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Analysis of an SIS epidemic model with treatment

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Abstract

An SIS epidemic model with saturated incidence rate and treatment is considered. According to different recovery rates, we use differential stability theory and qualitative theory to analyze the various kinds of endemic equilibria and disease-free equilibrium. Finally, we get complete configurations of different endemic equilibria and disease-free equilibrium.

Keywords: epidemic model; incidence rates; treatment; globally asymptotically stable

1 Introduction and model

Infectious diseases have tremendous influence on human life and will bring huge panic and disaster to mankind once out of control. Every year millions of human beings suffer from or die of various infectious diseases. In order to predict the spreading of infectious diseases, many epidemic models have been proposed and analyzed in recent years (see [1–13]). Some new conditions should be considered into SIS model to extend the results.

Li *et al.* (see [13]) studied an SIS model with bilinear incidence rate βSI and treatment. The model takes into account the medical conditions. The recovery of the infected rate is divided into natural and unnatural recovery rates. Because of the medical conditions, when the number of infected persons reaches a certain amount I_0, the unnatural recovery rate will be a fixed value δI_0. The study of this model should be divided into two cases to discuss with $I \leq I_0$ and $I > I_0$. In this paper, we study an SIS model with saturated incidence rate $\frac{\beta SI}{1+\alpha S}$ and treatment, and we extend some recent results.

By a saturated incidence rate $\frac{\beta SI}{1+\alpha S}$, we consider an SIS epidemic model which consists of the susceptible individuals $S(t)$, the infectious individuals $I(t)$ and the total population $N(t)$ at time t:

$$\begin{cases} \frac{dS}{dt} = A - dS - \frac{\beta SI}{1+\alpha S} + \gamma I + T(I), \\ \frac{dI}{dt} = \frac{\beta SI}{1+\alpha S} - (d + \varepsilon + \gamma)I - T(I), \end{cases} \tag{1.1}$$

$$N(t) = S(t) + I(t),$$

where $T(I) = \begin{cases} \delta I, & \text{if } 0 \leq I \leq I_0, \\ k, & \text{if } I \geq I_0 \end{cases}$ $(k = \delta I_0)$ is the rate at which infected individuals are treated; A is the recruitment rate of individuals (including newborns and immigrants) into the susceptible population; $\frac{\beta SI}{1+\alpha S}$ is the nonlinear incidence rate; d is the natural death rate;

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γ is the rate at which infected individuals are recovered; ε is the disease-related death rate, A, d, γ, δ, ε, α are all positive numbers.

Thus, if $0 \le I \le I_0$, model (1.1) implies

$$\begin{cases} \frac{dS}{dt} = A - dS - \frac{\beta SI}{1+\alpha S} + \gamma I + \delta I, \\ \frac{dI}{dt} = \frac{\beta SI}{1+\alpha S} - (d + \varepsilon + \gamma)I - \delta I. \end{cases} \tag{1.2}$$

If $I > I_0$, model (1.1) implies

$$\begin{cases} \frac{dS}{dt} = A - dS - \frac{\beta SI}{1+\alpha S} + \gamma I + k, \\ \frac{dI}{dt} = \frac{\beta SI}{1+\alpha S} - (d + \varepsilon + \gamma)I - k. \end{cases} \tag{1.3}$$

2 Existence of equilibria

Now, we study equilibria of model (1.1). Steady states of model (1.1) satisfy the following equations:

$$\begin{cases} A - dS - \frac{\beta SI}{1+\alpha S} + \gamma I + T(I) = 0, \\ \frac{\beta SI}{1+\alpha S} - (d + \varepsilon + \gamma)I - T(I) = 0. \end{cases} \tag{2.1}$$

We easily see that model (1.1) has a disease-free equilibrium $P_0(\frac{A}{d}, 0)$.

