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Existence of positive solutions for a discrete fractional boundary value problem

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Abstract

This paper is concerned with the existence of positive solutions to a discrete fractional boundary value problem. By using the Krasnosel'skii and Schaefer fixed point theorems, the existence results are established. Additionally, examples are provided to illustrate the effectiveness of the main results.

MSC: 26A33; 39A10; 47H07

Keywords: existence; positive solution; discrete; fractional boundary value problem

1 Introduction

Fractional differential equations have received increasing attention within the last ten years or so. The theory of fractional differential equations has been a new important mathematical branch due to its wide applications in different research areas and engineering, such as physics, chemistry, economics, control of dynamical *etc.* For more details, see [1–9] and the references therein. On the other hand, accompanied with the development of the theory for fractional calculus, fractional difference equations have attracted increasing attention slowly but steadily in the past three years or so. Some research papers have appeared, see [10–19]. For example, Atici and Eloe [10] analyzed the conjugate discrete fractional boundary value problem (FBVP) with delta derivative:

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), & t \in [0,b]_{\mathbf{N}_0}, \\ y(\nu-2) = y(\nu+b+1) = 0, & 1 < \nu \leq 2. \end{cases}$$

Goodrich [11] studied the discrete fractional boundary value problems:

$$\begin{cases} \Delta^{\nu} y(t) = \lambda f(t+\nu-1, y(t+\nu-1)), & t \in [0,T]_Z, \\ y(\nu-1) = y(\nu+T) + \sum_{i=1}^{N} F(t_i, y(t_i)), & 0 < \nu < 1. \end{cases}$$

In [12], Lv discussed the existence of solutions for discrete fractional boundary value problems with a p-Laplacian operator:

$$\begin{cases} \Delta_c^{\beta}[\phi_p(\Delta_c^{\alpha} u)](t) = f(t+\alpha+\beta-1, u(t+\alpha+\beta-1)), & t \in [0,b]_{\mathbf{N}_0}, \\ \Delta_c^{\alpha} u(t)|_{t=\beta-1} + \Delta_c^{\alpha} u(t)|_{t=\beta+b} = 0, \\ u(\alpha+\beta-2) + u(\alpha+\beta+b) = 0, & 0 < \alpha, \beta \leq 1, 1 < \alpha+\beta \leq 2. \end{cases}$$

They obtained a series of excellent results of discrete fractional boundary value problems. Motivated by the aforementioned works, in this paper we consider a discrete fractional boundary value problem (FBVP):

$$\begin{cases} \Delta^\nu y(t) = f(t+\nu-1, y(t+\nu-1)), & t \in [0,b]_{\mathbf{N}_0}, \\ y(\nu-2) = 0, & \Delta y(\nu-2) = \Delta y(\nu+b-1), \end{cases} \tag{1.1}$$

where $1 < \nu \leq 2$, Δ^ν denotes the Riemann-Liouville fractional difference operator, $\mathbf{N}_a = \{a, a+1, a+2, \ldots\}$ and $I_{\mathbf{N}_a} = I \cap \mathbf{N}_a$ for any number $a \in \mathbf{R}$ and each interval I of $\mathbf{R}$, $b \in \mathbf{N}_1$. We appeal to the convention that $\sum_{s=k}^{k-1} y(s) = 0$ for any $k \in \mathbf{N}_a$, where y is a function defined on $\mathbf{N}_a$. By using the Krasnosel'skii and Schaefer fixed point theorems, the existence results are established and two examples are also provided to illustrate the effectiveness of the main results.

The rest of the paper is organized as follows. In Section 2, we introduce some lemmas and definitions which will be used later. In Section 3, the existence of positive solutions for the boundary value problem (1.1) is investigated. In Section 4, two examples are provided to illustrate the effectiveness of the main results.

2 Basic definitions and preliminaries

Firstly we present here some necessary definitions and lemmas which are used throughout this paper.

Definition 2.1 [13, 14] Define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any t and ν for which the right-hand side is defined. If $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2 [15] The νth fractional sum of a function f, for $\nu > 0$, is defined to be

$$\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t;a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s) \tag{2.1}$$

for $t \in \{a+\nu, a+\nu+1, \ldots\} := \mathbf{N}_{a+\nu}$. Define the νth fractional difference for $\nu > 0$ by $\Delta^\nu f(t) := \Delta^N \Delta^{\nu-N} f(t)$, $t \in \mathbf{N}_{a+\nu}$ and $N \in \mathbf{N}$ satisfies $0 \leq N-1 < \nu \leq N$.

