Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Applications of quantum calculus on finite intervals to impulsive difference inclusions

Sotiris K Ntouyas[1] and Jessada Tariboon[2,3*]

*Correspondence: jessadat@kmutnb.ac.th
[2]Department of Mathematics, Faculty of Applied Science, Nonlinear Dynamic Analysis Research Center, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand
[3]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok, 10400, Thailand
Full list of author information is available at the end of the article

Abstract

Recently Tariboon and Ntouyas (Adv. Differ. Equ. 2013:282, 2013) introduced the notions of q_k-derivative and q_k-integral of a function on finite intervals. As applications existence and uniqueness results for initial value problems for first- and second-order impulsive q_k-difference equations was proved. In this paper, continuing the study of Tariboon and Ntouyas (Adv. Differ. Equ. 2013:282, 2013), we apply the quantum calculus to initial value problems for impulsive first- and second-order q_k-difference inclusions. We establish new existence results, when the right hand side is convex valued, by using the nonlinear alternative of Leray-Schauder type. Some illustrative examples are also presented.

MSC: 34A60; 26A33; 39A13; 34A37

Keywords: q_k-derivative; q_k-integral; impulsive q-difference inclusion

1 Introduction and preliminaries

In [1] the notions of q_k-derivative and q_k-integral of a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$, have been introduced and their basic properties was proved. As applications, existence and uniqueness results for initial value problems for first- and second-order impulsive q_k-difference equations was proved.

We recall the notions of q_k-derivative and q_k-integral on finite intervals. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows.

Definition 1.1 Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression

$$D_{q_k}f(t) = \frac{f(t) - f(q_k t + (1 - q_k)t_k)}{(1 - q_k)(t - t_k)}, \quad t \neq t_k, \qquad D_{q_k}f(t_k) = \lim_{t \to t_k} D_{q_k}f(t), \tag{1.1}$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k provided $D_{q_k}f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (1.1), then $D_{q_k}f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}. \tag{1.2}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 1.2 Let $f : J_k \to \mathbb{R}$ is a continuous function, we call the second-order q_k-derivative $D^2_{q_k} f$ provided $D_{q_k} f$ is q_k-differentiable on J_k with $D^2_{q_k} f = D_{q_k}(D_{q_k} f) : J_k \to \mathbb{R}$. Similarly, we define higher order q_k-derivative $D^n_{q_k} : J_k \to \mathbb{R}$.

The properties of q_k-derivative are discussed in [1].

Definition 1.3 Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by

$$\int_{t_k}^{t} f(s)\, d_{q_k} s = (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1 - q_k^n\bigr) t_k\bigr) \tag{1.3}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite q_k-integral is defined by

$$\begin{aligned}\int_{a}^{t} f(s)\, d_{q_k} s = \int_{t_k}^{t} f(s)\, d_{q_k} s - \int_{t_k}^{a} f(s)\, d_{q_k} s &= (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1 - q_k^n\bigr) t_k\bigr) \\ &\quad - (1 - q_k)(a - t_k) \sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n a + \bigl(1 - q_k^n\bigr) t_k\bigr).\end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (1.3) reduces to q-integral of a function $f(t)$, defined by $\int_0^t f(s)\, d_q s = (1 - q)t \sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0, \infty)$.

The book by Kac and Cheung [2] covers many of the fundamental aspects of the quantum calculus. In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [3–15] and the references cited therein.

Impulsive differential equations, that is, differential equations involving the impulse effect, appear as a natural description of observed evolution phenomena of several real world problems. For some monographs on the impulsive differential equations we refer to [16–18].

Here, we remark that the classical q-calculus cannot be considered in problems with impulses as the definition of q-derivative fails to work when there are impulse points $t_k \in (qt, t)$ for some $k \in \mathbb{N}$. On the other hand, this situation does not arise for impulsive problems on a q-time scale as the points t and $qt = \rho(t)$ are consecutive points, where $\rho : \mathbb{T} \to \mathbb{T}$ is the backward jump operator; see [19]. In [1], quantum calculus on finite intervals, the points t and $q_k t + (1 - q_k)t_k$ are considered only in an interval $[t_k, t_{k+1}]$. Therefore, the problems with impulses at fixed times can be considered in the framework of q_k-calculus.

In this paper, continuing the study of [1], we apply q_k-calculus to establish existence results for initial value problems for impulsive first- and second-order q_k-difference inclusions. In Section 3, we consider the following initial value problem for the first-order q_k-difference inclusion:

$$\begin{aligned}&D_{q_k} x(t) \in F\bigl(t, x(t)\bigr), \quad t \in J := [0, T], t \neq t_k, \\ &\Delta x(t_k) = I_k\bigl(x(t_k)\bigr), \quad k = 1, 2, \ldots, m, \\ &x(0) = x_0,\end{aligned} \tag{1.4}$$

where $x_0 \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued function, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$, $I_k \in C(\mathbb{R},\mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $k = 1,2,\dots,m$ and $0 < q_k < 1$ for $k = 0,1,2,\dots,m$.