If $0 < I \le I_0$, it follows from equation (2.1) that

$$\begin{cases} A - dS - \frac{\beta SI}{1+\alpha S} + \gamma I + \delta I = 0, \\ \frac{\beta SI}{1+\alpha S} - (d + \varepsilon + \gamma)I - \delta I = 0, \end{cases} \tag{2.2}$$

and if $I > I_0$, we get

$$\begin{cases} A - dS - \frac{\beta SI}{1+\alpha S} + \gamma I + k = 0, \\ \frac{\beta SI}{1+\alpha S} - (d + \varepsilon + \gamma)I - k = 0. \end{cases} \tag{2.3}$$

From two equations of (2.2), we have

$$S = \frac{A - (d + \varepsilon)I}{d}. \tag{2.4}$$

By substituting (2.4) into the second equation of (2.2), we obtain the following equations:

$$\begin{aligned} &\frac{\beta A - \beta(d + \varepsilon)I}{d + A\alpha - \alpha(d + \varepsilon)I} - (d + \varepsilon + \gamma + \delta) = 0, \\ &(d + \varepsilon)\bigl[-\beta + \alpha(d + \varepsilon + \gamma + \delta)\bigr]I = (d + A\alpha)(d + \varepsilon + \gamma + \delta) - \beta A. \end{aligned} \tag{2.5}$$

Let $R_0 = \frac{\beta A}{(d+A\alpha)(d+\varepsilon+\gamma+\delta)}$. We study equation (2.5) as follows.

If $\alpha(d + \varepsilon + \gamma + \delta) > \beta$, $R_0 < 1$ holds if and only if

$$I = \frac{(d + A\alpha)(d + \varepsilon + \gamma + \delta)(1 - R_0)}{(d + \varepsilon)[-\beta + \alpha(d + \varepsilon + \gamma + \delta)]} > 0$$

with

$$S = \frac{1}{d}\left[A - (d+\varepsilon)\frac{(d+A\alpha)(d+\varepsilon+\gamma+\delta)(1-R_0)}{(d+\varepsilon)[-\beta+\alpha(d+\varepsilon+\gamma+\delta)]}\right] = -\frac{d+\varepsilon+\gamma+\delta}{-\beta+\alpha(d+\varepsilon+\gamma+\delta)} < 0.$$

So this case need not be considered.

If $\alpha(d+\varepsilon+\gamma+\delta) < \beta$, $R_0 > 1$ holds if and only if

$$I = \frac{(d+A\alpha)(d+\varepsilon+\gamma+\delta)(R_0-1)}{(d+\varepsilon)[\beta-\alpha(d+\varepsilon+\gamma+\delta)]} > 0$$

with

$$S = \frac{d+\varepsilon+\gamma+\delta}{\beta-\alpha(d+\varepsilon+\gamma+\delta)} > 0.$$

Then we get a positive equilibrium $P_*(S_*, I_*)$ of (1.2), where

$$I_* = \frac{(d+A\alpha)(d+\varepsilon+\gamma+\delta)(R_0-1)}{(d+\varepsilon)[\beta-\alpha(d+\varepsilon+\gamma+\delta)]},$$
$$S_* = \frac{d+\varepsilon+\gamma+\delta}{\beta-\alpha(d+\varepsilon+\gamma+\delta)}.$$

Furthermore, if $0 < I_* \le I_0$, $P_*(S_*, I_*)$ is an endemic equilibrium of model (1.1) when

$$1 < R_0 \le 1 + \frac{(d+\varepsilon)[\beta-\alpha(d+\varepsilon+\gamma+\delta)]}{(d+A\alpha)(d+\varepsilon+\gamma+\delta)} I_0.$$

Define

$$n_0 = 1 + \frac{(d+\varepsilon)[\beta-\alpha(d+\varepsilon+\gamma+\delta)]}{(d+A\alpha)(d+\varepsilon+\gamma+\delta)} I_0.$$

Therefore model (1.1) has a disease-free equilibrium $P_0(\frac{A}{d}, 0)$ and has an endemic equilibrium $P_*(S_*, I_*)$ except the disease-free equilibrium $P_0(\frac{A}{d}, 0)$ when $1 < R_0 \le n_0$.