Lemma 2.3 [15] *Let t and ν be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then $\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}$.*

Lemma 2.4 [15] *Assume that $0 \leq N-1 < \nu \leq N$. Then*

$$\Delta^{-\nu} \Delta^\nu y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \cdots + C_N t^{\underline{\nu-N}} \tag{2.2}$$

for some $C_i \in \mathbf{R}$, with $1 \leq i \leq N$.

Lemma 2.5 (The nonlinear alternative of Leray and Schauder [20]) *Let $\mathbf{E}$ be a Banach space with $C \subseteq \mathbf{E}$ closed and convex. Let U be a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ be a continuous and compact mapping. Then either*

(a) *the mapping T has a fixed point in $\overline{U}$; or*
(b) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda T u$.*

Lemma 2.6 [13] *Let* $\mathbf{B}$ *be a Banach space and let* $\mathbf{K} \subseteq \mathbf{B}$ *be a cone. Assume that* Ω_1 *and* Ω_2 *are bounded open sets contained in* $\mathbf{B}$ *such that* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subseteq \Omega_2$. *Assume further that* $T : \mathbf{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathbf{K}$ *is a completely continuous operator. If either*

(i) $\|Ty\| \le \|y\|$ *for* $y \in \mathbf{K} \cap \partial\Omega_1$ *and* $\|Ty\| \ge \|y\|$ *for* $y \in \mathbf{K} \cap \partial\Omega_2$; *or*

(ii) $\|Ty\| \ge \|y\|$ *for* $y \in \mathbf{K} \cap \partial\Omega_1$ *and* $\|Ty\| \le \|y\|$ *for* $y \in \mathbf{K} \cap \partial\Omega_2$;

then T *has at least one fixed point in* $\mathbf{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We state next the structural assumptions that we impose on (1.1).

(H_1) Assume that the nonlinearity function $f : [\nu - 1, \nu + b - 1]_{\mathbf{N}_{\nu-1}} \times \mathbf{R} \to [0, +\infty)$ is continuous.

(H_2) Assume that there exist nonnegative continuous functions $a_1(t)$, $a_2(t)$, $t \in [\nu - 1, \nu + b - 1]_{\mathbf{N}_{\nu-1}}$ such that $|f(t,y)| \le a_1(t) + a_2(t)|y|$, $\forall t \in [\nu - 1, \nu + b - 1]_{\mathbf{N}_{\nu-1}}$, $y \in \mathbf{R}$.

(H_3) Assume that $\lim_{y\to 0^+} \frac{f(t,y)}{y} = 0$ uniformly for $t \in [\nu - 1, \nu + b - 1]_{\mathbf{N}_{\nu-1}}$.

(H_4) Assume that $\lim_{y\to +\infty} \frac{f(t,y)}{y} = +\infty$ uniformly for $t \in [\nu - 1, \nu + b - 1]_{\mathbf{N}_{\nu-1}}$.

3 Existence results

In this section, we will establish the existence of at least one positive solution for problem (1.1). At first, we state and prove some preliminary lemmas.

Lemma 3.1 *Let* $h : [\nu-1, \nu+b-1]_{\mathbf{N}_{\nu-1}} \to \mathbf{R}$ *be given. Then the unique solution of the discrete fractional boundary value problem*

$$\begin{cases} \Delta^\nu y(t) = h(t + \nu - 1), & t \in [0, b]_{\mathbf{N}_0}, \\ y(\nu - 2) = 0, & \Delta y(\nu - 2) = \Delta y(\nu + b - 1), \end{cases} \tag{3.1}$$

is

$$y(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} G(t,s) h(s + \nu - 1). \tag{3.2}$$

Here, for $(t,s) \in [\nu - 2, \nu + b]_{\mathbf{N}_{\nu-2}} \times [0, b]_{\mathbf{N}_0}$, $G(t,s)$ *is defined by*

$$G(t,s) = \begin{cases} \frac{(b+\nu-s-2)^{\underline{\nu-2}} t^{\underline{\nu-1}}}{\Gamma(\nu-1)-(b+\nu-1)^{\underline{\nu-2}}} + (t - s - 1)^{\underline{\nu-1}}, & 0 \le s \le t - \nu \le b, \\ \frac{(b+\nu-s-2)^{\underline{\nu-2}} t^{\underline{\nu-1}}}{\Gamma(\nu-1)-(b+\nu-1)^{\underline{\nu-2}}}, & 0 \le t - \nu < s \le b. \end{cases} \tag{3.3}$$

