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Circular summation formulas for theta function $\vartheta_4(z|\tau)$

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Abstract

In this paper, we obtain the circular summation formulas of the theta function $\vartheta_4(z|\tau)$ and show the corresponding alternating summations and inverse relations. Some applications are also considered.

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1 Introduction

Throughout this paper we take $q = e^{\pi i \tau}$, $\text{Im}(\tau) > 0$, $z \in \mathbb{C}$. The classical Jacobi's four theta functions $\vartheta_i(z|\tau)$, $i = 1, 2, 3, 4$, are defined as follows, with the notation of Tannery and Molk (see, e.g., [1–3]):

$$\vartheta_1(z|\tau) = -iq^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)} e^{(2n+1)iz}, \quad (1.1)$$

$$\vartheta_2(z|\tau) = q^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} q^{n(n+1)} e^{(2n+1)iz}, \quad (1.2)$$

$$\vartheta_3(z|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}, \quad (1.3)$$

$$\vartheta_4(z|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}. \quad (1.4)$$

From the Jacobi theta functions (1.1)-(1.4), via the direct calculation and applying the induction, we have the following properties, respectively:

$$\vartheta_1(z + n\pi|\tau) = (-1)^n \vartheta_1(z|\tau), \quad \vartheta_1(z + n\pi\tau|\tau) = (-1)^n q^{-n^2} e^{-2niz} \vartheta_1(z|\tau), \quad (1.5)$$

$$\vartheta_2(z + n\pi|\tau) = (-1)^n \vartheta_2(z|\tau), \quad \vartheta_2(z + n\pi\tau|\tau) = q^{-n^2} e^{-2niz} \vartheta_2(z|\tau), \quad (1.6)$$

$$\vartheta_3(z + n\pi|\tau) = \vartheta_3(z|\tau), \quad \vartheta_3(z + n\pi\tau|\tau) = q^{-n^2} e^{-2niz} \vartheta_3(z|\tau), \quad (1.7)$$

$$\vartheta_4(z + n\pi|\tau) = \vartheta_4(z|\tau), \quad \vartheta_4(z + n\pi\tau|\tau) = (-1)^n q^{-n^2} e^{-2niz} \vartheta_4(z|\tau). \quad (1.8)$$

From the Jacobi theta functions (1.1)-(1.4), via a direct calculation, we also have the following transformation formulas:

$$\vartheta_1\left(z + \frac{\pi}{2} \middle| \tau\right) = \vartheta_2(z|\tau), \quad \vartheta_1\left(z + \frac{\pi\tau}{2} \middle| \tau\right) = iq^{-\frac{1}{4}}e^{-iz}\vartheta_4(z|\tau), \quad (1.9)$$

$$\vartheta_2\left(z + \frac{\pi}{2} \middle| \tau\right) = -\vartheta_1(z|\tau), \quad \vartheta_2\left(z + \frac{\pi\tau}{2} \middle| \tau\right) = q^{-\frac{1}{4}}e^{-iz}\vartheta_3(z|\tau), \quad (1.10)$$

$$\vartheta_3\left(z + \frac{\pi}{2} \middle| \tau\right) = \vartheta_4(z|\tau), \quad \vartheta_3\left(z + \frac{\pi\tau}{2} \middle| \tau\right) = q^{-\frac{1}{4}}e^{-iz}\vartheta_2(z|\tau), \quad (1.11)$$

$$\vartheta_4\left(z + \frac{\pi}{2} \middle| \tau\right) = \vartheta_3(z|\tau), \quad \vartheta_4\left(z + \frac{\pi\tau}{2} \middle| \tau\right) = iq^{-\frac{1}{4}}e^{-iz}\vartheta_1(z|\tau). \quad (1.12)$$

From the above equations we easily obtain

$$\vartheta_1\left(z + \frac{\pi}{2} + \frac{\pi\tau}{2} \middle| \tau\right) = q^{-\frac{1}{4}}e^{-iz}\vartheta_3(z|\tau), \quad (1.13)$$

$$\vartheta_2\left(z + \frac{\pi}{2} + \frac{\pi\tau}{2} \middle| \tau\right) = -iq^{-\frac{1}{4}}e^{-iz}\vartheta_4(z|\tau),$$

$$\vartheta_3\left(z + \frac{\pi}{2} + \frac{\pi\tau}{2} \middle| \tau\right) = iq^{-\frac{1}{4}}e^{-iz}\vartheta_1(z|\tau), \quad (1.14)$$

$$\vartheta_4\left(z + \frac{\pi}{2} + \frac{\pi\tau}{2} \middle| \tau\right) = q^{-\frac{1}{4}}e^{-iz}\vartheta_2(z|\tau).$$

From (1.5)-(1.12), we can obtain the following lemmas.

Lemma 1.1 *For n any positive integer, we have*

$$\vartheta_1\left(z + \frac{n\pi}{2} \middle| \tau\right) = \begin{cases} i^n \vartheta_1(z|\tau), & n \text{ is even,} \\ i^{n-1} \vartheta_2(z|\tau), & n \text{ is odd,} \end{cases} \quad (1.15)$$

$$\vartheta_2\left(z + \frac{n\pi}{2} \middle| \tau\right) = \begin{cases} i^n \vartheta_2(z|\tau), & n \text{ is even,} \\ -i^{n-1} \vartheta_1(z|\tau), & n \text{ is odd,} \end{cases} \quad (1.16)$$

$$\vartheta_3\left(z + \frac{n\pi}{2} \middle| \tau\right) = \begin{cases} \vartheta_3(z|\tau), & n \text{ is even,} \\ \vartheta_4(z|\tau), & n \text{ is odd,} \end{cases} \quad (1.17)$$

$$\vartheta_4\left(z + \frac{n\pi}{2} \middle| \tau\right) = \begin{cases} \vartheta_4(z|\tau), & n \text{ is even,} \\ \vartheta_3(z|\tau), & n \text{ is odd.} \end{cases} \quad (1.18)$$

Lemma 1.2 *For n any positive integer, we have*

$$\vartheta_1\left(z + \frac{n\pi\tau}{2} \middle| \tau\right) = \begin{cases} i^n q^{-\frac{n^2}{4}} e^{-niz} \vartheta_1(z|\tau), & n \text{ is even,} \\ i^n q^{-\frac{n^2}{4}} e^{-niz} \vartheta_4(z|\tau), & n \text{ is odd,} \end{cases} \quad (1.19)$$

$$\vartheta_2\left(z + \frac{n\pi\tau}{2} \middle| \tau\right) = \begin{cases} q^{-\frac{n^2}{4}} e^{-niz} \vartheta_2(z|\tau), & n \text{ is even,} \\ q^{-\frac{n^2}{4}} e^{-niz} \vartheta_3(z|\tau), & n \text{ is odd,} \end{cases} \quad (1.20)$$

$$\vartheta_3\left(z + \frac{n\pi\tau}{2} \middle| \tau\right) = \begin{cases} q^{-\frac{n^2}{4}} e^{-niz} \vartheta_3(z|\tau), & n \text{ is even,} \\ q^{-\frac{n^2}{4}} e^{-niz} \vartheta_2(z|\tau), & n \text{ is odd,} \end{cases} \quad (1.21)$$

$$\vartheta_4\left(z + \frac{n\pi\tau}{2} \middle| \tau\right) = \begin{cases} i^n q^{-\frac{n^2}{4}} e^{-niz} \vartheta_4(z|\tau), & n \text{ is even,} \\ i^n q^{-\frac{n^2}{4}} e^{-niz} \vartheta_1(z|\tau), & n \text{ is odd.} \end{cases} \quad (1.22)$$

On p.54 in Ramanujan's lost notebook (see [4, p.54, Entry 9.1.1], or [5, p.337]), Ramanujan recorded the following claim (without proof), which is now well known as Ramanujan's circular summation. The appellation of 'circular summation' is due to Son (see [5, p.338]).

Theorem 1.3 (Ramanujan's circular summation) *For each positive integer n and $|ab| < 1$,*

$$\sum_{-n/2 < r \leq n/2} \left(\sum_{\substack{k=-\infty \\ k \equiv r \pmod{n}}}^{\infty} a^{k(k+1)/(2n)} b^{k(k-1)/(2n)} \right)^n = f(a, b) F_n(ab), \quad (1.23)$$

where

$$F_n(q) := 1 + 2nq^{(n-1)/2} + \cdots, \quad n \geq 3. \quad (1.24)$$

Ramanujan's theta function $f(a, b)$ is defined by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (1.25)$$

By the definition of Ramanujan's theta function above and routine calculations, we can rewrite Ramanujan's circular summation (1.23) as follows (see, for details [5, p.338]).