In Section 4, we study the existence of solutions for the following initial value problem for second-order impulsive q_k-difference inclusion:

$$\begin{aligned} &D_{q_k}^2 x(t) \in F\bigl(t,x(t)\bigr), \quad t \in J, t \neq t_k, \\ &\Delta x(t_k) = I_k\bigl(x(t_k)\bigr), \quad k = 1,2,\dots,m, \\ &D_{q_k} x\bigl(t_k^+\bigr) - D_{q_{k-1}} x(t_k) = I_k^*\bigl(x(t_k)\bigr), \quad k = 1,2,\dots,m, \\ &x(0) = \alpha, \qquad D_{q_0} x(0) = \beta, \end{aligned} \tag{1.5}$$

where $\alpha, \beta \in \mathbb{R}$ and $I_k, I_k^* \in C(\mathbb{R},\mathbb{R})$.

We establish new existence results, when the right hand side is convex valued by using the nonlinear alternative of Leray-Schauder type.

The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. In Section 3 we establish the existence result for first-order q_k-difference inclusions, while the existence result for second-order q_k-difference inclusions is presented in Section 4. Some illustrative examples are also presented.

2 Preliminaries

In this section we recall some basic concepts of multivalued analysis [20, 21].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$; is *bounded* on bounded sets if $G(\mathbb{B}) = \bigcup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (*i.e.* $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$); is called *upper semicontinuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$; is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$.

In the sequel, we denote by $\mathcal{C} = C([0,T],\mathbb{R})$ the space of all continuous functions from $[0,T] \to \mathbb{R}$ with norm $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. By $L^1([0,T],\mathbb{R})$ we denote the space of all functions f defined on $[0,T]$ such that $\|x\|_{L^1} = \int_0^T |x(t)|\,dt < \infty$.

For each $y \in \mathcal{C}$, define the set of selections of F by

$$S_{F,y} := \bigl\{v \in \mathcal{C} : v(t) \in F\bigl(t,y(t)\bigr) \text{ on } [0,T]\bigr\}.$$

Definition 2.1 A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory (in the sense of q_k-calculus) if $x \longmapsto F(t,x)$ is upper semicontinuous on J. Further a Carathéodory function F is called L^1-Carathéodory if there exists $\varphi_\alpha \in L^1(J,\mathbb{R}^+)$ such that $\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t)$ for all $\|x\| \leq \alpha$ on J for each $\alpha > 0$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps and a useful result regarding closed graphs.

Lemma 2.2 (Nonlinear alternative for Kakutani maps) [22] *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$*

is a upper semicontinuous compact map. Then either

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Lemma 2.3 ([23, 24]) *Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,\mathbb{R})$ to $C(J,\mathbb{R})$. Then the operator*

$$\Theta \circ S_F : C(J,\mathbb{R}) \to \mathcal{P}_{cp,c}\big(C(J,\mathbb{R})\big), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

Let $J = [0,T]$, $J_0 = [t_0,t_1]$, $J_k = (t_k,t_{k+1}]$ for $k = 1,2,\dots,m$. Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1,2,\dots,m\}$. $PC(J,\mathbb{R})$ is a Banach space with the norms $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

3 First-order impulsive q_k-difference inclusions

In this section, we study the existence of solutions for the first-order impulsive q_k-difference inclusion (1.4).

The following lemma was proved in [1].

Lemma 3.1 *If $y \in PC(J,\mathbb{R})$, then for any $t \in J_k$, $k = 0,1,2,\dots,m$, the solution of the problem*

$$\begin{aligned} &D_{q_k}x(t) = y(t), \quad t \in J, t \neq t_k, \\ &\Delta x(t_k) = I_k\big(x(t_k)\big), \quad k = 1,2,\dots,m, \\ &x(0) = x_0 \end{aligned} \tag{3.1}$$

is given by

$$x(t) = x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} y(s)\,d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\big(x(t_k)\big) + \int_{t_k}^{t} y(s)\,d_{q_k}s, \tag{3.2}$$

with $\sum_{0<0}(\cdot) = 0$.

Before studying the boundary value problem (1.4) let us begin by defining its solution.

Definition 3.2 A function $x \in PC(J,\mathbb{R})$ is said to be a solution of (1.4) if $x(0) = x_0$, $\Delta x(t_k) = I_k(x(t_k))$, $k = 1,2,\dots,m$, and there exists $f \in L^1(J,\mathbb{R})$ such that $f(t) \in F(t,x(t))$ on J and

$$x(t) = x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} f(s)\,d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\big(x(t_k)\big) + \int_{t_k}^{t} f(s)\,d_{q_k}s.$$

Theorem 3.3 *Assume that*:

(H_1) *$F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact and convex values*;

(H_2) *there exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C(J,\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \textit{for each } (t,x) \in J \times \mathbb{R};$$

(H_3) *there exist constants* c_k *such that* $|I_k(y)| \le c_k$, $k = 1,2,\dots,m$ *for each* $y \in \mathbb{R}$;

(H_4) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{|x_0| + T\psi(M)\|p\| + \sum_{k=1}^m c_k} > 1.$$

Then the initial value problem (1.4) *has at least one solution on* J.

Proof Define the operator $\mathcal{H} : PC(J,\mathbb{R}) \to \mathcal{P}(PC(J,\mathbb{R}))$ by

$$\mathcal{H}(x) = h \in PC(J,\mathbb{R}) : h(t) = x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\big(x(t_k)\big) + \int_{t_k}^{t} f(s)\, d_{q_k}s,$$

for $f \in S_{F,x}$.

