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Finite-time stability analysis of fractional singular time-delay systems

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Abstract
This paper studies the finite-time stability of fractional singular time-delay systems. First, by the method of the steps, we discuss the existence and uniqueness of the solutions for the equivalent systems to the fractional singular time-delay systems. Furthermore, we give the Mittag-Leffler estimation of the solutions for the equivalent systems and obtain the sufficient conditions of the finite-time stability for the original systems.
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1 Introduction

In the past 30 years or so, fractional calculus has attracted many physicists, mathematicians, and engineers, and notable contributions have been made to both the theory and the applications of fractional differential equations (see [1–9]). Moreover, the different techniques have been applied to investigate the stability of various fractional dynamical systems, such as the principle of contraction mappings [10], the Lyapunov direct method [11], linear matrix inequalities [12], Gronwall inequalities [13–16] and fixed-point theorems [17].

At the same time, we notice that large numbers of practical systems, such as economic systems, power systems and so on, are singular differential systems which are also named differential-algebraic systems or descriptor systems. Such systems have some particular properties including regularity and impulse behavior which does not need to be considered in normal systems. In [18–22], the authors discuss singular systems with or without delay and obtain some important results. However, in the previous literature, there are few results on the stability of fractional singular systems, especially the fractional singular systems with time delay. In this regard, it is necessary and important to study the stability problems for fractional singular dynamical systems. Motivated by this consideration, in this paper, we investigate the stability of fractional singular dynamical systems with state delay via the generalized Gronwall approach.

In this paper, we consider the following fractional singular time-delay system:

$$\begin{cases} E({}^cD^\alpha x(t)) = Ax(t) + Bx(t-\tau), & t \in [0,T], \\ x(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order $0 < \alpha \leq 1$; the vector function $x(t) \in R^n$ is a state vector; $A, B, E \in R^{n\times n}$ are constant matrices; $E \in R^{n\times n}$ is a singular matrix *i.e.* $\operatorname{rank}(E) = q < n$; the constant parameter $\tau > 0$ represents the delay argument and $\varphi(t)$ is a given sufficiently often differentiable function on $[-\tau, 0]$.

The organization of this paper is as follows. In Section 2, we summarize some notations and give preliminary results which will be used in this paper. In Section 3, we present our main results.

2 Preliminaries and lemmas

For completeness, in this section, we firstly demonstrate and study the definitions and some fundamental results of fractional calculus which can be found in [2–4].

Definition 2.1 (see [2]) The Euler gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad z \in \mathbb{C}, \tag{2.1}$$

where $\mathbb{C}$ denotes the complex plane.

Definition 2.2 (see [2]) The fractional integral of order α with the lower limit zero for any function $f(t) \in C([0, +\infty), R)$, $t \geq 0$ is defined as

$$I^\alpha f(t) = \lim_{\substack{h\to 0\\ nh=t}} h^\alpha \sum_{r=0}^{n} \begin{bmatrix} \alpha \\ r \end{bmatrix} f(t - rh) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta)\, d\theta, \quad \alpha > 0, \tag{2.2}$$

where $\left[\begin{smallmatrix} \alpha \\ r \end{smallmatrix}\right] = \frac{\alpha(\alpha+1)\cdots(\alpha+r-1)}{r!}$, $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 (see [2]) The Riemann-Liouville derivative of order α with the lower limit zero for any function $f(t) \in C([0, +\infty), R)$, $t \geq 0$ is defined as

$${}^lD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\theta)^{n-\alpha-1} f(\theta)\, d\theta, \quad n-1 < \alpha < n. \tag{2.3}$$

Definition 2.4 (see [2]) The Caputo derivative of order α for any function $f(t) \in C^n([0, +\infty), R)$, $t \geq 0$, is defined as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\theta)^{n-\alpha-1} f^{(n)}(\theta)\, d\theta = I^{n-\alpha} f^{(n)}(t), \quad n-1 < \alpha < n. \tag{2.4}$$

Remark 2.1 (see [2])

(i) The Laplace transform of the Caputo derivative is

$$L\{{}^cD^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \quad (n-1 < \alpha \leq n); \tag{2.5}$$

(ii) The Caputo fractional derivative is a linear operator satisfying the relation

$${}^cD^\alpha\big(\lambda f(t) + \mu g(t)\big) = \lambda\, {}^cD^\alpha f(t) + \mu\, {}^cD^\alpha g(t), \tag{2.6}$$

where λ and μ are scalars.