Proof Suppose that $y(t)$ defined on $[\nu - 2, \nu + b]_{\mathbf{N}_{\nu-2}}$ is a solution of (3.1). Using Lemma 2.4, for some constants $C_1, C_2 \in \mathbf{R}$, we have

$$y(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - s - 1)^{\underline{\nu-1}} h(s + \nu - 1) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}}, \quad t \in [\nu - 2, \nu + b]_{\mathbf{N}_{\nu-2}}. \tag{3.4}$$

By $y(\nu - 2) = 0$ and Definition 2.1, we obtain $C_2 = 0$.

Then, for all $t \in [\nu - 2, \nu + b - 1]_{\mathbf{N}_{\nu-2}}$, we obtain [21]

$$\Delta y(t) = \frac{1}{\Gamma(\nu - 1)} \sum_{s=0}^{t-(\nu-1)} (t - s - 1)^{\underline{\nu-2}} h(s + \nu - 1) + C_1(\nu - 1) t^{\underline{\nu-2}}. \tag{3.5}$$

In view of $\Delta y(\nu-2) = \Delta y(\nu+b-1)$, we have

$$C_1 = \frac{1}{\Gamma(\nu-1)[\Gamma(\nu)-(\nu-1)(b+\nu-1)^{\underline{\nu-2}}]}\sum_{s=0}^{b}(b+\nu-s-2)^{\underline{\nu-2}}h(s+\nu-1).$$

Substituting the values of C_1 and C_2 in (3.4), we have

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1) \\ &\quad + \sum_{s=0}^{b}\frac{(b+\nu-s-2)^{\underline{\nu-2}}t^{\underline{\nu-1}}}{\Gamma(\nu-1)[\Gamma(\nu)-(\nu-1)(b+\nu-1)^{\underline{\nu-2}}]}h(s+\nu-1) \\ &= \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b}G(t,s)h(s+\nu-1), \quad t\in[\nu-2,\nu+b-1]_{\mathbf{N}_{\nu-2}}. \end{aligned}$$

□

Lemma 3.2 *The function* $G(t,s)$ *given in* (3.3) *satisfies the following*:

(1) $0 \leq G(t,s) \leq \frac{D(\nu+b)^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}G(s+\nu-1,s)$, $(t,s)\in[\nu-2,\nu+b]_{\mathbf{N}_{\nu-2}}\times[0,b]_{\mathbf{N}_0}$;

(2) $\min_{t\in[\nu-1,\nu+b]_{\mathbf{N}_{\nu-1}}} G(t,s) \geq \frac{\Gamma(\nu)}{(s+\nu-1)^{\underline{\nu-1}}}G(s+\nu-1,s) > 0$.

Here,

$$\begin{aligned} D &= \max_{s\in[0,b]}\left\{1+\frac{\Gamma(\nu-1)-(\nu+b-1)^{\underline{\nu-2}}}{(\nu+b-s-2)^{\underline{\nu-2}}}\right\} \\ &= 1+\frac{\Gamma(\nu-1)-(\nu+b-1)^{\underline{\nu-2}}}{(\nu+b-2)^{\underline{\nu-2}}}. \end{aligned} \tag{3.6}$$

Proof First of all, (3.3) implies that $G(s+\nu-1,s) = \frac{(\nu+b-s-2)^{\underline{\nu-2}}(s+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu-1)-(\nu+b-1)^{\underline{\nu-2}}}$. Note that $\Gamma(\nu-1) - (\nu+b-1)^{\underline{\nu-2}} > 0$, we know $G(s+\nu-1,s) > 0$.

Second of all, by (3.3) and the definition of D in (3.6), we obtain

$$\begin{aligned} 0 \leq G(t,s) &\leq \frac{(b+\nu-s-2)^{\underline{\nu-2}}(\nu+b)^{\underline{\nu-1}}}{\Gamma(\nu-1)-(b+\nu-1)^{\underline{\nu-2}}} + (\nu+b)^{\underline{\nu-1}} \\ &\leq \frac{D(b+\nu)^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}G(s+\nu-1,s). \end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned} \min_{t\in[\nu-1,\nu+b]_{\mathbf{N}_{\nu-1}}} G(t,s) &= \min_{t\in[\nu-1,\nu+b]_{\mathbf{N}_{\nu-1}}} \frac{(b+\nu-s-2)^{\underline{\nu-2}}t^{\underline{\nu-1}}}{\Gamma(\nu-1)-(b+\nu-1)^{\underline{\nu-2}}} \\ &\geq \frac{(\nu-1)^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}\cdot\frac{(b+\nu-s-2)^{\underline{\nu-2}}(s+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu-1)-(\nu+b-1)^{\underline{\nu-2}}} \\ &= \frac{\Gamma(\nu)}{(s+\nu-1)^{\underline{\nu-1}}}G(s+\nu-1,s) > 0. \end{aligned} \tag{3.8}$$