We will show that $\mathcal{H}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\mathcal{H}$ *is convex for each* $x \in PC(J,\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that $\mathcal{H}$ *maps bounded sets (balls) into bounded sets in* $PC(J,\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C(J,\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $C(J,\mathbb{R})$. Then, for each $h \in \mathcal{H}(x)$, $x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$h(t) = x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\big(x(t_k)\big) + \int_{t_k}^{t} f(s)\, d_{q_k}s.$$

Then for $t \in J$ we have

$$\begin{aligned}
|h(t)| &\le |x_0| + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} |f(s)|\, d_{q_{k-1}}s + \sum_{0<t_k<t} \big|I_k\big(x(t_k)\big)\big| + \int_{t_k}^{t} |f(s)|\, d_{q_k}s \\
&\le |x_0| + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} p(s)\psi(\|x\|)\, d_{q_{k-1}}s + \sum_{k=1}^m c_k + \int_{t_k}^{t} p(s)\psi(\|x\|)\, d_{q_k}s \\
&\le |x_0| + \psi(\|x\|) \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} p(s)\, d_{q_{k-1}}s + \sum_{k=1}^m c_k + \psi(\|x\|) \int_{t_k}^{t} p(s)\, d_{q_k}s \\
&\le |x_0| + T\psi(\|x\|)\|p\| + \sum_{k=1}^m c_k.
\end{aligned}$$

Consequently,

$$\|h\| \le |x_0| + T\psi(\rho)\|p\| + \sum_{k=1}^m c_k.$$

Now we show that $\mathcal{H}$ *maps bounded sets into equicontinuous sets of* $PC(J,\mathbb{R})$. Let $\tau_1,\tau_2 \in J$, $\tau_1 < \tau_2$ with $\tau_1 \in J_\nu$, $\tau_2 \in J_u$, $\nu \le u$ for some $u,\nu \in \{0,1,2,\dots,m\}$ and $x \in B_\rho$. For each $h \in \mathcal{H}(x)$, we obtain

$$\begin{aligned}\bigl|h(\tau_2)-h(\tau_1)\bigr| &\le \left|\int_{t_u}^{\tau_2} f(s)\,d_{q_k}s - \int_{t_\nu}^{\tau_1} f(s)\,d_{q_k}s\right| + \left|\sum_{\tau_1<t_k<\tau_2} I_k\bigl(x(t_k)\bigr)\right| \\ &\quad + \left|\sum_{\tau_1<t_k<\tau_2}\int_{t_{k-1}}^{t_k} f(s)\,d_{q_{k-1}}s\right| \\ &\le \left|\int_{t_u}^{\tau_2} f(s)\,d_{q_k}s - \int_{t_\nu}^{\tau_1} f(s)\,d_{q_k}s\right| + \sum_{\tau_1<t_k<\tau_2}\bigl|I_k\bigl(x(t_k)\bigr)\bigr| \\ &\quad + \sum_{\tau_1<t_k<\tau_2}\int_{t_{k-1}}^{t_k}\bigl|f(s)\bigr|\,d_{q_{k-1}}s.\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $\tau_2-\tau_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{H}: PC(J,\mathbb{R}) \to \mathcal{P}(PC(J,\mathbb{R}))$ is completely continuous.

Since $\mathcal{H}$ is completely continuous, in order to prove that it is upper semicontinuous it is enough to prove that it has a closed graph. Thus, in our next step, we show that $\mathcal{H}$ *has a closed graph*. Let $x_n \to x_*$, $h_n \in \mathcal{H}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{H}(x_*)$. Associated with $h_n \in \mathcal{H}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$h_n(t) = x_0 + \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} f_n(s)\,d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\bigl(x_n(t_k)\bigr) + \int_{t_k}^{t} f_n(s)\,d_{q_k}s.$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that, for each $t \in J$,

$$h_*(t) = x_0 + \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} f_*(s)\,d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\bigl(x_*(t_k)\bigr) + \int_{t_k}^{t} f_*(s)\,d_{q_k}s.$$

Let us consider the linear operator $\Theta: L^1(J,\mathbb{R}) \to PC(J,\mathbb{R})$ given by

$$f \mapsto \Theta(f)(t) = x_0 + \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} f(s)\,d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\bigl(x(t_k)\bigr) + \int_{t_k}^{t} f(s)\,d_{q_k}s.$$

Observe that

$$\begin{aligned}\bigl\|h_n(t)-h_*(t)\bigr\| &= \left\|\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\bigl(f_n(u)-f_*(u)\bigr)\,d_{q_{k-1}}s + \sum_{0<t_k<t}\bigl|I_k\bigl(x_n(t_k)\bigr)-I_k\bigl(x_*(t_k)\bigr)\bigr|\right. \\ &\quad \left. + \int_{t_k}^{t}\bigl(f_n(u)-f_*(u)\bigr)\,d_{q_k}s\right\| \to 0,\end{aligned}$$

as $n \to \infty$.

