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The general meromorphic solutions of the Petviashvili equation

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Abstract

In this paper, we employ the complex method to first obtain all meromorphic exact solutions of complex Petviashvili equation, and then find all exact solutions of Petviashvili equation. The idea introduced in this paper can be applied to other non-linear evolution equations. Our results show that the complex method is simpler than other methods. Finally, we give some computer simulations to illustrate our main results.

MSC: Primary 30D35; secondary 34A05

Keywords: Petviashvili equation; exact solution; meromorphic function; elliptic function

1 Introduction and main results

In 2006 and 2008, Zhang *et al.* [1, 2] obtained abundant exact solutions of the Petviashvili equation by using the modified mapping method and the availability of symbolic computation. These solutions include the Jacobi elliptic function solutions, triangular function solutions, and soliton solutions. In this paper, we employ the complex method to obtain first all traveling meromorphic exact solutions of complex Petviashvili equation, and then find all exact solutions of the Petviashvili equation.

In order to state our main result, we need some concepts and some notation. A meromorphic function $w(z)$ means that $w(z)$ is holomorphic in the complex plane $\mathbb{C}$ except for poles. α, b, c, c_i, and c_{ij} are constants, which may be different from each other in different places. We say that a meromorphic function f belongs to the class W if f is an elliptic function, or a rational function of $e^{\alpha z}$ ($\alpha \in \mathbb{C}$), or a rational function of z.

The Petviashvili equation [1, 2] is

$$\frac{\partial}{\partial t}\left(\nabla^2\phi - \phi\right) + C_R(1+\phi)\frac{\partial \phi}{\partial t} + J\left(\phi, \nabla^2\phi\right) = 0,$$

where

$$\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}, \qquad J(A,B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x},$$

$$C_R = \frac{\beta L_R^2}{\sqrt{gH}}$$

are two-dimensional Laplace and Jacobian operators, respectively, C_R is the linear zero-dimensional phase velocity of Rossby wave, L_R is the characteristic length about x and y,

H is the average thickness of the fluid, and g is the acceleration of gravity; $\beta = \frac{\partial f}{\partial y}$, and $\frac{1}{f}$ is the characteristic value about t.

Substituting the traveling wave transformation

$$\phi = w(z), \quad z = kx + ly - \omega t$$

into the Petviashvili equation gives a non-linear ordinary differential equation

$$-\omega(k^2 + l^2)w''' + kC_R ww' + (\omega + kC_R)w' = 0,$$

and integrating it yields the auxiliary ordinary differential equation

$$-\omega(k^2 + l^2)w'' + \frac{1}{2}kC_R w^2 + (\omega + kC_R)w + d = 0, \tag{1}$$

where ω, k, l, C_R are constants.

Our main result is Theorem 1.

Theorem 1 *Equation* (1) *is integrable if and only if* $2dkC_R = -12\omega^2(k^2+l^2)^2 g_2 + (\omega + kC_k)^2$, $F^2 = 4E^3 - g_2E - g_3$, g_3, *and* E *are arbitrary. Furthermore, the general solutions of the Eq.* (1) *are of the following form.*

(I) *The elliptic general solutions are*

$$w_d(z) = \frac{12\omega(k^2 + l^2)}{kC_R}\left\{-\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + F}{\wp(z) - E}\right]^2\right\} - \frac{12\omega(k^2 + l^2)E}{kC_R} - 1 - \frac{\omega}{kC_R}, \tag{2}$$

if $2dkC_R = -12\omega^2(k^2 + l^2)^2 g_2 + (\omega + kC_k)^2$, $F^2 = 4E^3 - g_2E - g_3$, g_3 *and* E *are arbitrary.*

(II) *The simply periodic solutions are*

$$w_s(z) = \frac{3\omega(k^2 + l^2)}{kC_R}\alpha \coth^2\frac{\alpha}{2}(z - z_0) - \frac{2\omega(k^2 + l^2)}{kC_R}\alpha^2 - 1 - \frac{\omega}{kC_R}, \tag{3}$$

if $2dkC_R = -\omega^2(k^2 + l^2)^2\alpha^4 + (\omega + kC_k)^2$, $\alpha \neq 0$, $z_0 \in \mathbb{C}$.