By substituting (2.4) into the second equation of (2.3), we obtain the following equation:

$$(d+\varepsilon)\big[\beta-\alpha(d+\varepsilon+\gamma)\big]I^2 + \big[(A\alpha+d)(d+\varepsilon+\gamma) - \beta A - k\alpha(d+\varepsilon)\big]I + k(A\alpha+d) = 0. \quad (2.6)$$

Let $b = (A\alpha+d)(d+\varepsilon+\gamma) - \beta A - k\alpha(d+\varepsilon)$. We study equation (2.6) as follows.

If $\beta = \alpha(d+\varepsilon+\gamma)$, (2.6) has a positive root if $b < 0$, then

$$I = \frac{k(A\alpha+d)}{k\alpha(d+\varepsilon) - d(d+\varepsilon+\gamma)},$$
$$S = -\frac{k(d+\varepsilon) + A(d+\varepsilon+\gamma)}{k\alpha(d+\varepsilon) - d(d+\varepsilon+\gamma)} < 0.$$

So this case need not be considered.

If $\beta < \alpha(d+\varepsilon+\gamma)$, it follows from (2.6) that

$$(d+\varepsilon)\big[\alpha(d+\varepsilon+\gamma) - \beta\big]I^2 + \big[\beta A + k\alpha(d+\varepsilon) - (A\alpha+d)(d+\varepsilon+\gamma)\big]I - k(A\alpha+d) = 0. \quad (2.7)$$

Then

$$\Delta_1 = \bigl[\beta A + k\alpha(d+\varepsilon) - (A\alpha + d)(d+\varepsilon+\gamma)\bigr]^2 + 4k(d+\varepsilon)\bigl[\alpha(d+\varepsilon+\gamma) - \beta\bigr](A\alpha+d) > 0.$$

Denoting two roots of (2.7) by I_1 and I_2, we have

$$\begin{aligned} I_1 + I_2 &= -\frac{\beta A + k\alpha(d+\varepsilon) - (A\alpha+d)(d+\varepsilon+\gamma)}{(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]}, \\ I_1 \cdot I_2 &= \frac{-k(A\alpha+d)}{(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]} < 0. \end{aligned}$$

So (2.7) has only one positive root, denote it by I_1,

$$\begin{aligned} I_1 &= \frac{b + \sqrt{\Delta_1}}{2(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]}, \\ S_1 &= \frac{1}{d}\bigl[A - (d+\varepsilon)I_1\bigr]. \end{aligned}$$

Then $S_1 > 0$ holds only if

$$R_0 < \frac{(A\alpha - d)(d+\varepsilon+\gamma) + k\alpha(d+\varepsilon) - \sqrt{\Delta_1}}{(A\alpha+d)(d+\varepsilon+\gamma+\delta)}.$$

Define

$$n_1 = \frac{(A\alpha - d)(d+\varepsilon+\gamma) + k\alpha(d+\varepsilon) - \sqrt{\Delta_1}}{(A\alpha+d)(d+\varepsilon+\gamma+\delta)}.$$

The point $P_1(S_1, I_1)$ satisfies (2.3), then $I_1 > I_0$, *i.e.*,

$$\frac{b + \sqrt{\Delta_1}}{2(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]} > I_0,$$

we have

$$\sqrt{\Delta_1} > -b + 2(d+\varepsilon)\bigl[\alpha(d+\varepsilon+\gamma) - \beta\bigr]I_0. \tag{2.8}$$

Then $-b + 2(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]I_0 < 0$.

And

$$R_0 < 1 - \frac{\delta(A\alpha+d) + k\alpha(d+\varepsilon)}{(A\alpha+d)(d+\varepsilon+\gamma+\delta)} - \frac{2(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]I_0}{(d+\varepsilon+\gamma+\delta)(A\alpha+d)}.$$

Define

$$n_2 = 1 - \frac{\delta(A\alpha+d) + k\alpha(d+\varepsilon)}{(A\alpha+d)(d+\varepsilon+\gamma+\delta)} - \frac{2(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]I_0}{(d+\varepsilon+\gamma+\delta)(A\alpha+d)}.$$

Then (2.8) holds only if

$$\begin{cases} -b + 2(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]I_0 \geq 0, \\ \Delta_1 \geq \{-b + 2(d+\varepsilon)[\alpha(d+\varepsilon+\gamma)-\beta]I_0\}^2. \end{cases}$$