Lemma 2.1 (see [3]) *Let* $0 < \alpha < 1$, *then we have*

$$I^{\alpha}\left({}^{c}D^{\alpha}x(t)\right) = x(t) - x(0). \tag{2.7}$$

Definition 2.5 (see [4]) The Mittag-Leffler function in two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \tag{2.8}$$

where $\alpha > 0$, $\beta > 0$, and $z \in \mathbb{C}$.

Remark 2.2 (see [4])

(i) For $\beta = 1$, $E_{\alpha,1}(\lambda z^{\alpha}) = E_{\alpha}(\lambda z^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k (z^{\alpha})^k}{\Gamma(\alpha k+1)}$, and $E_{1,1}(z) = e^z$, $z \in \mathbb{C}$;

(ii) for $\beta = 1$, the matrix extension of the aforementioned Mittag-Leffler function has the following representation: $E_{\alpha}(At^{\alpha}) = \sum_{k=0}^{\infty} \frac{A^k (t^{\alpha})^k}{\Gamma(\alpha k+1)}$, $z \in \mathbb{C}$ and ${}^{c}D^{\alpha}E_{\alpha}(At^{\alpha}) = AE_{\alpha}(At^{\alpha})$;

(iii) we have the Laplace transform of the Mittag-Leffler function in two parameters

$$L\left\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}\left(\pm at^{\alpha}\right); s\right\} = \frac{k! s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}} \quad \left(\operatorname{Re}(s) > |a|^{1/\alpha}\right), \tag{2.9}$$

where $\operatorname{Re}(s)$ denotes the real parts of s.

Next, we introduce some fundamental definitions and lemmas about singular systems.

Definition 2.6 (see [18]) For any given two matrices $E, A \in R^{n\times n}$, the pencil (E, A) is called regular if there exists a constant scalar $\lambda \in \mathbb{C}$ such that $|\lambda E + A| \neq 0$, or the polynomial $|sE - A| \not\equiv 0$.

Lemma 2.2 (see [18]) *The pencil* (E, A) *is regular if and only if two nonsingular matrices* Q, P *may be chosen such that*

$$QEP = \operatorname{diag}(I_{n_1}, N), \qquad QAP = \operatorname{diag}(A_1, I_{n_2}), \tag{2.10}$$

where $n_1 + n_2 = n$; $A_1 \in R^{n_1\times n_1}$; $N \in R^{n_2\times n_2}$ *is nilpotent*; I_{n_1}, I_{n_2} *are identity matrices*.

Remark 2.3 (see [18])

(i) $N \in R^{n_2\times n_2}$ is nilpotent (the nilpotent index is denoted by h), and we have

$$N^h = 0 \quad \text{and} \quad N^{h-1} \neq 0, \tag{2.11}$$

where h is also called the index of the matrix pair (E, A);

(ii) the system (1.1) will be termed regular if the pencil (E, A) is regular.

In the following, we present the first equivalent form (FE1) of system (1.1) by the coordinate transformation, which is also called the standard decomposition of a singular system.

For convenience, we denote ${}^cD^\alpha x(t)$ by $x^{(\alpha)}(t)$, from Lemma 2.1 and Remark 2.4, we deduce the following statement. Assume that the system (1.1) is regular throughout this paper, there exist two nonsingular matrices Q and P such that the system (1.1) is a restricted system equivalent to

$$\begin{cases} \begin{cases} x_1^{(\alpha)}(t) = A_1x_1(t) + B_{11}x_1(t-\tau) + B_{12}x_2(t-\tau), & t \geq 0, \\ x_1(t) = \varphi_1(t), & -\tau \leq t \leq 0; \end{cases} & \text{(a)} \\ \begin{cases} Nx_2^{(\alpha)}(t) = x_2(t) + B_{21}x_1(t-\tau) + B_{22}x_2(t-\tau), & t \geq 0, \\ x_2(t) = \varphi_2(t), & -\tau \leq t \leq 0, \end{cases} & \text{(b)} \end{cases} \tag{2.12}$$