Thus, it follows by Lemma 2.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} f_*(s)\, d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\big(x_*(t_k)\big) + \int_{t_k}^{t} f_*(s)\, d_{q_k}s,$$

for some $f_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $x \notin \mathcal{H}(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\mathcal{H}(x)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in J$, we have

$$x(t) = x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}}s + \sum_{0<t_k<t} I_k\big(x(t_k)\big) + \int_{t_k}^{t} f(s)\, d_{q_k}s.$$

Repeating the computations of the second step, we have

$$\big|x(t)\big| \le |x_0| + T\psi\big(\|x\|\big)\|p\| + \sum_{k=1}^{m} c_k.$$

Consequently, we have

$$\frac{\|x\|}{|x_0| + T\psi\big(\|x\|\big)\|p\| + \sum_{k=1}^{m} c_k} \le 1.$$

In view of (H_4), there exists M such that $\|x\| \neq M$. Let us set

$$U = \big\{x \in PC(J,\mathbb{R}) : \|x\| < M\big\}.$$

Note that the operator $\mathcal{H} : \overline{U} \to \mathcal{P}(PC(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\mathcal{H}(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.2), we deduce that $\mathcal{H}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.4). This completes the proof. □

Example 3.4 Let us consider the following first-order initial value problem for impulsive q_k-difference inclusions:

$$\begin{aligned} &D_{\frac{1}{2+k}} x(t) \in F\big(t, x(t)\big), \quad t \in J = [0,1], t \neq t_k = \frac{k}{10}, \\ &\Delta x(t_k) = \frac{|x(t_k)|}{12 + |x(t_k)|}, \quad k = 1,2,\ldots,9, \\ &x(0) = 0. \end{aligned} \tag{3.3}$$

Here $q_k = 1/(2+k)$, $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, and $I_k(x) = |x|/(12+|x|)$. We find that $|I_k(x) - I_k(y)| \le (1/12)|x-y|$ and $|I_k(x)| \le 1$.

(a) Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t,x) = \left[\frac{|x|}{|x| + \sin^2 x + 1} + t + 1, e^{-x^2} + \frac{4}{5}t^2 + 3\right]. \tag{3.4}$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{|x|}{|x| + \sin^2 x + 1} + t + 1, e^{-x^2} + t^2 + 3\right) \leq 5, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq 5 = p(t)\psi(\|x\|), \quad x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|) = 5$. Further, using the condition (H_4) we find that $M > 14$. Therefore, all the conditions of Theorem 3.3 are satisfied. So, problem (3.3) with $F(t,x)$ given by (3.4) has at least one solution on $[0,1]$.

(b) If $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{(t+1)x^2}{x^2+1}, \frac{t|x|(\cos^2 x + 1)}{2(|x|+1)}\right]. \tag{3.5}$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{(t+1)x^2}{x^2+1}, \frac{t|x|(\cos^2 x + 1)}{2(|x|+1)}\right) \leq t + 1, \quad x \in \mathbb{R}.$$

Here $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq (t+1) = p(t)\psi(\|x\|)$, $x \in \mathbb{R}$, with $p(t) = t+1$, $\psi(\|x\|) = 1$. It is easy to verify that $M > 10.5$. Then, by Theorem 3.3, the problem (3.3) with $F(t,x)$ given by (3.5) has at least one solution on $[0,1]$.

4 Second-order impulsive q_k-difference inclusions

In this section, we study the existence of solutions for the second-order impulsive q_k-difference inclusion (1.5).

We recall the following lemma from [1].

Lemma 4.1 *If $y \in C(J,\mathbb{R})$, then for any $t \in J$, the solution of the problem*

$$\begin{aligned}
&D_{q_k}^2 x(t) = y(t), \quad t \in J, t \neq t_k, \\
&\Delta x(t_k) = I_k\big(x(t_k)\big), \quad k = 1,2,\ldots,m, \\
&D_{q_k} x\big(t_k^+\big) - D_{q_{k-1}} x(t_k) = I_k^*\big(x(t_k)\big), \quad k = 1,2,\ldots,m, \\
&x(0) = \alpha, \qquad D_{q_0} x(0) = \beta,
\end{aligned} \tag{4.1}$$

is given by

$$\begin{aligned}
x(t) = {} & \alpha + \beta t \\
& + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} \big(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\big) y(s)\, d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\
& + t\left[\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} fy(s)\, d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{0<t_k<t} t_k \left(\int_{t_{k-1}}^{t_k} y(s)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) \right) \\
&+ \int_{t_k}^{t} \big(t - q_k s - (1-q_k)t_k\big) y(s)\, d_{q_k} s, \qquad (4.2)
\end{aligned}
$$

with $\sum_{0<0}(\cdot) = 0$.

Definition 4.2 A function $x \in PC(J,\mathbb{R})$ is said to be a solution of (1.5) if $x(0) = x_0$, $D_{q_0}x(0) = \beta$, $\Delta x(t_k) = I_k(x(t_k))$, $D_{q_k}x(t_k^+) - D_{q_{k-1}}x(t_k) = I_k^*(x(t_k))$, $k = 1,2,\ldots,m$ and there exists $f \in L^1(J,\mathbb{R})$ such that $f(t) \in F(t,x(t))$ on J and

$$
\begin{aligned}
x(t) = {} & \alpha + \beta t \\
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \big(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\big) f(s)\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \right) \\
&+ t \left[\sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) \right) \right] \\
&- \sum_{0<t_k<t} t_k \left(\int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) \right) \\
&+ \int_{t_k}^{t} \big(t - q_k s - (1-q_k)t_k\big) f(s)\, d_{q_k} s, \qquad (4.3)
\end{aligned}
$$

with $\sum_{0<0}(\cdot) = 0$.