Theorem 1.4 (Ramanujan's circular summation) *Let $U_k = a^{k(k+1)/(2n)}$ and $V_k = b^{k(k-1)/(2n)}$. For each positive integer n and $|ab| < 1$,*

$$\sum_{k=0}^{n-1} U_k^n f^n\left(\frac{U_{n+k}}{U_k}, \frac{V_{n-k}}{V_k}\right) = f(a, b) F_n(ab), \quad (1.26)$$

where

$$F_n(q) := 1 + 2nq^{(n-1)/2} + \cdots, \quad n \geq 3. \quad (1.27)$$

If we are going to apply the transformation $\tau \mapsto -\frac{1}{\tau}$ to Ramanujan's identity, it will be convenient to convert Ramanujan's theorem into one involving the classical theta function $\vartheta_3(z|\tau)$ defined by (1.3). Surprisingly Chan *et al.* [6] prove that Theorem 1.5 below is equivalent to Theorem 1.3.

Theorem 1.5 (Ramanujan's circular summation) *For any positive integer $n \geq 2$,*

$$\sum_{k=0}^{n-1} q^{k^2} e^{2kiz} \vartheta_3^n(z + k\pi\tau | n\tau) = \vartheta_3(z|\tau) F_n(\tau). \quad (1.28)$$

When $n \geq 3$,

$$F_n(q) = 1 + 2nq^{n-1} + \dots \quad (1.29)$$

Chan *et al.* [6] also showed that Theorem 1.6 below is equivalent; we have Theorem 1.5 by applying the Jacobi imaginary transformation formula [3, p.475].

Theorem 1.6 (Ramanujan's circular summation) *For any positive integer n , there exists a quantity $G_n(\tau)$ such that*

$$\sum_{k=0}^{n-1} \vartheta_3^n \left(z + \frac{k\pi}{n} \middle| \tau \right) = G_n(\tau) \vartheta_3(nz|n\tau), \quad (1.30)$$

where

$$G_n(\tau) = \sqrt{n}(-i\tau)^{(1-n)/2} F_n \left(-\frac{1}{n\tau} \right). \quad (1.31)$$

Ramanujan's circular summation is an interesting subject in his notebook. On the subject of Ramanujan's circular summation and related identities of theta functions and their various extensions, a remarkably large number of investigations have appeared in the literature (see, for example, Andrews, Berndt, Rangachari, Ono, Ahlgren, Chua, Murayama, Son, Chan, Liu, Ng, Chan, Shen, Cai, Zhu, and Xu *et al.* [5–25]).

Recently, Zeng [23] extended Ramanujan's circular summation in the following form.

Theorem 1.7 *For any nonnegative integers n , k , a , and b with $a + b = k$, there exists a quantity $R_{33}(a, b; \frac{y}{ab}, \frac{\tau}{kn^2})$ such that*

$$\begin{aligned} \sum_{s=0}^{kn-1} \vartheta_3^a \left(\frac{z}{kn} + \frac{y}{a} + \frac{\pi s}{kn} \middle| \frac{\tau}{kn^2} \right) \vartheta_3^b \left(\frac{z}{kn} - \frac{y}{b} + \frac{\pi s}{kn} \middle| \frac{\tau}{kn^2} \right) \\ = R_{33} \left(a, b; \frac{y}{ab}, \frac{\tau}{kn^2} \right) \vartheta_3(z|\tau). \end{aligned} \quad (1.32)$$

Chan and Liu [12] further extended Theorem 1.7 in the following general form.

Theorem 1.8 *Suppose $y_1, y_2, \dots, y_n$ are n complex numbers such that $y_1 + y_2 + \dots + y_n = 0$; there exists a quantity $G_{m,n}(y_1, y_2, \dots, y_n|\tau)$ such that*

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = G_{m,n}(y_1, y_2, \dots, y_n|\tau) \vartheta_3(mnz|m^2n\tau), \quad (1.33)$$

In the present paper, motivated by [10, 12], and [6], by applying the theory and method of elliptic functions, we obtain the circular summation formulas of theta functions $\vartheta_4(z|\tau)$ and show the corresponding alternating summations and inverse relations. We also give some applications and derive some interesting identities of theta functions.

2 Ramanujan's circular summation formula for theta functions $\vartheta_4(z|\tau)$

In the present section, we obtain Ramanujan's circular summation formula for the theta functions $\vartheta_4(z|\tau)$. We now state our main result as follows.

Theorem 2.1 *Suppose that m, n are any positive integers, p is any integer; $y_1, y_2, \dots, y_n$ are any complex numbers.*

- When $y_1 + y_2 + \dots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2}$, we have

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4\left(z + y_j + \frac{k\pi}{mn} \middle| \tau\right) = R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_3(mnz|m^2n\tau). \quad (2.1)$$

- When $y_1 + y_2 + \dots + y_n = \frac{(2p+1)\pi}{2m} + \frac{n\pi}{2}$, we have

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4\left(z + y_j + \frac{k\pi}{mn} \middle| \tau\right) = R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_4(mnz|m^2n\tau). \quad (2.2)$$

Here

$$R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (2.3)$$

Proof Let $f(z)$ be the left-hand side of (2.1) with $z \mapsto \frac{z}{mn}$, $\tau \mapsto \frac{\tau}{m^2n}$. We have

$$f(z) = \sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4\left(\frac{z}{mn} + y_j + \frac{k\pi}{mn} \middle| \frac{\tau}{m^2n}\right). \quad (2.4)$$

By (1.8) for $n = 1$, we easily obtain

$$\begin{aligned} f(z + \pi) &= \sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4\left(\frac{z + \pi}{mn} + y_j + \frac{k\pi}{mn} \middle| \frac{\tau}{m^2n}\right) \\ &= \sum_{k=1}^{mn-1} \prod_{j=1}^n \vartheta_4\left(\frac{z}{mn} + y_j + \frac{k\pi}{mn} \middle| \frac{\tau}{m^2n}\right) + \prod_{j=1}^n \vartheta_4\left(\frac{z}{mn} + y_j \middle| \frac{\tau}{m^2n}\right). \end{aligned} \quad (2.5)$$

Comparing (2.4) and (2.5), we have

$$f(z) = f(z + \pi). \quad (2.6)$$

By (1.8), we obtain

$$\begin{aligned} f(z + \pi\tau) &= \sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4\left(\frac{z + \pi\tau}{mn} + y_j + \frac{k\pi}{mn} \middle| \frac{\tau}{m^2n}\right) \\ &= \sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4\left(\frac{z}{mn} + y_j + \frac{k\pi}{mn} + m\pi \frac{\tau}{m^2n} \middle| \frac{\tau}{m^2n}\right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{mn-1} \prod_{j=1}^n (-1)^m q^{-\frac{1}{n}} e^{-2im(\frac{z}{mn} + y_j + \frac{k\pi}{mn})} \vartheta_4 \left(\frac{z}{mn} + y_j + \frac{k\pi}{mn} \middle| \frac{\tau}{m^2 n} \right) \\
 &= (-1)^{mn} q^{-1} e^{-2iz} e^{-2im(y_1 + y_2 + \dots + y_n)} f(z).
 \end{aligned} \tag{2.7}$$

• When $y_1 + y_2 + \dots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2}$ in (2.7), we have

$$f(z + \pi\tau) = q^{-1} e^{-2iz} f(z). \tag{2.8}$$

We construct the function $\frac{f(z)}{\vartheta_3(z|\tau)}$, by (1.7) for $n = 1$, (2.6) and (2.8), we find that the function $\frac{f(z)}{\vartheta_3(z|\tau)}$ is an elliptic function with double periods π and $\pi\tau$ and has *only* a simple pole at $z = \frac{\pi}{2} + \frac{\pi\tau}{2}$ in the period parallelogram. Hence the function $\frac{f(z)}{\vartheta_3(z|\tau)}$ is a constant, say $C_{4,4}^{(1)}(y_1, y_2, \dots, y_n; \tau)$, and we have

$$f(z) = C_{4,4}^{(1)}(y_1, y_2, \dots, y_n; \tau) \vartheta_3(z|\tau),$$

or, equivalently,

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4 \left(\frac{z}{mn} + y_j + \frac{k\pi}{mn} \middle| \frac{\tau}{m^2 n} \right) = C_{4,4}^{(1)}(y_1, y_2, \dots, y_n; \tau) \vartheta_3(z|\tau). \tag{2.9}$$

Letting

$$z \mapsto mnz \quad \text{and} \quad \tau \mapsto m^2 n\tau$$

in (2.9), and then setting

$$R_{4,4}^{(1)}(y_1, y_2, \dots, y_n; m, n, p; \tau) = C_{4,4}^{(1)}(y_1, y_2, \dots, y_n; m^2 n\tau),$$

we arrive at (2.1).