with the coordinate transformation

$$\bar{x} = [x_1/x_2] = P^{-1}x, \qquad \bar{\varphi} = [\varphi_1/\varphi_2] = P^{-1}\varphi; \quad x_1, \varphi_1 \in R^{n_1}; x_2, \varphi_2 \in R^{n_2}, \tag{2.13}$$

and

$$QEP = \operatorname{diag}(I_{n_1}, N), \qquad QAP = \operatorname{diag}(A_1, I_{n_2}), \qquad QBP = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{2.14}$$

where $\varphi_1 \in R^{n_1}$; $\varphi_2 \in R^{n_2}$; $n_1 + n_2 = n$; $A_1, B_{11} \in R^{n_1 \times n_1}$; $N, B_{22} \in R^{n_2 \times n_2}$; $B_{12} \in R^{n_1 \times n_2}$; N is nilpotent.

The following definitions and lemmas will play important roles in our next analysis.

Definition 2.7 (see [23]) If X and Y are normed linear spaces, an operator $\mathbb{T}: X \to Y$ is linear if

$$\mathbb{T}(\alpha x_1 + \beta x_2) = \alpha\mathbb{T}(x_1) + \beta\mathbb{T}(x_2), \tag{2.15}$$

for all x_1, x_2 in X and scalars α and β.

Remark 2.4 (see [23]) We say the linear operator $\mathbb{T}$ is a bounded linear operator from X to Y if there is a finite constant C_0 such that $\|\mathbb{T}x\|_Y \leq C_0\|x\|_X$ for all x in X.

Lemma 2.3 (see [23]) *If $\mathbb{T}: X \to Y$ is a linear operator from a normed linear space X to a normed linear space Y, the following are equivalent*:

(i) $\mathbb{T}$ *is bounded*;
(ii) $\mathbb{T}$ *is continuous*;
(iii) $\mathbb{T}$ *is continuous at* 0.

Lemma 2.4 (see [24]; Generalized Gronwall Inequality) *Suppose $x(t)$, $a(t)$ are nonnegative and local integrable on $0 \leq t < T$; some $T \leq \infty$, and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$; $g(t) \leq C_1$, where C_1 is a constant, $\alpha > 0$ with*

$$x(t) \leq a(t) + g(t)\int_0^t (t-s)^{\alpha-1}x(s)\,ds \tag{2.16}$$

on this interval. Then

$$x(t) \le a(t) + g(t)\int_0^t \sum_{n=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\,ds, \quad 0 \le t < T. \tag{2.17}$$

Lemma 2.5 (see [24]) *Under the hypothesis of Theorem* 3.2, *let* $a(t)$ *be a nondecreasing function on* $[0, T)$. *Then*

$$x(t) \le a(t)E_\alpha\bigl(g(t)\Gamma(\alpha)t^\alpha\bigr), \tag{2.18}$$

where E_α *is the Mittag-Leffler function.*

3 Main results

In this section, we discuss some problems of the singular fractional time-delay system (1.1).

Let D^α be the Caputo fractional differential operator of order $0 < \alpha \le 1$, $D^{n\alpha}f(t) = \underbrace{D^\alpha D^\alpha \cdots D^\alpha}_{n} f(t)$ and $\mathbb{T} = (ND^\alpha - I)^{-1}$. It is not difficult to verify the following:

$$\mathbb{T} = -\bigl(I + ND^\alpha + N^2D^{2\alpha} + \cdots + N^{h-1}D^{(h-1)\alpha}\bigr), \tag{3.1}$$

where $I \in R^{n_2\times n_2}$ is an identity matrix.

Theorem 3.1 *The fractional differential operator* $\mathbb{T}$ *is bounded i.e. there exists a positive constant* M *such that for* $\forall x_2(t)$ *we have*

$$\bigl\|\mathbb{T}x_2(t)\bigr\| \le M\bigl\|x_2(t)\bigr\|. \tag{3.2}$$

Proof Obviously, the fractional differential operator $\mathbb{T}$ is linear. According to Lemma 2.3, we are only necessary to show that $\mathbb{T}$ is continuous at 0. Let any sequences $x_n(t) \to 0$, $y_n(t) \to y_0(t)$, and $y_n(t) = \mathbb{T}x_n(t)$, all we finally need to do is to show that $y_0(t) = 0$.