The proof of Lemma 3.2 is completed. □

Let $\mathbf{B}$ be the collection of all functions $y:[\nu-2,\nu+b]_{\mathbf{N}_{\nu-2}}\to\mathbf{R}$ with the norm $\|y\| = \max\{|y(t)| : t\in[\nu-2,\nu+b]_{\mathbf{N}_{\nu-2}}\}$.

Define the operator $T:\mathbf{B}\to\mathbf{B}$ by

$$Ty(t)=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{b}G(t,s)f\big(s+\nu-1,y(s+\nu-1)\big). \tag{3.9}$$

In view of the continuity of f, it is easy to know that T is continuous. Furthermore, it is not difficult to verify that T maps bounded sets into bounded sets and equi-continuous sets. Therefore, in the light of the well-known Arzelá-Ascoli theorem, we know that T is a compact operator (see [11, 12]).

Let $\mathbf{E}=\{y\in\mathbf{B}|y(t)\geq 0,t\in[\nu-2,\nu+b]_{\mathbf{N}_{\nu-2}}\}$ and set

$$A=\sum_{s=0}^{b}\frac{D(\nu+b)^{\underline{\nu-1}}G(s+\nu-1,s)}{\Gamma(\nu)(s+\nu-1)^{\underline{\nu-1}}}\|a_2\|,$$

$$B=\sum_{s=0}^{b}\frac{D(\nu+b)^{\underline{\nu-1}}G(s+\nu-1,s)}{\Gamma(\nu)(s+\nu-1)^{\underline{\nu-1}}}\|a_1\|.$$

We have the following theorem.

Theorem 3.3 *Assume that* (H_1) *and* (H_2) *hold. Then system* (1.1) *has at least one positive solution provided that*

$$\sum_{s=0}^{b}\frac{D(\nu+b)^{\underline{\nu-1}}G(s+\nu-1,s)}{\Gamma(\nu)(s+\nu-1)^{\underline{\nu-1}}}\|a_2\|<1. \tag{3.10}$$

Proof Let $\Omega=\{y\in\mathbf{E}|\|y\|<r\}$ with $r=\frac{B}{1-A}>0$. If $y\in\overline{\Omega}$, that is, $\|y\|\leq r$. From (H_1), (H_2) and (3.9), we have

$$\begin{aligned}\|Ty(t)\| &= \max_{t\in[\nu-2,\nu+b]_{\mathbf{N}_{\nu-2}}}\left|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{b}G(t,s)f\big(s+\nu-1,y(s+\nu-1)\big)\right| \\ &\leq \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b}\frac{D(\nu+b)^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}G(s+\nu-1,s)\big(|a_1(t)|+|a_2(t)||y(t)|\big) \\ &\leq \sum_{s=0}^{b}\frac{(\nu+b)^{\underline{\nu-1}}}{\Gamma(\nu)}\frac{DG(s+\nu-1,s)}{(s+\nu-1)^{\underline{\nu-1}}}\|a_1\| \\ &\quad+\sum_{s=0}^{b}\frac{(\nu+b)^{\underline{\nu-1}}}{\Gamma(\nu)}\frac{DG(s+\nu-1,s)}{(s+\nu-1)^{\underline{\nu-1}}}\|a_2\|\|y\| \\ &= B+A\|y\|\leq r,\end{aligned}$$

which shows that $Ty\in\overline{\Omega}$.