Setting

$$z \mapsto z + y_j + \frac{k\pi}{mn}$$

in (1.4), via the direct calculation, we obtain

$$\begin{aligned}
 \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) &= \sum_{r_1, \dots, r_n = -\infty}^{\infty} (-1)^{r_1 + \dots + r_n} q^{r_1^2 + \dots + r_n^2} \\
 &\quad \times e^{2(r_1 + \dots + r_n)iz} e^{2i(r_1 y_1 + \dots + r_n y_n)} e^{\frac{2k\pi i}{mn}(r_1 + \dots + r_n)}.
 \end{aligned} \tag{2.10}$$

Setting

$$z \mapsto mnz \quad \text{and} \quad \tau \mapsto m^2 n\tau$$

in (1.3), we get

$$\vartheta_3(mnz|m^2 n\tau) = \sum_{r=-\infty}^{\infty} q^{m^2 nr^2} e^{2mnriz}. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.1), we find that

$$\begin{aligned} & \sum_{r_1, \dots, r_n = -\infty}^{\infty} (-1)^{r_1 + \dots + r_n} q^{r_1^2 + \dots + r_n^2} e^{2(r_1 + \dots + r_n)iz} e^{2i(r_1 y_1 + \dots + r_n y_n)} \sum_{k=0}^{mn-1} e^{\frac{2k\pi i}{mn}(r_1 + \dots + r_n)} \\ &= R_{4,4}^{(1)}(y_1, y_2, \dots, y_n; m, n, p; \tau) \sum_{r=-\infty}^{\infty} q^{m^2 n r^2} e^{2mn r i z}. \end{aligned} \quad (2.12)$$

Equating the constants of both sides of (2.12), we get (2.3).

- When $y_1 + y_2 + \dots + y_n = \frac{(2p+1)\pi}{2m} + \frac{n\pi}{2}$ in (2.7), we have

$$f(z + \pi \tau) = -q^{-1} e^{-2iz} f(z). \quad (2.13)$$

We construct the function $\frac{f(z)}{\vartheta_4(z|\tau)}$, by (1.8) for $n = 1$, (2.6) and (2.13), we find that the function $\frac{f(z)}{\vartheta_4(z|\tau)}$ is an elliptic function with double periods π and $\pi \tau$ and has *only* a simple pole at $z = \frac{\pi \tau}{2}$ in the period parallelogram. Hence the function $\frac{f(z)}{\vartheta_4(z|\tau)}$ is a constant, say $C_{4,4}^{(2)}(y_1, y_2, \dots, y_n; \tau)$, and we have

$$f(z) = C_{4,4}^{(2)}(y_1, y_2, \dots, y_n; \tau) \vartheta_4(z|\tau),$$

or, equivalently,

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4\left(\frac{z}{mn} + y_j + \frac{k\pi}{mn} \middle| \frac{\tau}{m^2 n}\right) = C_{4,4}^{(2)}(y_1, y_2, \dots, y_n; \tau) \vartheta_4(z|\tau). \quad (2.14)$$

Letting

$$z \mapsto mnz \quad \text{and} \quad \tau \mapsto m^2 n \tau$$

in (2.14), and then setting

$$R_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n, p; \tau) = C_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m^2 n \tau),$$

we arrive at (2.2).

Similarly, we may obtain

$$R_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n, p; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (2.15)$$

Clearly,

$$R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) = R_{4,4}^{(1)}(y_1, y_2, \dots, y_n; m, n, p; \tau) = R_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n, p; \tau).$$

The proof is complete. $\square$

In the following we will prove that the following main result of Chan and Liu is a special case of Theorem 2.1.

Corollary 2.2 (Chan and Liu [12, p.1191, Theorem 4]) *Suppose that m, n are any positive integers; $y_1, y_2, \dots, y_n$ are any complex numbers and $y_1 + y_2 + \dots + y_n = 0$. We have*

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = R_{3,3}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_3(mnz | m^2 n \tau), \quad (2.16)$$

where

$$R_{3,3}(y_1, y_2, \dots, y_n; m, n; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (2.17)$$

Proof First, setting $y_1 + y_2 + \dots + y_n = 0$ in (2.1), we have $p = -\frac{mn}{2}$, noting that p is any integer, hence we say that mn is even. Setting

$$z \mapsto z + \frac{\pi}{2}$$

in (2.1) of Theorem 2.1 and applying the properties (1.12) and (1.17), we have

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = R_{3,3}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_3(mnz | m^2 n \tau). \quad (2.18)$$

Next taking $y_1 + y_2 + \dots + y_n = 0$ in (2.2), we have $p = -\frac{mn+1}{2}$, noting that p is any integer, hence we say that mn is odd. Setting

$$z \mapsto z + \frac{\pi}{2}$$

in (2.2) of Theorem 2.1 and applying the properties (1.12) and (1.18), this leads to the following:

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = R_{3,3}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_3(mnz | m^2 n \tau). \quad (2.19)$$

Obviously, by (2.18) and (2.19), for any positive integers mn , we obtain (2.16).

We easily see that the $R_{4,4}^{(1)}(y_1, y_2, \dots, y_n; m, n, p; \tau)$ and $R_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n, p; \tau)$ are independent of z , therefore, when $y_1 + y_2 + \dots + y_n = 0$, we find that

$$R_{3,3}(y_1, y_2, \dots, y_n; m, n; \tau) = R_{4,4}^{(1)}(y_1, y_2, \dots, y_n; m, n; \tau) = R_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n; \tau).$$

The proof is complete. $\square$

Remark 2.3 Obviously, Theorem 2.1 implies the well-known result (2.16), but we see that from (2.16) we do not obtain Theorem 2.1. However, we can reformulate the results of Chan and Liu as Theorem 2.1.

Suppose that m, n are any positive integers, p is any integer; $y_1, y_2, \dots, y_n$ are any complex numbers.

- When $y_1 + y_2 + \cdots + y_n = \frac{p\pi}{m}$, we have

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = R_{3,3}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_3(mnz | m^2 n \tau). \quad (2.20)$$

- When $y_1 + y_2 + \cdots + y_n = \frac{(2p+1)\pi}{2m}$, we have

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = R_{3,3}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_4(mnz | m^2 n \tau). \quad (2.21)$$

Here

$$R_{3,3}(y_1, y_2, \dots, y_n; m, n, p; \tau) = mn \sum_{\substack{r_1 + \cdots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \cdots + r_n^2} e^{2i(r_1 y_1 + \cdots + r_n y_n)}. \quad (2.22)$$

3 The imaginary transformations formulas for Theorem 2.1

In the present section, we derive the corresponding imaginary transformations formulas for Theorem 2.1.

Theorem 3.1 Suppose that m, n are any positive integers, p is any integer; $y_1, y_2, \dots, y_n$ are any complex numbers.

- When $y_1 + y_2 + \cdots + y_n = mnp + \frac{m^2 n^2}{2}$, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} q^{k^2 + 2kp + mnk} e^{(2k+2p+mn)iz} \prod_{j=1}^n \vartheta_2(mz + y_j \pi \tau + mk \pi \tau | m^2 n \tau) \\ &= F_{4,4}^{(1)}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_3(z | \tau). \end{aligned} \quad (3.1)$$

- When $y_1 + y_2 + \cdots + y_n = \frac{(2p+1)mn}{2} + \frac{m^2 n^2}{2}$, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} q^{k^2 + k(2p+1) + mnk} e^{(2k+2p+mn+1)iz} \prod_{j=1}^n \vartheta_2(mz + y_j \pi \tau + mk \pi \tau | m^2 n \tau) \\ &= F_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(z | \tau). \end{aligned} \quad (3.2)$$