(III) *The rational function solutions are*

$$w_R(z) = \frac{12\omega(k^2 + l^2)}{kC_R(z - z_0)^2} - 1 - \frac{\omega}{kC_R}, \tag{4}$$

if $2dkC_R = (\omega + kC_k)^2$, $z_0 \in \mathbb{C}$.

2 Preliminary lemmas and the complex method

In order to explain our complex method and give the proof of Theorem 1, we need some lemmas and results.

Lemma 1 [3, 4] *Let $k \in \mathbb{N}$, then any meromorphic solution w with at least one pole of the kth order Briot-Bouquet equation,*

$$F\left(w^{(k)}, w\right) = \sum_{i=0}^{n} P_i(w)\left(w^{(k)}\right)^i = 0,$$

belongs to W, where $P_i(w)$ are polynomials in w with constant coefficients.

Set $m \in \mathbf{N} := \{1, 2, 3, \ldots\}$, $r_j \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $r = (r_0, r_1, \ldots, r_m)$, $j = 0, 1, \ldots, m$. Define

$$M_r[w](z) := \left[w(z)\right]^{r_0}\left[w'(z)\right]^{r_1}\left[w''(z)\right]^{r_2} \cdots \left[w^{(m)}(z)\right]^{r_m}.$$

$p(r) := r_0 + r_1 + \cdots + r_m$ is called the degree of $M_r[w]$. The differential polynomial $P(w, w', \ldots, w^{(m)})$ is defined as follows:

$$P\left(w, w', \ldots, w^{(m)}\right) := \sum_{r \in I} a_r M_r[w],$$

where a_r are constants, and I is a finite index set. The total degree of $P(w, w', \ldots, w^{(m)})$ is defined by $\deg P(w, w', \ldots, w^{(m)}) := \max_{r \in I}\{p(r)\}$.

We will consider the following complex ordinary differential equations:

$$P\left(w, w', \ldots, w^{(m)}\right) = bw^n + c, \tag{5}$$

where $b \neq 0$, c are constants, $n \in \mathbf{N}$.

Definition 2 Let $p, q \in \mathbf{N}$. Suppose that Eq. (5) has a meromorphic solution w with at least one pole; then we say that Eq. (5) satisfies the weak $\langle p, q\rangle$ condition if substituting the Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k, \quad q > 0, c_{-q} \neq 0 \tag{6}$$

into Eq. (5) we can determine p distinct Laurent singular parts in the form below:

$$\sum_{k=-q}^{-1} c_k z^k.$$

Lemma 3 [5–7] *Let $p, l, m, n \in \mathbf{N}$, $\deg P(w, w^{(m)}) < n$. Suppose that the mth order Briot-Bouquet equation*

$$P\left(w^{(m)}, w\right) = bw^n + c \tag{7}$$

satisfies the weak $\langle p, q\rangle$ condition; then all meromorphic solutions w belong to the class W. If for some values of parameters such a solution w exists, then other meromorphic solutions *form a one-parametric family $w(z-z_0)$, $z_0 \in \mathbf{C}$. Furthermore each elliptic solution with a pole at $z=0$ can be written as*

$$w(z) = \sum_{i=1}^{l-1}\sum_{j=2}^{q} \frac{(-1)^j c_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+B_i}{\wp(z)-A_i}\right]^2 - \wp(z)\right)$$
$$+ \sum_{i=1}^{l-1}\frac{c_{-i1}}{2}\frac{\wp'(z)+B_i}{\wp(z)-A_i} + \sum_{j=2}^{q}\frac{(-1)^j c_{-lj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + c_0, \tag{8}$$

where c_{-ij} are given by Eq. (6), $B_i^2 = 4A_i^3 - g_2A_i - g_3$, *and* $\sum_{i=1}^{l} c_{-i1} = 0$.

Each rational function solution $w := R(z)$ is of the form

$$R(z) = \sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(z-z_i)^j} + c_0, \tag{9}$$

with l $(\leq p)$ distinct poles of multiplicity q.