Then

$$n_2 < R_0 \leq 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\gamma+\delta)} + \frac{k}{(d+\varepsilon+\gamma+\delta)I_0} + \frac{(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]I_0}{(A\alpha+d)(d+\varepsilon+\gamma+\delta)},$$

define

$$n_3 = 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\gamma+\delta)} + \frac{k}{(d+\varepsilon+\gamma+\delta)I_0} + \frac{(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]I_0}{(A\alpha+d)(d+\varepsilon+\gamma+\delta)}.$$

So, if $R_0 \leq n_3$ and $R_0 \leq n_1$, $P_1(S_1, I_1)$ is an endemic equilibrium, where

$$I_1 = \frac{b + \sqrt{\Delta_1}}{2(d+\varepsilon)[\alpha(d+\varepsilon+\gamma) - \beta]}, \qquad S_1 = \frac{1}{d}\bigl[A - (d+\varepsilon)I_1\bigr].$$

If $\beta > \alpha(d+\varepsilon+\gamma)$, it is easy to see that (2.6) has no positive root if $b \geq 0$.

If $b < 0$,

$$\Delta_2 = b^2 - 4\mathrm{k}(d+\varepsilon)(A\alpha+d)\bigl[\beta - \alpha(d+\varepsilon+\gamma)\bigr],$$
$$b = -R_0(d+\varepsilon+\gamma+\delta)(A\alpha+d) + (A\alpha+d)(d+\varepsilon+\delta+\gamma) - k\alpha(d+\varepsilon) - \delta(A\alpha+d).$$

Then $\Delta_2 \geq 0$ implies $b^2 \geq 4k(d+\varepsilon)(A\alpha+d)[\beta - \alpha(d+\varepsilon+\gamma)]$, we get

$$R_0 \leq 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} - \frac{2\sqrt{k(A\alpha+d)(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]}}{(A\alpha+d)(d+\varepsilon+\delta+\gamma)}$$

or

$$R_0 \geq 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{2\sqrt{k(A\alpha+d)(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]}}{(A\alpha+d)(d+\varepsilon+\delta+\gamma)}.$$

Define

$$n_4 = 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} - \frac{2\sqrt{k(A\alpha+d)(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]}}{(A\alpha+d)(d+\varepsilon+\delta+\gamma)},$$
$$n_5 = 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{2\sqrt{k(A\alpha+d)(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]}}{(A\alpha+d)(d+\varepsilon+\delta+\gamma)}.$$

At the same time, $b < 0$ holds if and only if $R_0 > 1 - \frac{\delta(A\alpha+d)+k\alpha(d+\varepsilon)}{(A\alpha+d)(d+\varepsilon+\delta+\gamma)}$.

Define

$$n_6 = 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)}.$$

Therefore, if $R_0 \geq n_5$, we have $b < 0$ and $\Delta_2 \geq 0$, then (2.6) has two positive roots I_2, I_3, where

$$I_2 = \frac{-b - \sqrt{\Delta_2}}{2(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]}, \qquad I_3 = \frac{-b + \sqrt{\Delta_2}}{2(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]}.$$

Then $S_i = \frac{1}{d}[A - (d+\varepsilon)I_i] > 0$ $(i = 2, 3)$ holds only if

$$R_0 < 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{2A[\beta - \alpha(d+\varepsilon+\gamma)] + \sqrt{\Delta_2}}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)},$$
$$R_0 < 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon) + \sqrt{\Delta_2}}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{2A[\beta - \alpha(d+\varepsilon+\gamma)]}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)}.$$

Define

$$n_7 = 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{2A[\beta - \alpha(d+\varepsilon+\gamma)] + \sqrt{\Delta_2}}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)},$$
$$n_8 = 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon) + \sqrt{\Delta_2}}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{2A[\beta - \alpha(d+\varepsilon+\gamma)]}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)}.$$

It is easy to see that $n_8 < n_7$, which implies that (2.6) has two positive equilibrium points $P_2(S_2, I_2)$, $P_3(S_3, I_3)$ if $R_0 < n_8$, (2.6) has only one positive equilibrium point $P_2(S_2, I_2)$ if $n_8 < R_0 < n_7$, (2.6) has no positive equilibrium point if $R_0 \geq n_7$.