Here

$$\begin{aligned} & F_{4,4}^{(1)}(y_1, y_2, \dots, y_n; m, n, p; \tau) \\ &= \frac{(-i\tau)^{\frac{1-n}{2}}}{(m^2 n)^{\frac{n}{2}}} q^{-\frac{y_1^2 + \cdots + y_n^2}{m^2 n}} R_{4,4}^{(1)} \left(\frac{y_1 \pi}{m^2 n}, \frac{y_2 \pi}{m^2 n}, \dots, \frac{y_n \pi}{m^2 n}; m, n, p; -\frac{1}{m^2 n \tau} \right), \end{aligned} \quad (3.3)$$

$$\begin{aligned} & F_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n, p; \tau) \\ &= \sum_{k=0}^{mn-1} q^{-(k + \frac{mn}{2})^2 + \frac{m^2 n^2}{2}} \sum_{\substack{r_1, \dots, r_n = -\infty \\ m(r_1 + \cdots + r_n) = mn + k + p}}^{\infty} q^{m^2 n(r_1^2 + \cdots + r_n^2) - 2(r_1 y_1 + \cdots + r_n y_n)}, \end{aligned} \quad (3.4)$$

$$F_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n, p; \tau) = \frac{(-i\tau)^{\frac{1-n}{2}}}{(m^2 n)^{\frac{n}{2}}} q^{-\frac{y_1^2 + \dots + y_n^2}{m^2 n}} R_{4,4}^{(2)}\left(\frac{y_1 \pi}{m^2 n}, \frac{y_2 \pi}{m^2 n}, \dots, \frac{y_n \pi}{m^2 n}; m, n, p; -\frac{1}{m^2 n \tau}\right), \quad (3.5)$$

$$F_{4,4}^{(2)}(y_1, y_2, \dots, y_n; m, n, p; \tau) = \sum_{k=0}^{mn-1} q^{-(k + \frac{mn}{2} - \frac{1}{2})^2 - mnk + \frac{m^2 n^2}{2}} \sum_{\substack{r_1, \dots, r_n = -\infty \\ m(r_1 + \dots + r_n) = mn + k + p}}^{\infty} q^{m^2 n(r_1^2 + \dots + r_n^2) - 2(r_1 y_1 + \dots + r_n y_n)}. \quad (3.6)$$

Proof In (2.1) making the transformations $\tau \mapsto -\frac{1}{m^2 n \tau}$, and then $z \mapsto \frac{z}{mn\tau}$ and $y_j \mapsto \frac{y_j \pi}{m^2 n}$ for $j = 1, 2, \dots, n$, (2.1) becomes

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4\left(\frac{mz + y_j \pi \tau + mk \pi \tau}{m^2 n \tau} \middle| -\frac{1}{m^2 n \tau}\right) = R_{4,4}^{(1)}\left(\frac{y_1 \pi}{m^2 n}, \dots, \frac{y_n \pi}{m^2 n}; m, n, p; -\frac{1}{m^2 n \tau}\right) \vartheta_3\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right). \quad (3.7)$$

Applying the imaginary transformation formulas (see, e.g., [1–3])

$$\vartheta_3\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) = \sqrt{-i\tau} e^{\frac{iz^2}{\pi\tau}} \vartheta_3(z|\tau) \quad \text{and} \quad \vartheta_4\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) = \sqrt{-i\tau} e^{\frac{iz^2}{\pi\tau}} \vartheta_2(z|\tau)$$

to (3.7), via the suitable substitutions of the variable z and τ and noting that $y_1 + y_2 + \dots + y_n = mnp + \frac{m^2 n^2}{2}$, and simplifying, we thus obtain (3.1) and (3.3). Applying the series expressions (1.3) and (1.4) in (3.1), via the direct calculation, we obtain (3.4).

In a similar manner, we use the imaginary transformation formula:

$$\vartheta_4\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) = \sqrt{-i\tau} e^{\frac{iz^2}{\pi\tau}} \vartheta_2(z|\tau),$$

and (2.2), and noting that $y_1 + y_2 + \dots + y_n = \frac{(2p+1)mn}{2} + \frac{m^2 n^2}{2}$, we can prove (3.2), (3.5), and (3.6). Therefore we complete the proof of Theorem 3.1. $\square$

4 The alternating Ramanujan's circular summation formula

In this section, we will obtain the corresponding alternating Ramanujan's circular summation formula from Theorem 2.1.

Theorem 4.1 *Suppose that m, n are any positive integers, p is any integer; $y_1, y_2, \dots, y_n$ are any complex numbers.*

- When $y_1 + y_2 + \dots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2}$, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_4\left(z + y_j + \frac{k\pi}{mn} \middle| \tau\right) \\ &= (-1)^p R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(mnz | m^2 n \tau), \quad m \text{ is even,} \end{aligned} \quad (4.1)$$

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_1 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^p R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(mnz | m^2 n \tau), \quad m \text{ is odd.} \end{aligned} \quad (4.2)$$

• When $y_1 + y_2 + \dots + y_n = \frac{(2p+1)\pi}{2m} + \frac{n\pi}{2}$, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^{p+1} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m \text{ is even,} \end{aligned} \quad (4.3)$$

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_1 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^{p+1} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m \text{ is odd.} \end{aligned} \quad (4.4)$$

Here

$$R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (4.5)$$

Proof Let

$$z \mapsto z + \frac{m\pi\tau}{2}$$

in Theorem 2.1.

By using (1.11), (1.12), and (1.22) we compute

$$\begin{aligned} & \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{mn} + \frac{m\pi\tau}{2} \middle| \tau \right) \\ &= \begin{cases} (-1)^k i^{mn} q^{-\frac{m^2 n}{4}} e^{-mniz} e^{-mi(y_1 + \dots + y_n)} \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right), & m \text{ is even,} \\ (-1)^k i^{mn} q^{-\frac{m^2 n}{4}} e^{-mniz} e^{-mi(y_1 + \dots + y_n)} \prod_{j=1}^n \vartheta_1 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right), & m \text{ is odd,} \end{cases} \end{aligned} \quad (4.6)$$

$$\vartheta_3 \left(mnz + \frac{m^2 n \pi \tau}{2} \middle| m^2 n \tau \right) = q^{-\frac{m^2 n}{4}} e^{-mniz} \vartheta_2(mnz | m^2 n \tau), \quad (4.7)$$

$$\vartheta_4 \left(mnz + \frac{m^2 n \pi \tau}{2} \middle| m^2 n \tau \right) = i q^{-\frac{m^2 n}{4}} e^{-mniz} \vartheta_1(mnz | m^2 n \tau). \quad (4.8)$$

Substituting (4.6) and (4.7) into (2.1) of Theorem 2.1 and simplifying, we get (4.1) of Theorem 4.1.

Substituting (4.6) and (4.8) into (2.2) of Theorem 2.1 and simplifying, we arrive at (4.3) of Theorem 4.1. This proof is complete. $\square$

Corollary 4.2 Suppose that m, n are any positive integers, p is any integer; $y_1, y_2, \dots, y_n$ are any complex numbers.

- When $y_1 + y_2 + \cdots + y_n = 0$, we have

$$\sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = i^{-mn} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(mnz | m^2 n \tau), \quad m \text{ is even}, \quad (4.9)$$

$$\sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_1 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = i^{-mn} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(mnz | m^2 n \tau),$$

$m \text{ is odd and } n \text{ is even.}$

(4.10)

- When $y_1 + y_2 + \cdots + y_n = \frac{\pi}{2m}$, we have

$$\sum_{k=0}^{mn-1} (-1)^{k+1} \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = i^{-mn} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m \text{ is even}, \quad (4.11)$$

$$\sum_{k=0}^{mn-1} (-1)^{k+1} \prod_{j=1}^n \vartheta_1 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = i^{-mn} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz | m^2 n \tau),$$

$m \text{ is odd and } n \text{ is even.}$

(4.12)

- When $y_1 + y_2 + \cdots + y_n = 0$ and m, n are odd, we have

$$\sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_1 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = i^{mn-1} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz | m^2 n \tau). \quad (4.13)$$

Here

$$R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) = mn \sum_{\substack{r_1 + \cdots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \cdots + r_n^2} e^{2i(r_1 y_1 + \cdots + r_n y_n)}. \quad (4.14)$$

Proof If mn is even, setting $p = -\frac{mn}{2}$ in $y_1 + y_2 + \cdots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2}$, then we have $y_1 + y_2 + \cdots + y_n = 0$ and $(-1)^p = i^{-mn}$. We directly deduce (4.9) of Corollary 4.2.

If mn is an even, setting $p = -\frac{mn}{2}$ in $y_1 + y_2 + \cdots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2} + \frac{\pi}{2}$, we have $y_1 + y_2 + \cdots + y_n = \frac{\pi}{2m}$. We get (4.11) of Corollary 4.2.