Each simply periodic solution is a rational function $R(\xi)$ of $\xi = e^{\alpha z}$ $(\alpha \in \mathbf{C})$. $R(\xi)$ has l $(\leq p)$ distinct poles of multiplicity q, and it is of the form

$$R(\xi) = \sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(\xi-\xi_i)^j} + c_0. \tag{10}$$

In order to give the representations of the elliptic solutions, we need some notation and results concerning the elliptic function [6].

Let ω_1, ω_2 be two given complex numbers such that $\operatorname{Im}\frac{\omega_1}{\omega_2} > 0$, let $L = L[2\omega_1, 2\omega_2]$ be discrete subset $L[2\omega_1, 2\omega_2] = \{\omega \mid \omega = 2n\omega_1 + 2m\omega_2, n, m \in \mathbf{Z}\}$, which is isomorphic to $\mathbf{Z}\times\mathbf{Z}$. The discriminant is $\Delta = \Delta(c_1, c_2) := c_1^3 - 27c_2^2$ and

$$s_n = s_n(L) := \sum_{n\geq 3, n\in N}\frac{1}{\omega^n}.$$

The Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ is a meromorphic function with double periods $2\omega_1$, $2\omega_2$, and satisfying the equation

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{11}$$

where $g_2 = 60s_4$, $g_3 = 140s_6$ and $\Delta(g_2, g_3) \neq 0$.

On changing Eq. (11) to the form

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z)-e_1\right)\left(\wp(z)-e_2\right)\left(\wp(z)-e_3\right), \tag{12}$$

we have $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_2)$, $e_3 = \wp(\omega_1+\omega_2)$.

Inversely, given two complex numbers g_2 and g_3 such that $\Delta(g_2, g_3) \neq 0$, there exists a Weierstrass elliptic function $\wp(z)$ with double periods $2\omega_1$, $2\omega_2$ such that the above holds.

Lemma 4 [6, 8] *The Weierstrass elliptic functions $\wp(z) := \wp(z, g_2, g_3)$ have two successive degeneracies and in addition we have the following.*

(I) *Degeneracy to simply periodic functions* (*i.e.*, *rational functions of one exponential* e^{kz}) *occurs according to*

$$\wp\left(z, 3d^2, -d^3\right) = 2d - \frac{3d}{2}\coth^2\sqrt{\frac{3d}{2}}z \tag{13}$$

if one root e_j *is double* ($\Delta(g_2, g_3) = 0$).

(II) *Degeneracy to rational functions of* z *occurs according to*

$$\wp(z, 0, 0) = \frac{1}{z^2}$$

if one root e_j *is triple* ($g_2 = g_3 = 0$).

(III) *The addition formula holds according to*

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2. \tag{14}$$

By the above lemmas, we can give a new method below, called, say, the complex method, to find exact solutions of some PDEs.

Step 1. *Substituting the transform* $T : u(x, y, t) \to w(z), (x, y, t) \to z$ *into a given PDE gives a non-linear ordinary differential equation* (5) *or* (7).

Step 2. *Substitute Eq.* (6) *into Eq.* (5) *or* (7) *to determine whether the weak* $\langle p, q\rangle$ *condition holds.*

Step 3. *By the indeterminate relations* (8)-(10) *we find the elliptic, rational, and simply periodic solutions* $w(z)$ *of Eq.* (5) *or* (7) *with pole at* $z = 0$, *respectively.*

Step 4. *By Lemmas* 1 *and* 3 *we obtain all meromorphic solutions* $w(z - z_0)$.

Step 5. *Substituting the inverse transform* T^{-1} *into these meromorphic solutions* $w(z - z_0)$, *we get all exact solutions* $u(x, t)$ *of the originally given PDE.*

3 Proof of Theorem 1

Substituting (6) into Eq. (1) we have $q = 2$, $p = 1$, $c_{-2} = \frac{12\omega(k^2+l^2)}{kC_R}$, $c_{-1} = 0$, $c_0 = -1 - \frac{\omega}{kC_R}$, $c_1 = 0$, $c_2 = \frac{(\omega+kC_R)^2-2dkC_R}{20\omega(k^2+l^2)kC_R}$, $c_3 = 0$, c_4 is arbitrary.