According to $y_n(t) = \mathbb{T}x_n(t) = (ND^\alpha - I)^{-1}x_n(t)$, we have

$$\bigl(ND^\alpha - I\bigr)y_n(t) = x_n(t), \tag{3.3}$$

and for $n \to \infty$,

$$\begin{aligned} &\bigl(ND^\alpha - I\bigr)y_0(t) = 0 \quad \Rightarrow \quad ND^\alpha y_0(t) = y_0(t) \\ &\quad \Rightarrow \quad y_0(t) = ND^\alpha y_0(t) = N^2D^{2\alpha}y_0(t) = \cdots = N^{h-1}D^{(h-1)\alpha}y_0(t) = N^hD^{h\alpha}y_0(t). \end{aligned} \tag{3.4}$$

Combining Remark 2.3 and (3.4) yields

$$y_0(t) = 0. \tag{3.5}$$

Therefore, the operator $\mathbb{T}$ is bounded. □

To give the solution of systems (2.12), let us define a new function.

Definition 3.1 (see [25]) Let α obey ($0 \leq \alpha < 1$), the function

$$\delta^{\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\delta(\theta)}{(t-\theta)^{\alpha}} \, d\theta \tag{3.6}$$

is called an $\alpha - \delta$ function, where $\delta(t)$ is the Dirac delta function.

Remark 3.1 (see [25]) The Laplace transformation of the $\alpha - \delta$ function is $L\{\delta^{\alpha}(t); s\} = s^{\alpha-1}$.

Theorem 3.2 *If the system* (1.1) *is regular, the solution for the system* (2.12) *exists uniquely.*

Proof From Remark 2.3, we know that the pencil (E, A) is regular if the system (1.1) is regular. By the coordinate transformation, the system (2.12) is equivalent to the system (1.1). For $t \in [0, \tau]$, then $t - \tau \in [-\tau, 0]$, the system (2.12)(a) may be written as

$$x_1^{(\alpha)}(t) = A_1 x_1(t) + B_{11}\varphi_1(t-\tau) + B_{12}\varphi_2(t-\tau), \quad t \in [0, \tau]. \tag{3.7}$$

Let $f_1(t) = B_{11}\varphi_1(t-\tau) + B_{12}\varphi_2(t-\tau)$. Obviously, if $f_1(t)$ is the known function, then (3.7) may be written as

$$x_1^{(\alpha)}(t) = A_1 x_1(t) + f_1(t). \tag{3.8}$$

Applying the Laplace transformation on both sides of (3.8) and using (2.5) yield

$$\begin{aligned} &s^{\alpha} X_1(s) - s^{\alpha-1} x_1(0) = A_1 X_1(s) + F_1(s), \\ &X_1(s) = \left(s^{\alpha} I - A_1\right)^{-1} s^{\alpha-1} x_1(0) + \left(s^{\alpha} I - A_1\right)^{-1} F_1(s). \end{aligned} \tag{3.9}$$

Applying the Laplace inverse transformation on both sides of (3.9) and using (2.9) yield

$$x_1(t) = E_{\alpha,1}\left(A_1 t^{\alpha}\right) x_1(0) + \int_0^t (t-\theta)^{\alpha-1} E_{\alpha,\alpha}\left(A_1 (t-\theta)^{\alpha}\right) f_1(\theta) \, d\theta, \quad t \in [0, \tau]. \tag{3.10}$$

As for the system (2.12)(b), it may be rewritten as

$$N x_2^{(\alpha)}(t) = x_2(t) + B_{21}\varphi_1(t-\tau) + B_{22}\varphi_2(t-\tau), \quad t \in [0, \tau]. \tag{3.11}$$

Similarly, let $f_2(t) = B_{21}\varphi_1(t-\tau) + B_{22}\varphi_2(t-\tau)$, and $f_2(t)$ is the known sufficiently often differentiable function, then (3.11) may be written as

$$N x_2^{(\alpha)}(t) = x_2(t) + f_2(t). \tag{3.12}$$

Taking the Laplace transformation on both sides of (3.12), we have

$$\left(s^{\alpha} N - I\right) X_2(s) = s^{\alpha-1} N x_2(0) + F_2(s),$$

$$\begin{aligned} X_2(s) &= \left(s^\alpha N - I\right)^{-1}\left(s^{\alpha-1}Nx_2(0) + F_2(s)\right) \\ &= -\sum_{i=0}^{h-1} N^i\left(s^\alpha\right)^i\left(s^{\alpha-1}Nx_2(0) + F_2(s)\right) \\ &= -\sum_{i=1}^{h} N^{i-1}s^{i\alpha-1}x_2(0) - \sum_{i=0}^{h-1} N^i s^{i\alpha}F_2(s). \end{aligned} \tag{3.13}$$