If mn is an odd, setting $p = -\frac{mn+1}{2}$ in $y_1 + y_2 + \cdots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2} + \frac{\pi}{2}$ and $(-1)^p = i^{-mn-1}$, we have $y_1 + y_2 + \cdots + y_n = 0$. We get (4.13) of Corollary 4.2. The proof is complete. $\square$

Corollary 4.3 Suppose that m, n are any positive integers. We have

$$\sum_{k=0}^{mn-1} (-1)^k \vartheta_4^n \left(z + \frac{k\pi}{mn} \middle| \tau \right) = i^{-mn} R_{4,4}(m, n; \tau) \vartheta_2(mnz | m^2 n \tau), \quad m \text{ is even}, \quad (4.15)$$

$$\sum_{k=0}^{mn-1} (-1)^k \vartheta_1^n \left(z + \frac{k\pi}{mn} \middle| \tau \right) = i^{-mn} R_{4,4}(m, n; \tau) \vartheta_2(mnz | m^2 n \tau), \quad m \text{ is odd and } n \text{ is even}, \quad (4.16)$$

$$\sum_{k=0}^{mn-1} (-1)^k \vartheta_1^n \left(z + \frac{k\pi}{mn} \middle| \tau \right) = i^{mn-1} R_{4,4}(m, n; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m, n \text{ are odd}, \quad (4.17)$$

$$\sum_{k=0}^{mn-1} (-1)^{k+1} \vartheta_4^n \left(z + \frac{\pi}{2mn} + \frac{k\pi}{mn} \middle| \tau \right) = i^{-mn} R_{4,4}(m, n; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m \text{ is even}, \quad (4.18)$$

$$\sum_{k=0}^{mn-1} (-1)^{k+1} \vartheta_1^n \left(z + \frac{\pi}{2mn} + \frac{k\pi}{mn} \middle| \tau \right) = i^{-mn} R_{4,4}(m, n; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m \text{ is odd and } n \text{ is even}, \quad (4.19)$$

where

$$R_{4,4}(m, n; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (4.20)$$

Proof We set $y_1 = y_2 = \dots = y_n = 0$ in (4.9), (4.10), and (4.13) of Corollary 4.2. We take $y_1 = y_2 = \dots = y_n = \frac{\pi}{2mn}$ in (4.11) and (4.12) of Corollary 4.2. $\square$

Remark 4.4 Taking $m = 1$ in (4.15) and (4.17), and noting that

$$R_{4,4}(n; \tau) = R_{4,4}(n; \tau) = n \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2},$$

we have

$$\sum_{k=0}^{n-1} (-1)^k \vartheta_3^n \left(z + \frac{k\pi}{n} \middle| \tau \right) = \begin{cases} i^{-n} R_{4,4}(n; \tau) \vartheta_2(nz | n \tau), & n \text{ is even,} \\ i^{1-n} R_{4,4}(n; \tau) \vartheta_1(nz | n \tau), & n \text{ is odd,} \end{cases} \quad (4.21)$$

which is just Chan's result (see [6, p.634, Theorem 4.2]). Hence (4.15) and (4.17) of Corollary 4.3 are the corresponding extension of Chan's result.

Remark 4.5 We note that Chan's result (see [6, p.632, Theorem 4.2]) has a misprint; we have here corrected this point in (4.21).

Remark 4.6 Setting $m = 1$ in (4.18), we have

$$\sum_{k=0}^{n-1} (-1)^k \vartheta_1^n \left(z + \frac{\pi}{2n} + \frac{k\pi}{n} \middle| \tau \right) = -i^{-n} R_{4,4}(n; \tau) \vartheta_1(nz | n \tau), \quad n \text{ is even.} \quad (4.22)$$

Setting $n = 1$ in (4.18), we have

$$\sum_{k=0}^{m-1} (-1)^k \vartheta_4 \left(z + \frac{\pi}{2m} + \frac{k\pi}{m} \middle| \tau \right) = -i^{-m} m \vartheta_1(mz|m^2\tau), \quad m \text{ is even.} \quad (4.23)$$

The above two formulas are the analogs of Boon's results (see [8, p.3440]).

Theorem 4.7 *Suppose that m, n are any positive integers, p is any integer; $y_1, y_2, \dots, y_n$ are any complex numbers.*

- When $y_1 + y_2 + \dots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2}$ and m is even, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^p i^{mn} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(mnz|m^2n\tau). \end{aligned} \quad (4.24)$$

- When $y_1 + y_2 + \dots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2}$ and m is odd and n is even, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_2 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^p i^{mn} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(mnz|m^2n\tau). \end{aligned} \quad (4.25)$$

- When $y_1 + y_2 + \dots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2}$ and m is odd and n is odd, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^{k+1} \prod_{j=1}^n \vartheta_2 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^p i^{mn+1} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz|m^2n\tau). \end{aligned} \quad (4.26)$$

- When $y_1 + y_2 + \dots + y_n = \frac{(2p+1)\pi}{2m} + \frac{n\pi}{2}$ and m is even, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^p i^{mn} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz|m^2n\tau). \end{aligned} \quad (4.27)$$

- When $y_1 + y_2 + \dots + y_n = \frac{(2p+1)\pi}{2m} + \frac{n\pi}{2}$ and m is odd and n is even, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_2 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^p i^{mn} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz|m^2n\tau). \end{aligned} \quad (4.28)$$

- When $y_1 + y_2 + \dots + y_n = \frac{(2p+1)\pi}{2m} + \frac{n\pi}{2}$ and m is odd and n is odd, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_2 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= (-1)^p i^{mn+1} R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(mnz|m^2n\tau). \end{aligned} \quad (4.29)$$

Here

$$R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (4.30)$$

Proof Let

$$z \mapsto z + \frac{\pi}{2} + \frac{m\pi\tau}{2}$$

in Theorem 2.1.

By using (1.9), (1.12), and (1.22) we have

$$\begin{aligned} & \prod_{j=1}^n \vartheta_4\left(z + y_j + \frac{k\pi}{mn} + \frac{\pi}{2} + \frac{m\pi\tau}{2} \middle| \tau\right) \\ &= \begin{cases} (-1)^k q^{-\frac{m^2 n}{4}} e^{-mniz} e^{-mi(y_1 + \dots + y_n)} \prod_{j=1}^n \vartheta_3\left(z + y_j + \frac{k\pi}{mn} \middle| \tau\right), & m \text{ is even,} \\ (-1)^{k+n} q^{-\frac{m^2 n}{4}} e^{-mniz} e^{-mi(y_1 + \dots + y_n)} \prod_{j=1}^n \vartheta_2\left(z + y_j + \frac{k\pi}{mn} \middle| \tau\right), & m \text{ is odd,} \end{cases} \end{aligned} \quad (4.31)$$

$$\begin{aligned} & \vartheta_3\left(mnz + \frac{m^2 n\pi\tau}{2} + \frac{mn\pi}{2} \middle| m^2 n\tau\right) \\ &= \begin{cases} q^{-\frac{m^2 n}{4}} e^{-mniz} \vartheta_2(mnz | m^2 n\tau), & mn \text{ is even,} \\ -iq^{-\frac{m^2 n}{4}} e^{-mniz} \vartheta_1(mnz | m^2 n\tau), & mn \text{ is odd,} \end{cases} \end{aligned} \quad (4.32)$$

$$\begin{aligned} & \vartheta_4\left(mnz + \frac{m^2 n\pi\tau}{2} + \frac{mn\pi}{2} \middle| m^2 n\tau\right) \\ &= \begin{cases} iq^{-\frac{m^2 n}{4}} e^{-mniz} \vartheta_1(mnz | m^2 n\tau), & mn \text{ is even,} \\ q^{-\frac{m^2 n}{4}} e^{-mniz} \vartheta_2(mnz | m^2 n\tau), & mn \text{ is odd.} \end{cases} \end{aligned} \quad (4.33)$$

Substituting (4.31) and (4.32) into (2.1) of Theorem 2.1 and simplifying, we get (4.24), (4.25), and (4.26) of Theorem 4.7.

Substituting (4.31) and (4.33) into (2.2) of Theorem 2.1 and simplifying, we arrive at (4.27), (4.28), and (4.29) of Theorem 4.7. This proof is complete. $\square$

Corollary 4.8 Suppose that m, n are any positive integers; $y_1, y_2, \dots, y_n$ are any complex numbers.

- When $y_1 + y_2 + \dots + y_n = 0$ and m is even, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_3\left(z + y_j + \frac{k\pi}{mn} \middle| \tau\right) \\ &= R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_2(mnz | m^2 n\tau). \end{aligned} \quad (4.34)$$

- When $y_1 + y_2 + \dots + y_n = 0$ and m is odd and n is even, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_2\left(z + y_j + \frac{k\pi}{mn} \middle| \tau\right) \\ &= R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_2(mnz | m^2 n\tau). \end{aligned} \quad (4.35)$$

- When $y_1 + y_2 + \cdots + y_n = 0$ and m is odd and n is odd, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^{k+1} \prod_{j=1}^n \vartheta_2 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_1(mnz | m^2 n \tau). \end{aligned} \quad (4.36)$$

- When $y_1 + y_2 + \cdots + y_n = \frac{\pi}{2m}$ and m is even, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_3 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_1(mnz | m^2 n \tau). \end{aligned} \quad (4.37)$$

- When $y_1 + y_2 + \cdots + y_n = \frac{\pi}{2m}$ and m is odd and n is even, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_2 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_1(mnz | m^2 n \tau). \end{aligned} \quad (4.38)$$

- When $y_1 + y_2 + \cdots + y_n = 0$ and m, n are odd, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \prod_{j=1}^n \vartheta_2 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_2(mnz | m^2 n \tau). \end{aligned} \quad (4.39)$$

Here

$$R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) = mn \sum_{\substack{r_1 + \cdots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \cdots + r_n^2} e^{2i(r_1 y_1 + \cdots + r_n y_n)}. \quad (4.40)$$

Proof If mn is an even, setting $p = -\frac{mn}{2}$ in $y_1 + y_2 + \cdots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2}$, then we have $y_1 + y_2 + \cdots + y_n = 0$ and $(-1)^p = i^{-mn}$. We directly deduce (4.34) and (4.35) of Corollary 4.8 from Theorem 4.7.