Hence, Eq. (1) satisfies the weak $\langle 1, 2\rangle$ condition and is a second-order Briot-Bouquet differential equation. Obviously, Eq. (1) satisfies the dominant condition. So, by Lemma 3, we know that all meromorphic solutions of Eq. (1) belong to W. Now we will give the forms of all meromorphic solutions of Eq. (1).

By Eq. (9), we infer that the indeterminate rational solutions of Eq. (1) with pole at $z = 0$ have the form of

$$R_1(z) = \frac{c_2}{z^2} + \frac{c_1}{z} + c_{10}.$$

Substituting $R_1(z)$ into Eq. (1), we get

$$R_1(z) = \frac{12\omega(k^2 + l^2)}{kC_R z^2} - 1 - \frac{\omega}{kC_R}.$$

Here $2dkC_R = (\omega + kC_R)^2$. Thus for all rational solutions of Eq. (1)

$$R(z) = \frac{12\omega(k^2 + l^2)}{kC_R(z - z_0)^2} - 1 - \frac{\omega}{kC_R}, \tag{15}$$

where $2dkC_R = (\omega + kC_R)^2$, $z_0 \in \mathbb{C}$.

In order to have simply periodic solutions, set $\xi = e^{\alpha z}$, put $w = R(\xi)$ into Eq. (1); then

$$-\omega(k^2 + l^2)\big[\xi R' + \xi^2 R''\big] + (w + kC_R)R + \frac{1}{2}kC_R R^2 + d = 0. \tag{16}$$

Substituting

$$R_2(\xi) = \frac{c_2}{(\xi - 1)^2} + \frac{c_1}{\xi - 1} + c_{10}$$

into Eq. (16), we obtain

$$R_2(z) = \frac{12\omega(k^2 + l^2)\alpha^2}{kC_R(\xi - 1)^2} + \frac{12\omega(k^2 + l^2)\alpha^2}{kC_R(\xi - 1)} + \frac{\omega(k^2 + l^2)\alpha^2}{kC_R} - 1 - \frac{\omega}{kC_R}.$$

Here $2dkC_R = (\omega + kC_R)^2 - \omega^2(k^2 - l^2)^2\alpha^2$. Substituting $\xi = e^{\alpha z}$ into the above relation, and then we get simply periodic solutions of Eq. (1) with pole at $z = 0$:

$$\begin{aligned} w_{s0}(z) &= \frac{12\omega(k^2 + l^2)\alpha^2}{kC_R(e^{\alpha z} - 1)^2} + \frac{12\omega(k^2 + l^2)\alpha^2}{kC_R(e^{\alpha z} - 1)} + \frac{\omega(k^2 + l^2)\alpha^2}{kC_R} - 1 - \frac{\omega}{kC_R} \\ &= \frac{12\omega(k^2 + l^2)\alpha^2 e^{\alpha z}}{kC_R(e^{\alpha z} - 1)^2} + \frac{\omega(k^2 + l^2)\alpha^2}{kC_R} - 1 - \frac{\omega}{kC_R} \\ &= w_s(z) = \frac{3\omega(k^2 + l^2)}{kC_R}\alpha \coth^2\frac{\alpha}{2}(z) - \frac{2\omega(k^2 + l^2)}{kC_R}\alpha^2 - 1 - \frac{\omega}{kC_R}. \end{aligned}$$

So all simply periodic solutions of Eq. (1) are obtained by

$$w_s(z) = \frac{3\omega(k^2 + l^2)}{kC_R}\alpha \coth^2\frac{\alpha}{2}(z - z_0) - \frac{2\omega(k^2 + l^2)}{kC_R}\alpha^2 - 1 - \frac{\omega}{kC_R}, \tag{17}$$

where $2dkC_R = -\omega^2(k^2 + l^2)^2\alpha^4 + (\omega + kC_k)^2$, $\alpha \neq 0$, $z_0 \in \mathbb{C}$.