Consider the eigenvalue problem

$$y=\lambda Ty,\quad \lambda\in(0,1). \tag{3.11}$$

Assume that y is a solution of (3.11), we obtain

$$\|y\| = \|\lambda Ty\| < \|Ty\| \leq r. \tag{3.12}$$

It shows that $y \notin \partial\Omega$. By Lemma 2.5, T has a fixed point in $\overline{\Omega}$. The proof is completed. □

We define the cone $\mathbf{K} \subseteq \mathbf{B}$ by

$$\mathbf{K} = \left\{ y \in \mathbf{E} : \min_{t \in [\nu-1,\nu+b]_{\mathbf{N}_{\nu-1}}} y(t) \geq \frac{\Gamma(\nu)}{D(\nu+b)^{\underline{\nu-1}}} \|y\| \right\}. \tag{3.13}$$

Lemma 3.4 *Let T be the operator defined in* (3.9) *and* $\mathbf{K}$ *be the cone defined in* (3.13). *Then* $T : \mathbf{K} \to \mathbf{K}$.

Proof Note that for each $t \in [\nu-2, \nu+b]_{\mathbf{N}_{\nu-2}}$, we have

$$\begin{aligned}
\|Ty(t)\| &= \max_{t \in [\nu-2,\nu+b]_{\mathbf{N}_{\nu-2}}} \left| \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} G(t,s) f\big(s+\nu-1, y(s+\nu-1)\big) \right| \\
&\leq \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} \frac{D(\nu+b)^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} G(s+\nu-1,s) \big| f\big(s+\nu-1, y(s+\nu-1)\big) \big| \\
&= \sum_{s=0}^{b} \frac{D(\nu+b)^{\underline{\nu-1}} G(s+\nu-1,s)}{\Gamma(\nu)(s+\nu-1)^{\underline{\nu-1}}} \big| f\big(s+\nu-1, y(s+\nu-1)\big) \big|.
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
&\min_{t \in [\nu-1,\nu+b]_{\mathbf{N}_{\nu-1}}} (Ty)(t) \\
&\quad = \min_{t \in [\nu-1,\nu+b]_{\mathbf{N}_{\nu-1}}} \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} G(t,s) f\big(s+\nu-1, y(s+\nu-1)\big) \\
&\quad \geq \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} \frac{\Gamma(\nu)}{(s+\nu-1)^{\underline{\nu-1}}} G(s+\nu-1,s) f\big(s+\nu-1, y(s+\nu-1)\big) \\
&\quad = \frac{\Gamma(\nu)}{D(\nu+b)^{\underline{\nu-1}}} \sum_{s=0}^{b} \frac{D(\nu+b)^{\underline{\nu-1}} G(s+\nu-1,s)}{\Gamma(\nu)(s+\nu-1)^{\underline{\nu-1}}} f\big(s+\nu-1, y(s+\nu-1)\big) \\
&\quad \geq \frac{\Gamma(\nu)}{D(\nu+b)^{\underline{\nu-1}}} \|Ty\|.
\end{aligned}$$

The conclusion of Lemma 3.4 holds. □

Theorem 3.5 *Suppose that conditions* (H_1), (H_3) *and* (H_4) *hold. Then problem* (1.1) *has at least one positive solution.*

Proof We have already shown $T(\mathbf{K}) \subseteq \mathbf{K}$ in Lemma 3.4. By condition (H_3), we can select $\eta_1 > 0$ sufficiently small so that both $|f(t,y)| \leq \eta_1 \|y\|$ and $\eta_1 \sum_{s=0}^{b} \frac{(\nu+b)^{\underline{\nu-1}}}{\Gamma(\nu)} \frac{DG(s+\nu-1,s)}{(s+\nu-1)^{\underline{\nu-1}}} < 1$ hold for all $t \in [\nu-1, \nu+b-1]_{\mathbf{N}_{\nu-1}}$ and $0 < y < r_1$, where $r_1 := r_1(\eta_1)$.

Let $\Omega_1 = \{y \in \mathbf{B} : \|y\| < r_1\}$. Then, for $y \in \partial\Omega_1 \cap \mathbf{K}$, we have

$$\begin{aligned}\|Ty(t)\| &= \max_{t\in[\nu-2,\nu+b]_{\mathbf{N}_{\nu-2}}} \left|\frac{1}{\Gamma(\nu)}\sum_{s=0}^{b} G(t,s) f\bigl(s+\nu-1, y(s+\nu-1)\bigr)\right| \\ &\le \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b} \frac{D(\nu+b)^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} G(s+\nu-1,s)\bigl|f\bigl(s+\nu-1, y(s+\nu-1)\bigr)\bigr| \\ &\le \sum_{s=0}^{b} \frac{D(\nu+b)^{\underline{\nu-1}} G(s+\nu-1,s)}{\Gamma(\nu)(s+\nu-1)^{\underline{\nu-1}}} \eta_1\|y\| < \|y\|.\end{aligned}$$

It implies that T is a cone contraction on $y \in \partial\Omega_1 \cap \mathbf{K}$.