If mn is an even, setting $p = -\frac{mn}{2}$ in $y_1 + y_2 + \cdots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2} + \frac{\pi}{2m}$, we have $y_1 + y_2 + \cdots + y_n = \frac{\pi}{2m}$. We get (4.37), (4.38), and (4.36) of Corollary 4.8 from Theorem 4.7.

If mn is an odd, setting $p = -\frac{mn+1}{2}$ in $y_1 + y_2 + \cdots + y_n = \frac{p\pi}{m} + \frac{n\pi}{2} + \frac{\pi}{2m}$ and $(-1)^p = i^{-mn-1}$, we have $y_1 + y_2 + \cdots + y_n = 0$. We get (4.39) of Corollary 4.8 from Theorem 4.7. The proof is complete. $\square$

Remark 4.9 Equation (4.34) of Corollary 4.8 is just the main result of Zhu (see [24, p.120, Theorem 1.7]). Of course, we need to make some transformations $r_j \mapsto r_j - \frac{m}{2}$,

$j = 1, 2, \dots, n$, in (4.40); we readily deduce that

$$R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) = mnq^{-\frac{m^2n}{4}} \sum_{\substack{r_1 + \dots + r_n = \frac{mn}{2} \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}, \quad (4.41)$$

which just is the corresponding result of Zhu (see [24, p.120, (16) of Theorem 1.7]).

Corollary 4.10 *Suppose that m, n are any positive integers. We have*

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \vartheta_3^n \left(z + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(m, n; \tau) \vartheta_2(mnz | m^2 n \tau), \quad m \text{ is even,} \end{aligned} \quad (4.42)$$

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \vartheta_2^n \left(z + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(m, n; \tau) \vartheta_2(mnz | m^2 n \tau), \quad m \text{ is odd and } n \text{ is even,} \end{aligned} \quad (4.43)$$

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \vartheta_2^n \left(z + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(m, n; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m, n \text{ are odd,} \end{aligned} \quad (4.44)$$

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \vartheta_3^n \left(z + \frac{\pi}{2mn} + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(m, n; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m \text{ is even,} \end{aligned} \quad (4.45)$$

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \vartheta_2^n \left(z + \frac{\pi}{2mn} + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_1(mnz | m^2 n \tau), \quad m \text{ is odd and } n \text{ is even,} \end{aligned} \quad (4.46)$$

$$\begin{aligned} & \sum_{k=0}^{mn-1} (-1)^k \vartheta_2^n \left(z + \frac{k\pi}{mn} \middle| \tau \right) \\ &= R_{4,4}(y_1, y_2, \dots, y_n; m, n, p; \tau) \vartheta_2(mnz | m^2 n \tau), \quad m, n \text{ are odd,} \end{aligned} \quad (4.47)$$

where

$$R_{4,4}(m, n; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2}. \quad (4.48)$$

Proof We set $y_1 = y_2 = \dots = y_n = 0$ in (4.34), (4.35), and (4.39) of Corollary 4.8. We take $y_1 = y_2 = \dots = y_n = \frac{\pi}{2mn}$ in (4.37) and (4.38) of Corollary 4.8. We deduce Corollary 4.10. $\square$

Remark 4.11 Equation (4.42) of Corollary 4.10 is an extension of Boon's result. On taking $n = 1$ in (4.42), and noting that $R_{4,4}(m, 1; \tau) = m$, we have

$$\sum_{k=0}^{m-1} (-1)^k \vartheta_3 \left(z + \frac{k\pi}{m} \middle| \tau \right) = m \vartheta_2(mz | m^2 \tau), \quad m \text{ is even,} \quad (4.49)$$

which is just Boon's result (see [8, p.3440, the second identity of (8)]).

Remark 4.12 Equation (4.45) of Corollary 4.10 is also an extension of Boon's result. On taking $n = 1$ in (4.45), and noting that $R_{4,4}(m, 1; \tau) = m$, we have

$$\sum_{k=0}^{m-1} (-1)^k \vartheta_3 \left(z + \frac{\pi}{2m} + \frac{k\pi}{m} \middle| \tau \right) = R_{4,4}(m, 1; \tau) \vartheta_1(mz | m^2 \tau), \quad m \text{ is even,} \quad (4.50)$$

which is just Boon's result (see [8, p.3440, the first identity of (8)]).

Remark 4.13 Equation (4.42) of Corollary 4.10 is an alternating summation of Ramanujan's circular summation. Taking $m = 2$ in (4.42), we have

$$\sum_{k=0}^{2n-1} (-1)^k \vartheta_3^n \left(z + \frac{k\pi}{2n} \middle| \tau \right) = R_{4,4}(n; \tau) \vartheta_2(2nz | 4n\tau). \quad (4.51)$$

Remark 4.14 We note that Boon's result (see [8, p.3440, the first identity of (8)]) has a misprint, we have here corrected this point in our formula (4.50).

Remark 4.15 If we make the transformations $z \mapsto z + \frac{\pi}{2}$, $z \mapsto z + \frac{mn\pi\tau}{2}$, $z \mapsto z + \frac{\pi}{2} + \frac{mn\pi\tau}{2}$, and using the transformation formulas (1.9)-(1.14) for Theorem 3.1, we can also obtain the corresponding alternating circular summation formulas.

5 The inverse formulas for Ramanujan's circular summation formula

In [15], Liu obtained the following two results.

Lemma 5.1 (see [15, p.1978, Theorem 1.1]) *Suppose that n is a positive integer and $f(z|\tau)$ is an entire function of z satisfying the functional equations*

$$f(z|\tau) = f(z + \pi|\tau) = q^n e^{2niz} f(z + \pi\tau|\tau). \quad (5.1)$$

Then for any positive integer m , there exists a constant $a_m^{(1)}$ independent of z such that

$$\sum_{k=0}^{n-1} e^{-\frac{2mkin}{n}} f \left(z + \frac{k\pi}{n} \middle| \tau \right) = na_m^{(1)} e^{2miz} \vartheta_3(nz + m\pi\tau | n\tau), \quad (5.2)$$

$$f(z|\tau) = \sum_{m=0}^{n-1} a_m^{(1)} e^{2miz} \vartheta_3(nz + m\pi\tau | n\tau). \quad (5.3)$$

Lemma 5.2 (see [15, p.1978, Theorem 1.2]) *Suppose that n is a positive integer and $f(z|\tau)$ is an entire function of z satisfying the functional equations*

$$f(z|\tau) = (-1)^n f(z + \pi|\tau) = -q^n e^{2niz} f(z + \pi\tau|\tau). \quad (5.4)$$

Then for any positive integer m , there exists a constant $a_m^{(2)}$ independent of z such that

$$\sum_{k=0}^{n-1} (-1)^k e^{-\frac{2mk\pi}{n}} f\left(z + \frac{k\pi}{n} \middle| \tau\right) = na_m^{(2)} e^{2miz} \vartheta_1(nz + m\pi\tau | n\tau), \quad (5.5)$$

$$f(z|\tau) = \sum_{m=0}^{n-1} a_m^{(2)} e^{2miz} \vartheta_1(nz + m\pi\tau | n\tau). \quad (5.6)$$

Liu also found many new identities of the theta functions from Lemma 5.1 and Lemma 5.2.

Below we give some new inverse relations for theta function $\vartheta_4(z|\tau)$ by using the results of Liu.

Theorem 5.3 Suppose that m, n are any positive integers, p is any integer; $y_1, y_2, \dots, y_n$ are any complex numbers.