From Eq. (8) of Lemma 3, we have indeterminate relations of the elliptic solutions of Eq. (1) with pole at $z = 0$,

$$w_{d0}(z) = c_{-2}\wp(z) + c_0.$$

Putting $w_{d0}(z)$ into Eq. (1), we obtain

$$w_{d0}(z) = \frac{12\omega(k^2 + l^2)}{kC_R}\wp(z) - 1 - \frac{\omega}{kC_R}.$$

Here $42dkC_R = -12\omega^2(k^2 + l^2)^2 g_2 + (\omega + kC_k)^2$. Therefore, for all elliptic solutions of Eq. (1)

$$w_d(z) = \frac{12\omega(k^2 + l^2)}{kC_R}\wp(z - z_0) - 1 - \frac{\omega}{kC_R},$$

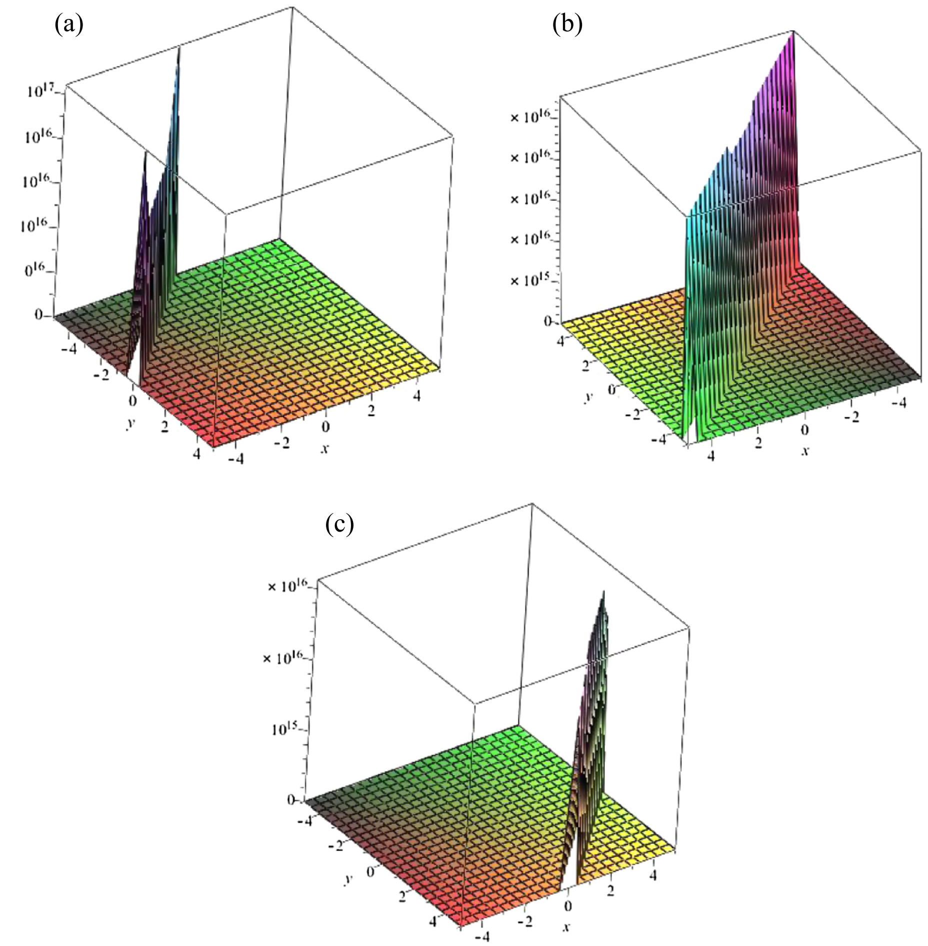

Figure 1 The solution of the Petviashvili equation corresponding to w_R, (a) $t = -5$, (b) $t = 0$, (c) $t = 5$.

where $z_0 \in \mathbb{C}$. Making use of the addition formula of Lemma 4, we rewrite it in the form

$$w_d(z) = \frac{12\omega(k^2 + l^2)}{kC_R}\left\{-\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + F}{\wp(z) - E}\right]^2\right\} - \frac{12\omega(k^2 + l^2)E}{kC_R} - 1 - \frac{\omega}{kC_R}. \tag{18}$$

Here, $2dkC_R = -12\omega^2(k^2 + l^2)^2 g_2 + (\omega + kC_k)^2$, $F^2 = 4E^3 - g_2E - g_3$, g_3, and E are arbitrary.

