_3(n) < 10.
\end{gathered}
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Acknowledgements
The authors would like to thank the referees for their valuable suggestions and comments.

Received: 17 July 2014 Accepted: 25 September 2014 Published: 13 Oct 2014

References

1. Anderson, DR: Discrete third-order three-point right-focal boundary value problems. Comput. Math. Appl. **45**, 861-871 (2003)
2. Anderson, D, Avery, RI: Multiple positive solutions to a third-order discrete focal boundary value problem. Comput. Math. Appl. **42**, 333-340 (2001)
3. Avery, RI, Peterson, AC: Three positive fixed points of nonlinear operators on ordered Banach spaces. Comput. Math. Appl. **42**, 313-322 (2001)
4. Aykut, N: Existence of positive solutions for boundary value problems of second-order functional difference equations. Comput. Math. Appl. **48**, 517-527 (2004)
5. Cai, X, Yu, J: Existence theorems for second-order discrete boundary value problems. J. Math. Anal. Appl. **320**, 649-661 (2006)
6. Cheung, W-S, Ren, J, Wong, PJY, Zhao, D: Multiple positive solutions for discrete nonlocal boundary value problems. J. Math. Anal. Appl. **330**, 900-915 (2007)
7. Graef, JR, Henderson, J: Double solutions of boundary value problems for $2m$th-order differential equations and difference equations. Comput. Math. Appl. **45**, 873-885 (2003)
8. He, Z: On the existence of positive solutions of p-Laplacian difference equations. J. Comput. Appl. Math. **161**, 193-201 (2003)
9. Yaslan Karaca, I: Discrete third-order three-point boundary value problem. J. Comput. Appl. Math. **205**, 458-468 (2007)
10. Liu, Y: Existence results for positive solutions of non-homogeneous BVPs for second order difference equations with one-dimensional p-Laplacian. J. Korean Math. Soc. **47**, 135-163 (2010)
11. Xia, J, Debnath, L, Jiang, H, Liu, Y: Three positive solutions of Sturm-Liouville type multi-point BVPs for second order p-Laplacian difference equations. Bull. Pure Appl. Math. **4**, 266-287 (2010)
12. Liu, Y: Studies on nonhomogeneous multi-point BVPs of difference equations with one-dimensional p-Laplacian. Mediterr. J. Math. **8**, 577-602 (2011)
13. Xia, J, Liu, Y: Positive solutions of BVPs for infinite difference equations with one-dimensional p-Laplacian. Miskolc Math. Notes **13**, 149-176 (2012)
14. Bian, L-H, He, X-P, Sun, H-R: Multiple positive solutions of m-point BVPs for third-order p-Laplacian dynamic equations on time scales. Adv. Differ. Equ. **2009**, Article ID 262857 (2009)
15. Xu, F, Meng, Z: The existence of positive solutions for third-order p-Laplacian m-point boundary value problems with sign changing nonlinearity on time scales. Adv. Differ. Equ. **2009**, Article ID 169321 (2009)
16. Han, W, Liu, M: Existence and uniqueness of a nontrivial solution for a class of third-order nonlinear p-Laplacian m-point eigenvalue problems on time scales. Nonlinear Anal. **70**(5), 1877-1889 (2009)
17. Wu, LM, Yang, J: Positive solutions to third-order nonlinear p-Laplacian functional dynamic equations on time scales. J. Lanzhou Univ. Nat. Sci. **46**(6), 100-104 (2010)
18. Song, C, Gao, X: Positive solutions for third-order p-Laplacian functional dynamic equations on time scales. Bound. Value Probl. **2011**, Article ID 585378 (2011)
19. Bai, Z, Ge, W: Existence of three positive solutions for some second-order boundary value problems. Comput. Math. Appl. **48**, 699-707 (2004)
20. Avery, RI: A generalization of the Leggett-Williams fixed point theorem. Math. Sci. Res. Hot-Line **3**, 9-14 (1999)

10.1186/1687-1847-2014-263
Cite this article as: Fen and Karaca: **Existence of positive solutions for boundary value problems of p-Laplacian difference equations.** *Advances in Difference Equations* 2014, 2014:263