- When $y_1 + y_2 + \dots + y_n = p\pi + \frac{n\pi}{2}$, we have

$$\sum_{k=0}^{n-1} e^{-\frac{2imk\pi}{n}} \prod_{j=1}^n \vartheta_4\left(z + y_j + \frac{k\pi}{n} \middle| \tau\right) = na_m^{(1)} e^{2miz} \vartheta_3(nz + m\pi\tau | n\tau), \quad (5.7)$$

$$\prod_{j=1}^n \vartheta_4(z + y_j | \tau) = \sum_{m=0}^{n-1} a_m^{(1)} e^{2miz} \vartheta_3(nz + m\pi\tau | n\tau). \quad (5.8)$$

- When $y_1 + y_2 + \dots + y_n = p\pi + \frac{n\pi}{2} + \frac{\pi}{2}$ and n is even, we have

$$\sum_{k=0}^{n-1} (-1)^k e^{-\frac{2imk\pi}{n}} \prod_{j=1}^n \vartheta_4\left(z + y_j + \frac{k\pi}{n} \middle| \tau\right) = na_m^{(2)} e^{2miz} \vartheta_1(nz + m\pi\tau | n\tau), \quad (5.9)$$

$$\prod_{j=1}^n \vartheta_4(z + y_j | \tau) = \sum_{m=0}^{n-1} a_m^{(2)} e^{2miz} \vartheta_1(nz + m\pi\tau | n\tau). \quad (5.10)$$

Here

$$a_m^{(1)} = (-1)^m \sum_{\substack{r_1, \dots, r_n = -\infty \\ r_1 + \dots + r_n = m}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}, \quad (5.11)$$

$$a_m^{(2)} = (-1)^m i^{n+1} q^{-m - \frac{n}{4}} \sum_{\substack{r_1, \dots, r_n = -\infty \\ 2r_1 + \dots + 2r_n = 2m + n}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (5.12)$$

Proof Let

$$f(z|\tau) = \prod_{j=1}^n \vartheta_4(z + y_j | \tau).$$

We easily obtain by using the first and second identities of (1.8) for $n = 1$, respectively:

$$f(z + \pi | \tau) = \prod_{j=1}^n \vartheta_4(z + \pi + y_j | \tau) = \prod_{j=1}^n \vartheta_4(z + y_j | \tau) = f(z | \tau), \quad (5.13)$$

$$f(z + \pi \tau | \tau) = \prod_{j=1}^n \vartheta_4(z + \pi \tau + y_j | \tau) = (-1)^n q^{-n} e^{-2niz} e^{-2i(y_1 + y_2 + \dots + y_n)} f(z | \tau). \quad (5.14)$$

- When $y_1 + y_2 + \dots + y_n = p\pi + \frac{n\pi}{2}$ in (5.14). From (5.13) and (5.14), we have

$$f(z | \tau) = f(z + \pi | \tau) = q^n e^{2niz} f(z + \pi \tau), \quad (5.15)$$

which satisfies condition (5.1). By Lemma 5.1 we obtain (5.7) and (5.8) of Theorem 5.3 at once.

Next we compute the constant $a_m^{(1)}$ independent of z . We apply the definitions (1.3) and (1.4).

Setting

$$z \mapsto nz + m\pi \tau \quad \text{and} \quad \tau \mapsto n\tau$$

in (1.3), we get

$$\vartheta_3(nz + m\pi \tau | n\tau) = \sum_{r=-\infty}^{\infty} q^{nr^2} e^{2ri(nz + m\pi \tau)}. \quad (5.16)$$

Setting

$$z \mapsto z + y_j + \frac{k\pi}{n}$$

in (1.4), we get

$$\begin{aligned} & \prod_{j=1}^n \vartheta_4\left(z + y_j + \frac{k\pi}{n} \middle| \tau\right) \\ &= \sum_{r_1, \dots, r_n = -\infty}^{\infty} (-1)^{r_1 + \dots + r_n} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)} e^{\frac{k\pi i}{n}(r_1 + \dots + r_n)} e^{2i(r_1 + \dots + r_n)z}. \end{aligned} \quad (5.17)$$

Substituting (5.16) and (5.17) into (5.7), we find that

$$\begin{aligned} & \sum_{k=0}^{n-1} e^{-\frac{2imk\pi}{n}} \sum_{r_1, \dots, r_n = -\infty}^{\infty} (-1)^{r_1 + \dots + r_n} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)} e^{\frac{k\pi i}{n}(r_1 + \dots + r_n)} e^{2i(r_1 + \dots + r_n)z} \\ &= na_m e^{2miz} \sum_{r=-\infty}^{\infty} q^{nr^2} e^{2ri(nz + m\pi \tau)}. \end{aligned} \quad (5.18)$$

Comparing the constants of both sides of (5.18) and noting that $y_1 + y_2 + \dots + y_n = p\pi + \frac{n\pi}{2}$, we obtain (5.11) of Theorem 5.3.

- When $y_1 + y_2 + \dots + y_n = p\pi + \frac{n\pi}{2} + \frac{\pi}{2}$ in (5.14) and n is *only* even. From (5.13) and (5.14), we have

$$f(z | \tau) = (-1)^n f(z + \pi | \tau) = -q^n e^{2niz} f(z + \pi \tau), \quad (5.19)$$

which satisfies condition (5.4) of Lemma 5.2. From Lemma 5.2 we obtain (5.9) and (5.10) of Theorem 5.3 immediately. In a similar way we can prove (5.12). The proof is complete. $\square$

Corollary 5.4 *Suppose that m, n are any positive integers; $y_1, y_2, \dots, y_n$ are any complex numbers.*

- When $y_1 + y_2 + \dots + y_n = 0$ and n is even, we have

$$\sum_{k=0}^{n-1} \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{n} \middle| \tau \right) = na_0^{(1)} \vartheta_3(nz|n\tau), \quad (5.20)$$

$$\prod_{j=1}^n \vartheta_4(z + y_j|\tau) = \sum_{m=0}^{n-1} a_m^{(1)} e^{2miz} \vartheta_3(nz + m\pi\tau|n\tau). \quad (5.21)$$

- When $y_1 + y_2 + \dots + y_n = \frac{\pi}{2}$ and n is even, we have

$$\sum_{k=0}^{n-1} (-1)^k \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{n} \middle| \tau \right) = na_0^{(2)} \vartheta_1(nz|n\tau), \quad (5.22)$$

$$\prod_{j=1}^n \vartheta_4(z + y_j|\tau) = \sum_{m=0}^{n-1} a_m^{(2)} e^{2miz} \vartheta_1(nz + m\pi\tau|n\tau). \quad (5.23)$$

Here

$$a_0^{(1)} = \sum_{\substack{r_1, \dots, r_n = -\infty \\ r_1 + \dots + r_n = 0}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}, \quad (5.24)$$

$$a_0^{(2)} = i^{n+1} q^{-\frac{n}{4}} \sum_{\substack{r_1, \dots, r_n = -\infty \\ 2r_1 + \dots + 2r_n = n}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}, \quad (5.25)$$

$$a_m^{(1)} = (-1)^m \sum_{\substack{r_1, \dots, r_n = -\infty \\ r_1 + \dots + r_n = m}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}, \quad (5.26)$$

$$a_m^{(2)} = (-1)^m i^{n+1} q^{-m - \frac{n}{4}} \sum_{\substack{r_1, \dots, r_n = -\infty \\ 2r_1 + \dots + 2r_n = 2m + n}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (5.27)$$

Proof If n is an even, setting $p = -\frac{n}{2}$ in $y_1 + y_2 + \dots + y_n = p\pi + \frac{n\pi}{2}$, then we have $y_1 + y_2 + \dots + y_n = 0$. We directly deduce (5.20) (setting $m = 0$) and (5.21) of Corollary 5.4 from Theorem 5.3.

If n is an even, setting $p = -\frac{n}{2}$ in $y_1 + y_2 + \dots + y_n = p\pi + \frac{n\pi}{2} + \frac{\pi}{2}$, we have $y_1 + y_2 + \dots + y_n = \frac{\pi}{2}$. We get (5.22) (setting $m = 0$) and (5.23) of Corollary 5.4 from Theorem 5.3. The proof is complete. $\square$

Setting $n = 2$ in (5.20) noting that $y_1 + y_2 = 0$ and $a_0^{(1)} = \vartheta_3(2y_1|2\tau)$, we have

$$\vartheta_3(z + y_1|\tau) \vartheta_3(z - y_1|\tau) + \vartheta_4(z + y_1|\tau) \vartheta_4(z - y_1|\tau) = 2\vartheta_3(2y_1|2\tau) \vartheta_3(2z|2\tau). \quad (5.28)$$

Setting $n = 2$ in (5.21), noting that $y_1 + y_2 = 0$ and $a_0^{(1)} = \vartheta_3(2y_1|2\tau)$, $a_1^{(1)} = -e^{-2iy_1} \times \vartheta_3(2y_1|2\tau)$, we have

$$\begin{aligned} & \vartheta_4(z + y_1|\tau)\vartheta_4(z - y_1|\tau) \\ &= \vartheta_3(2y_1|2\tau)\vartheta_3(2z|2\tau) - e^{2i(z-y_1)}\vartheta_3(2y_1|2\tau)\vartheta_3(2z + \pi\tau|2\tau). \end{aligned} \quad (5.29)$$

Setting $n = 2$ in (5.22) noting that $y_1 + y_2 = \frac{\pi}{2}$ and $a_0^{(2)} = \vartheta_3(2y_1|2\tau)$, we have

$$\sum_{k=0}^{n-1} (-1)^k \prod_{j=1}^n \vartheta_4\left(z + y_j + \frac{k\pi}{n} \middle| \tau\right) = na_0^{(2)} \vartheta_1(nz|n\tau), \quad (5.30)$$

$$\prod_{j=1}^n \vartheta_4(z + y_j|\tau) = \sum_{m=0}^{n-1} a_m^{(2)} e^{2miz} \vartheta_1(nz + m\pi\tau|n\tau), \quad (5.31)$$

$$a_m^{(2)} = (-1)^m i^{n+1} q^{-m-\frac{n}{4}} \sum_{\substack{r_1, \dots, r_n = -\infty \\ 2r_1 + \dots + 2r_n = 2m+n}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (5.32)$$

Corollary 5.5 Assume m is any positive integer.