Theorem 4.3 *Assume that* (H_1), (H_2) *hold. In addition we suppose that*:

(A_1) *there exist constants* c_k, c_k^* *such that* $|I_k(x)| \le c_k$, $|I_k^*(y)| \le c_k^*$, $k = 1,2,\ldots,m$ *for each* $x, y \in \mathbb{R}$;

(A_2) *there exists a constant* $M > 0$ *such that*

$$
\frac{M}{|\alpha| + |\beta|T + \|p\|\psi(M)\Lambda_1 + \sum_{k=1}^{m}[c_k + c_k^*(T + t_k)]} > 1,
$$

where

$$
\Lambda_1 = \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + \sum_{k=1}^{m} (T + t_k)(t_k - t_{k-1}). \qquad (4.4)
$$

Then the initial value problem (1.5) *has at least one solution on* J.

Proof Define the operator $\mathcal{H} : PC(J,\mathbb{R}) \to \mathcal{P}(PC(J,\mathbb{R}))$ by

$$
\begin{aligned}
\mathcal{H}(x) = h \in PC(J,\mathbb{R}) : h(t) = {} & \alpha + \beta t + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \big(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\big) f(s)\, d_{q_{k-1}} s \right. \\
&\left. + I_k\big(x(t_k)\big) \right) + t \left[\sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) \right) \right]
\end{aligned}
$$

$$- \sum_{0<t_k<t} t_k \biggl(\int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}} s + I_k^*\bigl(x(t_k)\bigr) \biggr)$$
$$+ \int_{t_k}^{t} \bigl(t - q_k s - (1-q_k)t_k\bigr) f(s)\, d_{q_k} s,$$

for $f \in S_{F,x}$.

We will show that $\mathcal{H}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\mathcal{H}$ *is convex for each* $x \in PC(J,\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that $\mathcal{H}$ *maps bounded sets (balls) into bounded sets in* $PC(J,\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in PC(J,\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $PC(J,\mathbb{R})$. Then, for each $h \in \mathcal{H}(x)$, $x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \alpha + \beta t \\ &\quad + \sum_{0<t_k<t} \biggl(\int_{t_{k-1}}^{t_k} \bigl(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\bigr) f(s)\, d_{q_{k-1}} s + I_k\bigl(x(t_k)\bigr) \biggr) \\ &\quad + t \biggl[\sum_{0<t_k<t} \biggl(\int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}} s + I_k^*\bigl(x(t_k)\bigr) \biggr) \biggr] \\ &\quad - \sum_{0<t_k<t} t_k \biggl(\int_{t_{k-1}}^{t_k} f(s)\, d_{q_{k-1}} s + I_k^*\bigl(x(t_k)\bigr) \biggr) \\ &\quad + \int_{t_k}^{t} \bigl(t - q_k s - (1-q_k)t_k\bigr) f(s)\, d_{q_k} s. \end{aligned}$$

Then for $t \in J$ we have

$$\begin{aligned} \bigl|h(t)\bigr| &\le |\alpha| + |\beta| t \\ &\quad + \sum_{0<t_k<t} \biggl(\int_{t_{k-1}}^{t_k} \bigl(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\bigr) \bigl|f(s)\bigr|\, d_{q_{k-1}} s + \bigl|I_k\bigl(x(t_k)\bigr)\bigr| \biggr) \\ &\quad + t \biggl[\sum_{0<t_k<t} \biggl(\int_{t_{k-1}}^{t_k} \bigl|f(s)\bigr|\, d_{q_{k-1}} s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| \biggr) \biggr] \\ &\quad + \sum_{0<t_k<t} t_k \biggl(\int_{t_{k-1}}^{t_k} \bigl|f(s)\bigr|\, d_{q_{k-1}} s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| \biggr) \\ &\quad + \int_{t_k}^{t} \bigl(t - q_k s - (1-q_k)t_k\bigr) \bigl|f(s)\bigr|\, d_{q_k} s \\ &\le |\alpha| + |\beta| T \\ &\quad + \sum_{0<t_k<T} \biggl(\int_{t_{k-1}}^{t_k} \bigl(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\bigr) p(s)\psi\bigl(\|x\|\bigr)\, d_{q_{k-1}} s + \bigl|I_k\bigl(x(t_k)\bigr)\bigr| \biggr) \\ &\quad + T \biggl[\sum_{0<t_k<T} \biggl(\int_{t_{k-1}}^{t_k} p(s)\psi\bigl(\|x\|\bigr)\, d_{q_{k-1}} s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| \biggr) \biggr] \\ &\quad + \sum_{0<t_k<T} t_k \biggl(\int_{t_{k-1}}^{t_k} p(s)\psi\bigl(\|x\|\bigr)\, d_{q_{k-1}} s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| \biggr) \end{aligned}$$

$$\begin{aligned}
&+\int_{t_m}^{T}\big(T-q_m s-(1-q_m)t_m\big)p(s)\psi\big(\|x\|\big)\,d_{q_m}s\\
&=|\alpha|+|\beta|T+\sum_{k=1}^{m}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\|p\|\psi\big(\|x\|\big)+c_k\right)\\
&\quad+T\left[\sum_{k=1}^{m}\big(\|p\|\psi\big(\|x\|\big)(t_k-t_{k-1})+c_k^*\big)\right]\\
&\quad+\sum_{k=1}^{m}t_k\big(\|p\|\psi\big(\|x\|\big)(t_k-t_{k-1})+c_k^*\big)+\frac{(T-t_m)^2}{1+q_m}\|p\|\psi\big(\|x\|\big)\\
&=|\alpha|+|\beta|T+\|p\|\psi\big(\|x\|\big)\left\{\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\sum_{k=1}^{m}(T+t_k)(t_k-t_{k-1})\right\}\\
&\quad+\sum_{k=1}^{m}\big[c_k+c_k^*(T+t_k)\big].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|h\|\le{}&|\alpha|+|\beta|T+\|p\|\psi(\rho)\left\{\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\sum_{k=1}^{m}(T+t_k)(t_k-t_{k-1})\right\}\\
&+\sum_{k=1}^{m}\big[c_k+c_k^*(T+t_k)\big].
\end{aligned}$$