According to Remark 3.1, the inverse Laplace transformation of $X_2(s)$ yields

$$x_2(s) = -\sum_{i=1}^{h} N^{i-1}\delta^{i\alpha}x_2(0) - \sum_{i=0}^{h-1} N^i D^{i\alpha}f_2(t), \quad t \in [0,\tau]. \tag{3.14}$$

Obviously, by the method of steps, once the solution $\bar{x}(t)$ of the system (2.12) on $[0,\tau]$ is known, continuing the above process, we can easily obtain the solution $\bar{x}(t)$ of the system (2.12) on $[\tau,2\tau],[2\tau,3\tau],\ldots$. Thus the solution $\bar{x}(t)$ of the system (2.12) on $[0,T]$ exists uniquely. □

Furthermore, we give the following theorems as regards the Mittag-Leffler estimation of the solution and finite-time stability for this singular system.

Let us denote by $C([a,b])$ the space of all continuous real functions defined on $[a,b]$ and by $C([a,b],R^n)$ the Banach space of continuous functions mapping the interval $[a,b]$ into R^n with the topology of uniform convergence. Let $C = C([-\tau,0],R^n)$, $[a,b] = [-\tau,0]$, and designate the norm of an element φ in C by

$$\|\varphi\| = \sup_{-\tau\le t\le 0}\left\|\varphi(t)\right\|. \tag{3.15}$$

Let $X = C([-\tau,T],R^n)$ and $x(t) = \varphi(t)$, $t \in [-\tau,0]$ be equipped with the norm

$$\left\|x(t)\right\| := \sup_{0\le t\le T} x(t), \qquad \|x_t\| := \left\|x(t+\theta)\right\| := \sup_{-\tau\le\theta\le 0}\left\|x(t+\theta)\right\|, \quad \forall x \in X. \tag{3.16}$$

Definition 3.2 (see [15]) The system given by (1.1) satisfying the initial condition $x(t) = \varphi(t)$, for $t \in [-\tau,0]$ is finite-time stable w.r.t. $\{t_0,\delta,\epsilon,J\}$, $\delta < \epsilon$, $J = [t_0,t_0+T]$ if and only if

$$\|\varphi\| < \delta \tag{3.17}$$

implies

$$\left\|x(t)\right\| < \epsilon, \quad \forall t \in J. \tag{3.18}$$

Theorem 3.3 *If $\bar{x}(t) = [x_1(t)/x_2(t)]$ is a solution of the system* (2.12), *then there exist positive constants a and b such that*

(i) $a = 1 + M\|B_{21}\| + M\|B_{22}\|$;
(ii) $b > \|A_1\| + \|B_{11}\| + \|B_{12}\|$;
(iii) $\|\bar{x}(t)\| \le a\|\bar{\varphi}\|E_\alpha(bt^\alpha)$, $\forall t \in J = [0,T]$.

Proof According to Lemma 2.1, the system (2.12)(a) may be rewritten in the form of the equivalent Volterra integral equation

$$\begin{aligned} x_1(t) &= x_1(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl[A_1x_1(s) + B_{11}x_1(s-\tau) + B_{12}x_2(s-\tau)\bigr]\,ds \\ &= \varphi_1(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A_1x_1(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl[B_{11}x_1(s-\tau) + B_{12}x_2(s-\tau)\bigr]\,ds, \quad t \ge 0. \end{aligned} \tag{3.19}$$

Using the appropriate property of the norm $\|\cdot\|$ on (3.19), it follows that

$$\begin{aligned} \|x_1(t)\| &\le \|\varphi_1(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\|A_1\|\|x_1(s)\|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\bigl[\|B_{11}\|\|x_1(s-\tau)\| + \|B_{12}\|\|x_2(s-\tau)\|\bigr]\,ds \\ &\le \|\bar{\varphi}\| + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\|A_1\|\|\bar{x}(s)\|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\bigl[\|B_{11}\|\|\bar{x}(s-\tau)\| + \|B_{12}\|\|\bar{x}(s-\tau)\|\bigr]\,ds, \quad t \ge 0. \end{aligned} \tag{3.20}$$