Now, we consider the conditions for $I_i > I_0$ $(i = 2, 3)$.

$$\begin{aligned} I_2 > I_0 \quad &\Rightarrow \quad -b - \sqrt{\Delta_2} > 2(d+\varepsilon)\bigl[\beta - \alpha(d+\varepsilon+\gamma)\bigr]I_0 \\ &\Rightarrow \quad b + 2(d+\varepsilon)\bigl[\beta - \alpha(d+\varepsilon+\gamma)\bigr]I_0 < 0. \end{aligned}$$

Then

$$R_0 > 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{2(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]I_0}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)}.$$

Define

$$n_9 = 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{2(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]I_0}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)}.$$

Furthermore,

$$-b - \sqrt{\Delta_2} > 2(d+\varepsilon)\bigl[\beta - \alpha(d+\varepsilon+\gamma)\bigr]I_0 \quad \Rightarrow \quad \bigl\{b + 2(d+\varepsilon)\bigl[\beta - \alpha(d+\varepsilon+\gamma)\bigr]I_0\bigr\}^2 > \Delta,$$

i.e.,

$$R_0 < 1 - \frac{\delta(A\alpha + d) + k\alpha(d+\varepsilon)}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)} + \frac{k}{(d+\varepsilon+\delta+\gamma)I_0} + \frac{(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]I_0}{(A\alpha + d)(d+\varepsilon+\delta+\gamma)}.$$

Therefore, if $n_9 < R_0 < n_3$, $I_2 > I_0$ holds.

Similarly, if $I_3 > I_0$,

$$b + 2(d+\varepsilon)\bigl[\beta - \alpha(d+\varepsilon+\gamma)\bigr]I_0 \leq 0$$

or

$$\begin{cases} b + 2(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]I_0 > 0, \\ \Delta > \{2(d+\varepsilon)[\beta - \alpha(d+\varepsilon+\gamma)]I_0 + b\}^2, \end{cases}$$

we get $R_0 \leq n_9$ or $R_0 > \max(n_3, n_9)$.

From the above discussion, we get the following conclusions.

Theorem 2.1 *If $R_0 < 1$, model* (1.2) *has only one disease-free equilibrium $P_0(\frac{A}{d}, 0)$; if $R_0 > 1$, model* (1.2) *has a unique endemic equilibrium $P_*(S_*, I_*)$ except the disease-free equilibrium $P_0(\frac{A}{d}, 0)$; if $1 < R_0 < n_0$, $P_*(S_*, I_*)$ is a unique endemic equilibrium of model* (1.1).

Theorem 2.2 *If $\beta < \alpha(d + \varepsilon + \gamma)$, then $P_1(S_1, I_1)$ is a unique endemic equilibrium of model* (1.3) *if $R_0 \le n_1$; $P_1(S_1, I_1)$ is a unique endemic equilibrium of model* (1.1) *if $R_0 \le n_1$ and $R_0 \le n_2$.*

If $\beta > \alpha(d + \varepsilon + \gamma)$, model (1.3) *has two positive equilibrium points $P_2(S_2, I_2)$, $P_3(S_3, I_3)$ if $R_0 < n_8$; model* (1.3) *has only one positive equilibrium point $P_2(S_2, I_2)$ if $n_8 < R_0 < n_7$; model* (1.3) *has no positive point if $R_0 \ge n_7$; $P_2(S_2, I_2)$ is an endemic equilibrium of model* (1.1) *if $n_9 < R_0 < n_3$; $P_3(S_3, I_3)$ is an endemic equilibrium of model* (1.1) *if $R_0 < n_9$ or $n_9 < R_0 < n_3$.*

If $\beta = \alpha(d + \varepsilon + \gamma)$, model (1.3) *has no endemic equilibrium.*