On the other hand, from condition (H_4), we may select a number $\eta_2 > 0$ such that both $|f(t,y)| > \eta_2\|y\|$ and $\eta_2 \sum_{s=0}^{b} \frac{G(s+\nu-1,s)}{(s+\nu-1)^{\underline{\nu-1}}} > 1$ hold for all $t \in [\nu-1, \nu+b-1]_{\mathbf{N}_{\nu-1}}$ and $0 < y < r_2$, where $r_2 := r_2(\eta_2)$ and $r_2 > r_1 > 0$. Define $\Omega_2 = \{y \in \mathbf{B} : \|y\| < r_2\}$, we obtain

$$\begin{aligned}\|Ty(t)\| &= \max_{t\in[\nu-2,\nu+b]_{\mathbf{N}_{\nu-2}}} \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b} G(t,s)\bigl|f\bigl(s+\nu-1, y(s+\nu-1)\bigr)\bigr| \\ &\ge \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b} \min_{t\in[\nu-1,\nu+b]_{\mathbf{N}_{\nu-1}}} G(t,s)\bigl|f\bigl(s+\nu-1, y(s+\nu-1)\bigr)\bigr| \\ &> \sum_{s=0}^{b} \frac{G(s+\nu-1,s)}{(s+\nu-1)^{\underline{\nu-1}}} \eta_2\|y\| > \|y\|,\end{aligned}$$

whenever $y \in \partial\Omega_2 \cap \mathbf{K}$, so that T is a cone expansion on $\partial\Omega_2 \cap \mathbf{K}$.

In summary, we may invoke Lemma 2.6 to deduce the existence of a function $y_0 \in \mathbf{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $Ty_0 = y_0$, where y_0 is a positive solution to problem (1.1). The proof is completed. □

4 Example

Example 4.1 Consider the fractional difference boundary value problem

$$\begin{cases}\Delta^{\frac{3}{2}} y(t) = f(t+\frac{1}{2}, y(t+\frac{1}{2})), & t \in [0,3]_{\mathbf{N}_0}, \\ y(-\frac{1}{2}) = 0, & \Delta y(-\frac{1}{2}) = \Delta y(\frac{7}{2}).\end{cases} \tag{4.1}$$

Set $a_1(t) = 1$, $a_2(t) = \frac{t}{30}$, $f(t,y) = \frac{t|y|}{30} + |\sin t|$, $t \in [\frac{1}{2}, \frac{7}{2}]_{\mathbf{N}_{\frac{1}{2}}}$. We have

$$\bigl|f(t,y)\bigr| \le 1 + \frac{t}{30}|y|.$$

By a simple computation, we can obtain $D \approx 1.7750$, $\sum_{s=0}^{3} \frac{(\frac{9}{2})^{\underline{\frac{1}{2}}} G(s+\frac{1}{2},s)}{\Gamma(\frac{3}{2})(s+\frac{1}{2})^{\underline{\frac{1}{2}}}} \approx 4.7688$, $\|a_2\| = \frac{7}{60}$. Therefore, $A \approx 0.9875 < 1$. The conditions of Theorem 3.3 hold, the boundary value problem (4.1) has at least one positive solution.

Example 4.2 Consider the fractional difference boundary value problem

$$\begin{cases} \Delta^{\frac{3}{2}} y(t) = f(t+\frac{1}{2}, y(t+\frac{1}{2})), & t \in [0,11]_{\mathbf{N}_0}, \\ y(-\frac{1}{2}) = 0, & \Delta y(-\frac{1}{2}) = \Delta y(\frac{23}{2}). \end{cases} \tag{4.2}$$

Set $f(t,y) = ty^2$, $t \in [\frac{1}{2}, \frac{23}{2}]_{\mathbf{N}_{\frac{1}{2}}}$. We have

$$(1) \quad \lim_{y\to 0^+} \frac{f(t,y)}{y} = \lim_{y\to 0^+} \frac{ty^2}{y} = 0, \qquad (2) \quad \lim_{y\to +\infty} \frac{f(t,y)}{y} = \lim_{y\to +\infty} \frac{ty^2}{y} = +\infty.$$

The conditions of Theorem 3.5 hold, the boundary value problem (4.2) has at least one positive solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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