Now we show that $\mathcal{H}$ *maps bounded sets into equicontinuous sets of* $PC(J,\mathbb{R})$. Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$ with $\tau_1\in J_u$, $\tau_2\in J_v$, $u\le v$ for some $u,v\in\{0,1,2,\dots,m\}$ and $x\in B_\rho$. For each $h\in\mathcal{H}(x)$, we obtain

$$\begin{aligned}
\big|h(\tau_2)-h(\tau_1)\big|\le{}&|\beta||\tau_2-\tau_1|\\
&+\sum_{\tau_1<t_k<\tau_2}\left(\int_{t_{k-1}}^{t_k}\big(t_k-q_{k-1}s-(1-q_{k-1})t_{k-1}\big)\big|f(s)\big|\,d_{q_{k-1}}s+\big|I_k\big(x(t_k)\big)\big|\right)\\
&+|\tau_2-\tau_1|\left[\sum_{0<t_k<\tau_1}\left(\int_{t_{k-1}}^{t_k}\big|f(s)\big|\,d_{q_{k-1}}s+\big|I_k^*\big(x(t_k)\big)\big|\right)\right]\\
&+\tau_2\left[\sum_{\tau_1<t_k<\tau_2}\left(\int_{t_{k-1}}^{t_k}\big|f(s)\big|\,d_{q_{k-1}}s+\big|I_k^*\big(x(t_k)\big)\big|\right)\right]\\
&+\sum_{\tau_1<t_k<\tau_2}t_k\left(\int_{t_{k-1}}^{t_k}\big|f(s)\big|\,d_{q_{k-1}}s+\big|I_k^*\big(x(t_k)\big)\big|\right)\\
&+\left|\int_{t_v}^{\tau_2}\big(\tau_2-q_k s-(1-q_k)t_k\big)\big|f(s)\big|\,d_{q_k}s\right.\\
&\left.-\int_{t_u}^{\tau_1}\big(\tau_1-q_k s-(1-q_k)t_k\big)\big|f(s)\big|\,d_{q_k}s\right|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $\tau_2-\tau_1\to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{H}:PC(J,\mathbb{R})\to\mathcal{P}(PC(J,\mathbb{R}))$ is completely continuous.

Since $\mathcal{H}$ is completely continuous, in order to prove that it is upper semicontinuous it is enough to prove that it has a closed graph. Thus, in our next step, we show that $\mathcal{H}$ *has a closed graph.* Let $x_n \to x_*$, $h_n \in \mathcal{H}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{H}(x_*)$. Associated with $h_n \in \mathcal{H}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$\begin{aligned} h_n(t) = \alpha + \beta t &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\big(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\big)f_n(s)\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\ &+ t\left[\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f_n(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)\right] \\ &- \sum_{0<t_k<t} t_k\left(\int_{t_{k-1}}^{t_k} f_n(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right) \\ &+ \int_{t_k}^{t}\big(t - q_k s - (1-q_k)t_k\big)f_n(s)\,d_{q_k}s. \end{aligned}$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) = \alpha + \beta t &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\big(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\big)f_*(s)\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\ &+ t\left[\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f_*(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)\right] \\ &- \sum_{0<t_k<t} t_k\left(\int_{t_{k-1}}^{t_k} f_*(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right) \\ &+ \int_{t_k}^{t}\big(t - q_k s - (1-q_k)t_k\big)f_*(s)\,d_{q_k}s. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1(J,\mathbb{R}) \to PC(J,\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) = \alpha + \beta t &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\big(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\big)f(s)\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\ &+ t\left[\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)\right] \\ &- \sum_{0<t_k<t} t_k\left(\int_{t_{k-1}}^{t_k} f(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right) \\ &+ \int_{t_k}^{t}\big(t - q_k s - (1-q_k)t_k\big)f(s)\,d_{q_k}s. \end{aligned}$$

Observe that

$$\begin{aligned} \big\|h_n(t) - h_*(t)\big\| = \Bigg\|&\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\big(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\big)\big(f_n(u) - f_*(u)\big)\,d_{q_{k-1}}s \\ &+ \sum_{0<t_k<t}\big|I_k\big(x_n(t_k)\big) - I_k\big(x_*(t_k)\big)\big| \end{aligned}$$

$$
\begin{aligned}
&+T\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\big(f_n(u)-f_*(u)\big)\,d_{q_{k-1}}s\\
&+T\sum_{0<t_k<t}\big|I_k^*\big(x_n(t_k)\big)-I_k^*\big(x_*(t_k)\big)\big|\\
&+\sum_{0<t_k<t}t_k\int_{t_{k-1}}^{t_k}\big(f_n(u)-f_*(u)\big)\,d_{q_{k-1}}s\\
&+\sum_{0<t_k<t}\big|I_k^*\big(x_n(t_k)\big)-I_k^*\big(x_*(t_k)\big)\big|\\
&+\int_{t_k}^{t}\big(t-q_ks-(1-q_k)t_k\big)\big(f_n(u)-f_*(u)\big)\,d_{q_k}s\Big\|\to 0,
\end{aligned}
$$

as $n\to\infty$.