As for the system (2.12)(b), we have

$$\begin{aligned} &ND^{\alpha}x_2(t) = x_2(t) + B_{21}x_1(t-\tau) + B_{22}x_2(t-\tau), \\ &x_2(t) = \bigl(ND^{\alpha} - I\bigr)^{-1}\bigl[B_{21}x_1(t-\tau) + B_{22}x_2(t-\tau)\bigr] \\ &\qquad = \mathbb{T}B_{21}x_1(t-\tau) + \mathbb{T}B_{22}x_2(t-\tau). \end{aligned} \tag{3.21}$$

Applying the appropriate property of the norm $\|\cdot\|$ and Theorem 3.1, we have

$$\begin{aligned} \|x_2(t)\| &\le M\|B_{21}\|\|x_1(t-\tau)\| + M\|B_{22}\|\|x_2(t-\tau)\| \\ &\le \bigl(M\|B_{21}\| + M\|B_{22}\|\bigr)\|\bar{x}(t-\tau)\|, \quad t \ge 0. \end{aligned} \tag{3.22}$$

Combining (3.20) and (3.22) yields

$$\begin{aligned} \|\bar{x}(t)\| &\le \|x_1(t)\| + \|x_2(t)\| \\ &\le \|\bar{\varphi}\| + \bigl(M\|B_{21}\| + M\|B_{22}\|\bigr)\|\bar{x}(t-\tau)\| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\|A_1\|\|\bar{x}(s)\|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\bigl(\|B_{11}\| + \|B_{12}\|\bigr)\|\bar{x}(s-\tau)\|\,ds, \quad t \ge 0. \end{aligned} \tag{3.23}$$

For $0 \le t \le \tau$, $\|\bar{x}(t-\tau)\| \le \|\bar{\varphi}\|$, (3.23) can be written as

$$\begin{aligned}\|\bar{x}(t)\| \le{}& \bigl(1 + M\|B_{21}\| + M\|B_{22}\|\bigr)\|\bar{\varphi}\| \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\|A_1\|\|\bar{x}(s)\|\,ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\bigl(\|B_{11}\| + \|B_{12}\|\bigr)\|\bar{x}(s-\tau)\|\,ds, \quad 0 \le t \le \tau. \end{aligned} \tag{3.24}$$

From Definition 2.2, we know that $I^\alpha f(t)$ is an increasing function of t, if $f(t) > 0$. So $\frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\|A_1\|\|\bar{x}(s)\|\,ds$ and $\frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}[\|B_{11}\| + \|B_{12}\|]\|\bar{x}(s-\tau)\|\,ds$ are both increasing functions with regard to t. Taking into account (3.24) and (3.16) yields

$$\begin{aligned}\|\bar{x}_t\| \le{}& \bigl(1 + M\|B_{21}\| + M\|B_{22}\|\bigr)\|\bar{\varphi}\| \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\|A_1\|\|\bar{x}_s\|\,ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\bigl(\|B_{11}\| + \|B_{12}\|\bigr)\|\bar{x}_s\|\,ds \\ \le{}& \bigl(1 + M\|B_{21}\| + M\|B_{22}\|\bigr)\|\bar{\varphi}\| \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\bigl(\|A_1\| + \|B_{11}\| + \|B_{12}\|\bigr)\|\bar{x}_s\|\,ds, \quad 0 \le t \le \tau. \end{aligned} \tag{3.25}$$

Let $a = 1 + M\|B_{21}\| + M\|B_{22}\|$ and $b_0 = \|A_1\| + \|B_{11}\| + \|B_{12}\|$, we can see the function $a(t)$ in Lemma 2.5 to be

$$a(t) = \bigl(1 + M\|B_{21}\| + M\|B_{22}\|\bigr)\|\bar{\varphi}\| = a\|\bar{\varphi}\|, \tag{3.26}$$

obviously, it is nondecreasing.