3 Stability of equilibria

Theorem 3.1 *The disease-free equilibrium $P_0(\frac{A}{d}, 0)$ is stable if $R_0 < 1$ and is a saddle point if $R_0 > 1$; the endemic equilibrium $P_*(S_*, I_*)$ is a stable node if it exists; the endemic equilibrium $P_1(S_1, I_1)$ is a stable node if it exists; if the endemic equilibrium points $P_2(S_2, I_2)$, $P_3(S_3, I_3)$ exist, then $P_2(S_2, I_2)$ is a stable node of model* (1.1) *if $\frac{-b-\sqrt{\Delta_2}}{2(d+\alpha)[\beta-\alpha(d+\alpha+\gamma)]}D_1 + D_2 > 0$ and $P_3(S_3, I_3)$ is a stable node of model* (1.1) *if $\frac{-b+\sqrt{\Delta_2}}{2(d+\alpha)[\beta-\alpha(d+\alpha+\gamma)]}D_1 + D_2 > 0$.*

Proof The Jacobi matrix of model (1.2) is

$$J = \begin{bmatrix} -d - \frac{\beta I}{(1+\alpha S)^2} & -\frac{\beta S}{1+\alpha S} + \delta + \gamma \\ \frac{\beta I}{(1+\alpha S)^2} & \frac{\beta S}{1+\alpha S} - (d + \varepsilon + \delta + \gamma) \end{bmatrix},$$

then

$$J(P_0) = \begin{bmatrix} -d & -\frac{\beta A}{d+\alpha A} + \gamma + \delta \\ 0 & \frac{\beta A}{d+\alpha A} - (d + \varepsilon + \gamma + \delta) \end{bmatrix},$$

$$\det J(P_0) = d(d + \varepsilon + \delta + \gamma)(1 - R_0),$$

$$\operatorname{tr} J(P_0) = -d - (d + \varepsilon + \delta + \gamma)(1 - R_0).$$

Thus, $P_0(\frac{A}{d}, 0)$ is a stable node if $R_0 < 1$, and is a saddle point if $R_0 > 1$.

For $P_*(S_*, I_*)$,

$$J(P_*) = \begin{bmatrix} -d - \frac{\beta I_*}{(1+\alpha S_*)^2} & -\frac{\beta S_*}{1+\alpha S_*} + \delta + \gamma \\ \frac{\beta I_*}{(1+\alpha S_*)^2} & \frac{\beta S_*}{1+\alpha S_*} - (d + \varepsilon + \delta + \gamma) \end{bmatrix} = \begin{bmatrix} -d - \frac{\beta I_*}{(1+\alpha S_*)^2} & -(d + \varepsilon) \\ \frac{\beta I_*}{(1+\alpha S_*)^2} & 0 \end{bmatrix},$$

$$\operatorname{tr} J(P_*) = -d - \frac{\beta I_*}{(1 + \alpha S_*)^2} < 0,$$

$$\det J(P_*) = (d + \varepsilon)\frac{\beta I_*}{(1 + \alpha S_*)^2} > 0.$$

So, $P_*(S_*, I_*)$ is a stable node if it exists.

The Jacobi matrix of model (1.3) is

$$J = \begin{bmatrix} -d - \frac{\beta I}{(1+\alpha S)^2} & -\frac{\beta S}{1+\alpha S} + \gamma \\ \frac{\beta I}{(1+\alpha S)^2} & \frac{\beta S}{1+\alpha S} - (d+\varepsilon+\gamma) \end{bmatrix}.$$

Then

$$\det J(P_1) = d\alpha\bigl[\alpha(d+\varepsilon+\gamma) - \beta\bigr]S_1^2 + 2d\bigl[\alpha(d+\varepsilon+\gamma) - \beta\bigr]S_1 + d(d+\varepsilon+\gamma) + \beta A.$$

If $\beta < \alpha(d+\varepsilon+\gamma)$, $P_1(S_1, I_1)$ does not exist, then $\det J(P_1) > 0$.

Because

$$\operatorname{tr} J(P_1) = \frac{1}{(1+\alpha S_1)^2}\bigl[-\beta I_1 + \beta S_1(1+\alpha S_1) - (1+\alpha S_1)^2(2d+\varepsilon+\gamma)\bigr] < 0,$$

then $P_1(S_1, I_1)$ is a stable node if it exists.