4 Computer simulations for new solutions

In this section, we give some computer simulations to illustrate our main results. Here we take the new rational solutions $w_R(z)$ and simply periodic solutions $w_s(z)$ to further analyze their properties by Figures 1 and 2.

(1) Take $k = l = 1$, $\omega = C_R = -1$, $d = 2$, $z_0 = 0$ in $w_R(z)$.

(2) Take $k = l = 1$, $\omega = C_R = -1$, $d = 2$, $z_0 = 0$, $\alpha = 1$ in $w_s(z)$.

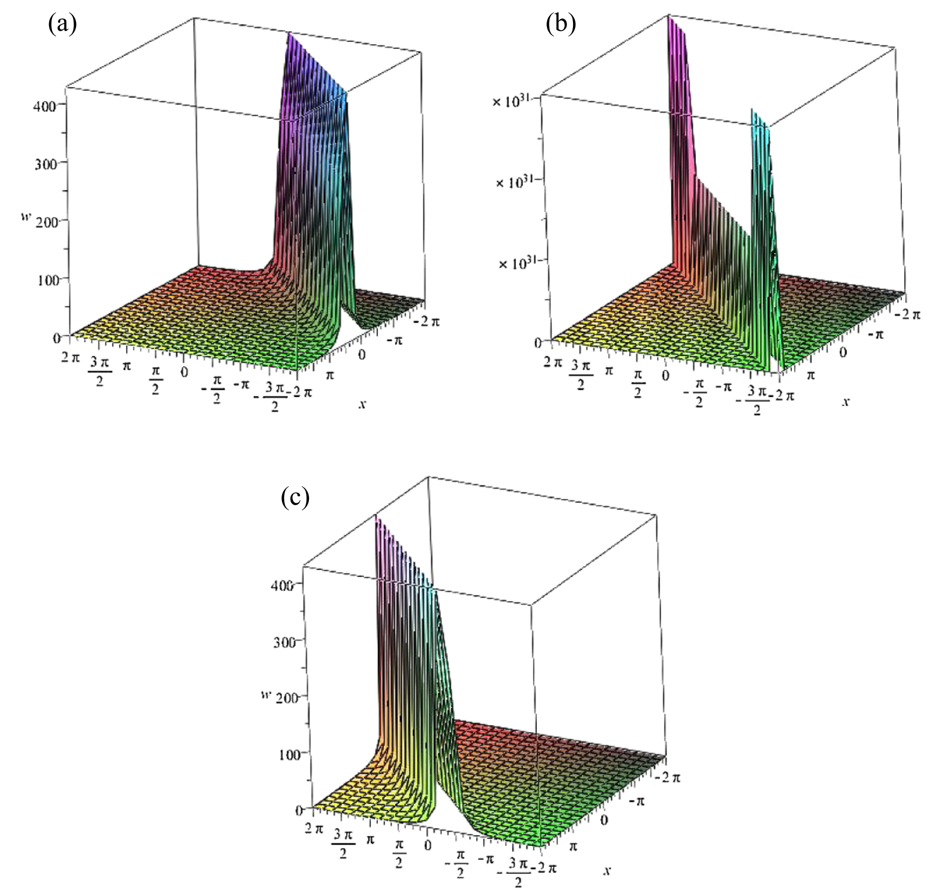

Figure 2 The solution of the Petviashvili equation corresponding to w_s, (a) $t = -5$, (b) $t = 0$, (c) $t = 5$.

5 Conclusions

The complex method is a very important tool in finding the traveling wave exact solutions of non-linear evolution equations such as the Petviashvili equation. In this paper, we employ the complex method to obtain all meromorphic exact solutions of the complex variant Eq. (1); then we find all traveling wave exact solutions of the Petviashvili equation. The idea introduced in this paper can be applied to other non-linear evolution equations. Our result shows that the complex method is simpler than other methods.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

ZH and WY carried out the design of the study and performed the analysis. JL participated in its design and coordination. All authors read and approved the final manuscript.

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