Thus, it follows by Lemma 2.3 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$
\begin{aligned}
h_*(t)={}&\alpha+\beta t+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\big(t_k-q_{k-1}s-(1-q_{k-1})t_{k-1}\big)f_*(s)\,d_{q_{k-1}}s+I_k\big(x(t_k)\big)\right)\\
&+t\left[\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}f_*(s)\,d_{q_{k-1}}s+I_k^*\big(x(t_k)\big)\right)\right]\\
&-\sum_{0<t_k<t}t_k\left(\int_{t_{k-1}}^{t_k}f_*(s)\,d_{q_{k-1}}s+I_k^*\big(x(t_k)\big)\right)\\
&+\int_{t_k}^{t}\big(t-q_ks-(1-q_k)t_k\big)f_*(s)\,d_{q_k}s,
\end{aligned}
$$

for some $f_*\in S_{F,x_*}$.

Finally, we show there exists an open set $U\subseteq C(J,\mathbb{R})$ with $x\notin\mathcal{H}(x)$ for any $\lambda\in(0,1)$ and all $x\in\partial U$. Let $\lambda\in(0,1)$ and $x\in\lambda\mathcal{H}(x)$. Then there exists $f\in L^1(J,\mathbb{R})$ with $f\in S_{F,x}$ such that, for $t\in J$, we have

$$
\begin{aligned}
x(t)={}&\alpha+\beta t+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\big(t_k-q_{k-1}s-(1-q_{k-1})t_{k-1}\big)f(s)\,d_{q_{k-1}}s+I_k\big(x(t_k)\big)\right)\\
&+t\left[\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}f(s)\,d_{q_{k-1}}s+I_k^*\big(x(t_k)\big)\right)\right]\\
&-\sum_{0<t_k<t}t_k\left(\int_{t_{k-1}}^{t_k}f(s)\,d_{q_{k-1}}s+I_k^*\big(x(t_k)\big)\right)\\
&+\int_{t_k}^{t}\big(t-q_ks-(1-q_k)t_k\big)f(s)\,d_{q_k}s.
\end{aligned}
$$

Repeating the computations of the second step, we have

$$
\begin{aligned}
\big|x(t)\big|\le{}&|\alpha|+|\beta|T+\|p\|\psi\big(\|x\|\big)\left\{\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\sum_{k=1}^{m}(T+t_k)(t_k-t_{k-1})\right\}\\
&+\sum_{k=1}^{m}\big[c_k+c_k^*(T+t_k)\big].
\end{aligned}
$$

Consequently, we have

$$\frac{\|x\|}{|\alpha| + |\beta|T + \|p\|\psi(\|x\|)\Lambda_1 + \sum_{k=1}^{m}[c_k + c_k^*(T + t_k)]} \leq 1.$$

In view of (A_2), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in PC(J, \mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\mathcal{H} : \overline{U} \to \mathcal{P}(PC(J, \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\mathcal{H}(x)$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.2), we deduce that $\mathcal{H}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.4). This completes the proof. □

Example 4.4 Let us consider the following second-order impulsive q_k-difference inclusion with initial conditions:

$$\begin{cases} D^2_{\frac{2}{3+k}} x(t) \in F(t, x(t)), & t \in J = [0, 1], t \neq t_k = \frac{k}{10}, \\ \Delta x(t_k) = \frac{|x(t_k)|}{15(6+|x(t_k)|)}, & k = 1, 2, \ldots, 9, \\ D_{\frac{2}{3+k}} x(t_k^+) - D_{\frac{2}{3+k-1}} x(t_k) = \frac{|x(t_k)|}{19(3+|x(t_k)|)}, & k = 1, 2, \ldots, 9, \\ x(0) = 0, & D_{\frac{2}{3}} x(0) = 0. \end{cases} \tag{4.5}$$

Here $q_k = 2/(3 + k)$, $k = 0, 1, 2, \ldots, 9$, $m = 9$, $T = 1$, $\alpha = 0$, $\beta = 0$, $I_k(x) = |x|/(15(6 + |x|))$, and $I_k^*(x) = |x|/(19(3 + |x|))$. We find that $|I_k(x) - I_k(y)| \leq (1/90)|x - y|$, $|I_k^*(x) - I_k^*(y)| \leq (1/57)|x - y|$, and $I_k(x) \leq 1/15$, $I_k^*(x) \leq 1/19$; and we have

$$\Lambda_1 = \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + \sum_{k=1}^{m} (T + t_k)(t_k - t_{k-1}) \approx 1.42663542.$$

(a) Let $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t, x) = \left[\frac{|x|}{|x| + \sin^2 x + 1} + t + 1, e^{-x^2} + \frac{4}{5}t^2 + 3\right]. \tag{4.6}$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{|x|}{|x| + \sin^2 x + 1} + t + 1, e^{-x^2} + t^2 + 3\right) \leq 5, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \leq 5 = p(t)\psi(\|x\|), \quad x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|) = 5$. Further, using the condition (A_2) we find

$$\frac{M}{5\Lambda_1 + \sum_{k=1}^{9}[\frac{1}{15} + \frac{1}{19}(1 + t_k)]} > 1,$$

which implies $M > 8.44370316$. Therefore, all the conditions of Theorem 4.3 are satisfied. So, problem (4.5) with $F(t,x)$ given by (4.6) has at least one solution on $[0,1]$.