Consider points $P_2(S_2, I_2)$, $P_3(S_3, I_3)$,

$$\begin{aligned}
J(P_2) &= \begin{bmatrix} -d - \frac{\beta I_2}{(1+\alpha S_2)^2} & -\frac{\beta S_2}{1+\alpha S_2} + \gamma \\ \frac{\beta I_2}{(1+\alpha S_2)^2} & \frac{\beta S_2}{1+\alpha S_2} - (d+\varepsilon+\gamma) \end{bmatrix}, \\
\operatorname{tr}(P_2) &= -d - \frac{\beta I_2}{(1+\alpha S_2)^2} - (d+\varepsilon+\gamma) + \frac{\beta S_2}{1+\alpha S_2} \\
&= \frac{1}{(1+\alpha S_2)^2(d+\varepsilon)}\bigl[-\beta A + \beta d S_2 + \beta S_2(1+\alpha S_2)(d+\varepsilon) \\
&\quad - (1+\alpha S_2)^2(2d+\varepsilon+\gamma)(d+\varepsilon)\bigr] \\
&\le \bigl(\beta(d+\varepsilon)S_2 + \beta\alpha(d+\varepsilon)S_2^2 - \bigl[(2d+\varepsilon+\gamma)(d+\varepsilon) + 2\alpha(2d+\varepsilon+\gamma)(d+\varepsilon)S_2 \\
&\quad + (2d+\varepsilon+\gamma)(d+\varepsilon)\alpha^2 S_2^2\bigr]\bigr)/\bigl((1+\alpha S_2)^2(d+\varepsilon)\bigr)
\end{aligned}$$

If $\beta > \alpha(d+\varepsilon+\gamma)$, then $\operatorname{tr}(P_2) < 0$,

$$\begin{aligned}
\det J(P_2) &= d\alpha\bigl[\alpha(d+\varepsilon+\gamma) - \beta\bigr]S_2^2 + 2d\bigl[\alpha(d+\varepsilon+\gamma) - \beta\bigr]S_2 + d(d+\varepsilon+\gamma) + \beta A \\
&= d\alpha\bigl[\alpha(d+\varepsilon+\gamma) - \beta\bigr]\frac{[A-(d+\varepsilon)I_2]^2}{d^2} \\
&\quad + 2d\bigl[\alpha(d+\varepsilon+\gamma) - \beta\bigr]\frac{A-(d+\varepsilon)I_2}{d} + d(d+\varepsilon+\gamma) + \beta A.
\end{aligned}$$

Because I_2 satisfies equation (2.6), we get

$$\det J(P_2) = D_1 I_1 + D_2 = \frac{-b - \sqrt{\Delta_2}}{2(d+\alpha)[\beta - \alpha(d+\alpha+\gamma)]}D_1 + D_2,$$

where

$$\begin{aligned}
D_1 &= \alpha(A\alpha + d)(d+\alpha+\gamma) - k\alpha^2(d+\alpha) - \alpha\beta A \\
&\quad + 2\alpha(d+\alpha)A + 2d(d+\alpha)\bigl[\beta - \alpha(d+\alpha+\gamma)\bigr], \\
D_2 &= -\alpha\bigl[\beta - \alpha(d+\alpha+\gamma)\bigr]A^2 - 2d\bigl[\beta - \alpha(d+\alpha+\gamma)\bigr]A + d^2(d+\alpha+\gamma) + k(A\alpha + d).
\end{aligned}$$

If $\frac{-b-\sqrt{\Delta_2}}{2(d+\alpha)[\beta-\alpha(d+\alpha+\gamma)]}D_1+D_2>0$, $\det J(P_2)>0$ holds, then $P_2(S_2,I_2)$ is a stable node if $\beta>\alpha(d+\varepsilon+\gamma)$ and $\frac{-b-\sqrt{\Delta_2}}{2(d+\alpha)[\beta-\alpha(d+\alpha+\gamma)]}D_1+D_2>0$. Similarly, $P_3(S_3,I_3)$ is a stable node if $\beta>\alpha(d+\varepsilon+\gamma)$ and $\frac{-b+\sqrt{\Delta_2}}{2(d+\alpha)[\beta-\alpha(d+\alpha+\gamma)]}D_1+D_2>0$. This completes the proof. □

Theorem 3.2 *If $\delta<d$, there is no limit cycle of model* (1.1).