An application of the corollary of the generalized Gronwall inequality (2.18) yields

$$\|\bar{x}_t\| \le a\|\bar{\varphi}\|E_\alpha\bigl(b_0 t^\alpha\bigr), \quad 0 \le t \le \tau. \tag{3.27}$$

Similarly, the same argument implies the following estimate:

$$\|\bar{x}_t\| \le a\|\bar{x}_{\tau_0}\|E_\alpha\bigl(b_0(t-\tau_0)^\alpha\bigr), \quad \tau_0 \le t \le \tau_0 + \tau, \tau_0 \ge 0. \tag{3.28}$$

From Definition 2.5, we know that the Mittag-Leffler function $E_\alpha(t)$ is an increasing function with regard to t. Therefore, there exists $b > b_0$ such that $E_\alpha(b\tau^\alpha) > E_\alpha(b_0\tau^\alpha)$ and $\frac{E_\alpha(b(t-\tau)^\alpha)E_\alpha(b_0\tau^\alpha)}{E_\alpha(bt^\alpha)} < \frac{1}{a}$.

Equations (3.27) and (3.28) suggest the following general expression:

$$\|\bar{x}_t\| \le a\|\bar{\varphi}\|E_\alpha\bigl(bt^\alpha\bigr), \quad 0 \le t \le n\tau \le T. \tag{3.29}$$

To prove (3.29) by induction we have to show that it holds for $n = 1$ because of (3.27) and if it holds for $n = k$, then it also holds for $n = k + 1$. Indeed, for $t \in [\tau, (k+1)\tau]$, so that $t - \tau \in [0, k\tau]$, on the one hand, using (3.28), we have

$$\|\bar{x}_t\| \le a\|\bar{x}_{t-\tau}\|E_\alpha\bigl(b_0\tau^\alpha\bigr). \tag{3.30}$$

On the other hand, using (3.29) we obtain

$$\|\bar{x}_{t-\tau}\| \leq a\|\bar{\varphi}\|E_\alpha\bigl(b(t-\tau)^\alpha\bigr). \tag{3.31}$$

Taking into account (3.30) and (3.31) we conclude that

$$\begin{aligned}\|\bar{x}_t\| &\leq a\bigl[a\|\bar{\varphi}\|E_\alpha\bigl(b(t-\tau)^\alpha\bigr)\bigr]E_\alpha\bigl(b_0\tau^\alpha\bigr)\\ &= a\|\bar{\varphi}\|E_\alpha\bigl(bt^\alpha\bigr)\frac{aE_\alpha(b(t-\tau)^\alpha)E_\alpha(b_0\tau^\alpha)}{E_\alpha(bt^\alpha)}\\ &\leq a\|\bar{\varphi}\|E_\alpha\bigl(bt^\alpha\bigr).\end{aligned} \tag{3.32}$$

That is,

$$\bigl\|\bar{x}(t)\bigr\| \leq \|\bar{x}_t\| \leq a\|\bar{\varphi}\|E_\alpha\bigl(bt^\alpha\bigr). \tag{3.33}$$

The proof is completed. □

Theorem 3.4 *The fractional singular time-delay system given by* (1.1) *is finite-time stable w.r.t.* $\{0,\delta,\epsilon,J\}$, $\delta<\epsilon$, *if the following condition is satisfied*:

$$a\|P\|^2E_\alpha\bigl(bt^\alpha\bigr) \leq \frac{\epsilon}{\delta}, \quad \forall t\in J=[0,T]. \tag{3.34}$$

Proof From the coordinate transformation (2.13), we have

$$x(t)=P\bar{x}(t)=P\bigl[x_1(t)/x_2(t)\bigr], \qquad \varphi(t)=P\bar{\varphi}(t)=P\bigl[\varphi_1(t)/\varphi_2(t)\bigr]. \tag{3.35}$$

From Theorem 3.3 we obtain

$$\bigl\|x(t)\bigr\| \leq \|P\|\bigl\|\bar{x}(t)\bigr\| \leq a\|P\|^2\|\varphi\|E_\alpha\bigl(bt^\alpha\bigr). \tag{3.36}$$

Hence, using Definition 3.2 and the basic condition of Theorem 3.4, it follows that

$$\bigl\|x(t)\bigr\| < \epsilon, \quad \forall t\in J=[0,T]. \tag{3.37}$$

The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the manuscript. Both of them read and approved the final manuscript.

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