(b) If $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{(t+1)x^2}{x^2+1}, \frac{t|x|(\cos^2 x + 1)}{2(|x|+1)} \right]. \tag{4.7}$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{(t+1)x^2}{x^2+1}, \frac{t|x|(\cos^2 x + 1)}{2(|x|+1)} \right) \le t+1, \quad x \in \mathbb{R}.$$

Here $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le (t+1) = p(t)\psi(\|x\|)$, $x \in \mathbb{R}$, with $p(t) = t+1$, $\psi(\|x\|) = 1$. It is easy to verify that $M > 3.45047945$. Then, by Theorem 4.3, the problem (4.5) with $F(t,x)$ given by (4.7) has at least one solution on $[0,1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this article. They read and approved the final manuscript.

Author details

[1]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece. [2]Department of Mathematics, Faculty of Applied Science, Nonlinear Dynamic Analysis Research Center, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand. [3]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok, 10400, Thailand.

Authors' information

Sotiris K Ntouyas is a member of Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group at King Abdulaziz University, Jeddah, Saudi Arabia.

Acknowledgements

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

Received: 16 June 2014 Accepted: 30 September 2014 Published: 13 Oct 2014

References

1. Tariboon, J, Ntouyas, SK: Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. **2013**, Article ID 282 (2013)
2. Kac, V, Cheung, P: Quantum Calculus. Springer, New York (2002)
3. Bangerezako, G: Variational *q*-calculus. J. Math. Anal. Appl. **289**, 650-665 (2004)
4. Dobrogowska, A, Odzijewicz, A: Second order *q*-difference equations solvable by factorization method. J. Comput. Appl. Math. **193**, 319-346 (2006)
5. Gasper, G, Rahman, M: Some systems of multivariable orthogonal *q*-Racah polynomials. Ramanujan J. **13**, 389-405 (2007)
6. Ismail, MEH, Simeonov, P: *q*-Difference operators for orthogonal polynomials. J. Comput. Appl. Math. **233**, 749-761 (2009)
7. Bohner, M, Guseinov, GS: The *h*-Laplace and *q*-Laplace transforms. J. Math. Anal. Appl. **365**, 75-92 (2010)
8. El-Shahed, M, Hassan, HA: Positive solutions of *q*-difference equation. Proc. Am. Math. Soc. **138**, 1733-1738 (2010)
9. Ahmad, B: Boundary-value problems for nonlinear third-order *q*-difference equations. Electron. J. Differ. Equ. **2011**, 94 (2011)
10. Ahmad, B, Alsaedi, A, Ntouyas, SK: A study of second-order *q*-difference equations with boundary conditions. Adv. Differ. Equ. **2012**, Article ID 35 (2012)
11. Ahmad, B, Ntouyas, SK, Purnaras, IK: Existence results for nonlinear *q*-difference equations with nonlocal boundary conditions. Commun. Appl. Nonlinear Anal. **19**, 59-72 (2012)
12. Ahmad, B, Nieto, JJ: On nonlocal boundary value problems of nonlinear *q*-difference equations. Adv. Differ. Equ. **2012**, Article ID 81 (2012)
13. Ahmad, B, Ntouyas, SK: Boundary value problems for *q*-difference inclusions. Abstr. Appl. Anal. **2011**, Article ID 292860 (2011)
14. Zhou, W, Liu, H: Existence solutions for boundary value problem of nonlinear fractional *q*-difference equations. Adv. Differ. Equ. **2013**, Article ID 113 (2013)
15. Yu, C, Wang, J: Existence of solutions for nonlinear second-order *q*-difference equations with first-order *q*-derivatives. Adv. Differ. Equ. **2013**, Article ID 124 (2013)
16. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)

17. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
18. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions, vol. 2. Hindawi Publishing Corporation, New York (2006)
19. Bohner, M, Peterson, A: Dynamic Equations on Time Scales. An Introduction with Applications. Birkhäuser Boston, Boston (2001)
20. Deimling, K: Multivalued Differential Equations. de Gruyter, Berlin (1992)
21. Hu, S, Papageorgiou, N: Handbook of Multivalued Analysis. Vol. I. Theory. Kluwer Academic, Dordrecht (1997)
22. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2005)
23. Lasota, A, Opial, Z: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781-786 (1965)
24. Frigon, M: Théorèmes d'existence de solutions d'inclusions différentielles. In: Granas, A, Frigon, M (eds.) Topological Methods in Differential Equations and Inclusions. NATO ASI Series C, vol. 472, pp. 51-87. Kluwer Academic, Dordrecht (1995)

10.1186/1687-1847-2014-262
Cite this article as: Ntouyas and Tariboon: Applications of quantum calculus on finite intervals to impulsive difference inclusions. *Advances in Difference Equations* 2014, 2014:262