Proof Consider the Dulac function $D=\frac{1}{I}$. Note that

$$P=A-dS-\frac{\beta SI}{1+\alpha S}+\gamma I+T(I),$$
$$Q=\frac{\beta SI}{1+\alpha S}-(d+\varepsilon+\gamma)I-T(I).$$

If $0<I\le I_0$,

$$\frac{\partial(DP)}{\partial S}+\frac{\partial(DQ)}{\partial I}=-\frac{d}{I}-\frac{\beta}{(1+\alpha S)^2}<0.$$

If $I>I_0$, because $k=\delta I_0$,

$$\frac{\partial(DP)}{\partial S}+\frac{\partial(DQ)}{\partial I}=-\frac{d}{I}-\frac{\beta}{(1+\alpha S)^2}+\frac{k}{I^2}=\frac{1}{I}\left(\frac{\delta I_0}{I}-d\right)-\frac{\beta}{(1+\alpha S)^2}<\frac{1}{I}(\delta-d)-\frac{\beta}{(1+\alpha S)^2}.$$

Thus $\frac{\partial(DP)}{\partial S}+\frac{\partial(DQ)}{\partial I}<0$ if $\delta<d$.

Then there is no limit cycle of model (1.1) if $\delta<d$. This completes the proof. □

Theorem 3.3 *There is no limit cycle of model* (1.1) *if $\beta\le\alpha(2d+\varepsilon+\gamma)$.*

Proof If $I\le I_0$, consider the Dulac function $D=\frac{1}{I}$. Note that

$$P=A-dS-\frac{\beta SI}{1+\alpha S}+\gamma I+T(I),$$
$$Q=\frac{\beta SI}{1+\alpha S}-(d+\varepsilon+\gamma)I-T(I),$$
$$\frac{\partial(DP)}{\partial S}+\frac{\partial(DQ)}{\partial I}=-\frac{d}{I}-\frac{\beta}{(1+\alpha S)^2}<0.$$

If $I>I_0$,

$$\begin{aligned}\frac{\partial(P)}{\partial S}+\frac{\partial(Q)}{\partial I}&=-d-\frac{\beta I}{(1+\alpha S)^2}+\frac{\beta S}{(1+\alpha S)}-(d+\varepsilon+\gamma)\\&=\frac{1}{(1+\alpha S)^2}\bigl[-d(1+\alpha S)^2-\beta I+\beta S(1+\alpha S)-(d+\varepsilon+\gamma)(1+\alpha S)^2\bigr]\\&=\frac{1}{(1+\alpha S)^2}\bigl[-\beta I-(2d+\varepsilon+\gamma)(1+\alpha S)^2+\beta S(1+\alpha S)\bigr],\end{aligned}$$
$$\frac{(2d+\varepsilon+\gamma)(1+\alpha S)^2}{\beta S(1+\alpha S)}=\frac{(2d+\varepsilon+\gamma)(1+\alpha S)}{\beta S}=\frac{(2d+\varepsilon+\gamma)+\alpha(2d+\varepsilon+\gamma)S}{\beta S}.$$

Thus $\frac{\partial(DP)}{\partial S}+\frac{\partial(DQ)}{\partial I}<0$ if $\beta\le\alpha(2d+\varepsilon+\gamma)$.

Then there is no limit cycle of model (1.1) if $\beta \le \alpha(2d + \varepsilon + \gamma)$. This completes the proof. □

4 Numerical simulation and conclusion

With different A, d, γ, δ, ε, α, it is easy to test and verify the above results, so numerical simulation is omitted. In this paper, we study an SIS model with saturated incidence rate $\frac{\beta SI}{1+\alpha S}$ and treatment. We get some relatively complex conclusions by stability theory and qualitative theory of differential equations. These conclusions will help policy makers to make decisions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors are entirely responsible for this research. The authors read and approved the final manuscript.

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