- When n is even, we have

$$\sum_{k=0}^{n-1} \vartheta_4^n\left(z + \frac{k\pi}{n} \middle| \tau\right) = na_0^{(1)} \vartheta_3(nz|n\tau), \quad (5.33)$$

$$\vartheta_4^n(z|\tau) = \sum_{m=0}^{n-1} a_m^{(1)} e^{2miz} \vartheta_3(nz + m\pi\tau|n\tau). \quad (5.34)$$

- When n is even, we have

$$\sum_{k=0}^{n-1} (-1)^k \vartheta_4^n\left(z + \frac{\pi}{2n} + \frac{k\pi}{n} \middle| \tau\right) = na_0^{(2)} \vartheta_1(nz|n\tau), \quad (5.35)$$

$$\vartheta_4^n\left(z + \frac{\pi}{2n} \middle| \tau\right) = \sum_{m=0}^{n-1} a_m^{(2)} e^{2miz} \vartheta_1(nz + m\pi\tau|n\tau). \quad (5.36)$$

Here

$$a_0^{(1)} = \sum_{\substack{r_1, \dots, r_n = -\infty \\ r_1 + \dots + r_n = 0}}^{\infty} q^{r_1^2 + \dots + r_n^2}, \quad (5.37)$$

$$a_0^{(2)} = i^n q^{-\frac{n}{4}} \sum_{\substack{r_1, \dots, r_n = -\infty \\ 2r_1 + \dots + 2r_n = n}}^{\infty} q^{r_1^2 + \dots + r_n^2}, \quad (5.38)$$

$$a_m^{(1)} = (-1)^m \sum_{\substack{r_1, \dots, r_n = -\infty \\ r_1 + \dots + r_n = m}}^{\infty} q^{r_1^2 + \dots + r_n^2}, \quad (5.39)$$

$$a_m^{(2)} = (-1)^m i^n q^{-m-\frac{n}{4}} e^{\frac{m\pi i}{n}} \sum_{\substack{r_1, \dots, r_n = -\infty \\ 2r_1 + \dots + 2r_n = 2m+n}}^{\infty} q^{r_1^2 + \dots + r_n^2}. \quad (5.40)$$

Proof We set $y_1 = y_2 = \dots = y_n = 0$ in (5.20) and (5.21) of Corollary 5.4. We take $y_1 = y_2 = \dots = y_n = \frac{\pi}{2n}$ in (5.22) and (5.31) of Corollary 5.4. We obtain Corollary 5.5. $\square$

6 Some applications

The well-known cubic theta function $a(\tau)$ is defined (see [9]) by

$$a(\tau) = \sum_{r_1, r_2 = -\infty}^{\infty} q^{r_1^2 + r_1 r_2 + r_2^2}. \quad (6.1)$$

We define the multiple theta series $a(y_1, y_2 | \tau)$ as

$$a(y_1, y_2 | \tau) = \sum_{r_1, r_2 = -\infty}^{\infty} q^{r_1^2 + r_1 r_2 + r_2^2} e^{2i(r_1 y_1 + r_2 y_2)}. \quad (6.2)$$

Obviously, $a(\tau) = a(0, 0 | \tau)$.

We give some applications of Theorem 2.1 below.

Corollary 6.1 *Suppose that m, n are any positive integers; $y_1, y_2, \dots, y_n$ are any complex numbers.*

- When $y_1 + y_2 + \dots + y_n = 0$, we have

$$\begin{aligned} & \sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) \\ &= \begin{cases} R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_3(mnz | m^2 n \tau), & mn \text{ is even,} \\ R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_4(mnz | m^2 n \tau), & mn \text{ is odd.} \end{cases} \end{aligned} \quad (6.3)$$

- When $y_1 + y_2 + \dots + y_n = \frac{\pi}{2m}$ and mn is even, we have

$$\sum_{k=0}^{mn-1} \prod_{j=1}^n \vartheta_4 \left(z + y_j + \frac{k\pi}{mn} \middle| \tau \right) = R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) \vartheta_4(mnz | m^2 n \tau), \quad (6.4)$$

where

$$R_{4,4}(y_1, y_2, \dots, y_n; m, n; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2i(r_1 y_1 + \dots + r_n y_n)}. \quad (6.5)$$

Proof When mn is even, taking $p = -\frac{mn}{2}$ in (2.1); when mn is odd, taking $p = -\frac{mn+1}{2}$ in (2.2), we easily get (6.3).

When mn is even, taking $p = -\frac{mn}{2}$ in (2.2), we easily obtain (6.4). $\square$

Corollary 6.2 *Suppose that m, n are any positive integers, a, b are nonnegative integers and $a + b = n$; y is any complex numbers.*

$$\begin{aligned} & \sum_{k=0}^{mn-1} \vartheta_4^a \left(z + by + \frac{k\pi}{mn} \middle| \tau \right) \vartheta_4^b \left(z - ay + \frac{k\pi}{mn} \middle| \tau \right) \\ &= \begin{cases} R_{4,4}(y; m, n, a, b; \tau) \vartheta_3(mnz | m^2 n \tau), & mn \text{ is even,} \\ R_{4,4}(y; m, n, a, b; \tau) \vartheta_4(mnz | m^2 n \tau), & mn \text{ is odd.} \end{cases} \end{aligned} \quad (6.6)$$

- When mn is even, we have

$$\sum_{k=0}^{mn-1} \vartheta_4^n \left(z + \frac{\pi}{2mn} + \frac{k\pi}{mn} \middle| \tau \right) = R_{4,4}(m, n; \tau) \vartheta_4(mnz | m^2 n \tau). \quad (6.7)$$

Here

$$R_{4,4}(y; m, n, a, b; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2} e^{2niy(r_1 + \dots + r_n)}, \quad (6.8)$$

$$R_{4,4}(m, n; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2}. \quad (6.9)$$

Proof When $y_1 + y_2 + \dots + y_n = 0$, taking $y_1 = y_2 = \dots = y_a = by$ and $y_{a+1} = y_{a+2} = \dots = y_n = -ay$, $a + b = n$ in Corollary 6.1, we have (6.6).

When $y_1 + y_2 + \dots + y_n = \frac{\pi}{2m}$, taking $y_1 = y_2 = \dots = y_n = \frac{\pi}{2mn}$, in Corollary 6.1, we have (6.7). $\square$

Remark 6.3 Let $z \mapsto z + \frac{\pi}{2}$ in (6.7) and using the transformation formulas (1.8) and (1.12), we have

$$\sum_{k=0}^{mn-1} \vartheta_3^n \left(z + \frac{\pi}{2mn} + \frac{k\pi}{mn} \middle| \tau \right) = R_{4,4}(m, n; \tau) \vartheta_4(mnz | m^2 n \tau), \quad (6.10)$$

which is a new formula; it is also an extension of Boon's result.

If we take $n = 1$ in (6.10) and (6.9), we deduce

$$\sum_{k=0}^{m-1} \vartheta_3 \left(z + \frac{(2k+1)\pi}{2m} \middle| \tau \right) = m \vartheta_4(mz | m^2 \tau), \quad (6.11)$$

which is just Boon's result (see [8, p.3440]).

Remark 6.4 We here have corrected a misprint concerning the formulas of $\vartheta_1(nz | n\tau)$ and $\vartheta_4(nz | n\tau)$ in [8, p.3440].

Taking $y_1 = y_2 = \dots = y_n = 0$ in (6.3), we have the following.

Corollary 6.5 Suppose that m, n are any positive integers. We have

$$\sum_{k=0}^{mn-1} \vartheta_4^n \left(z + \frac{k\pi}{mn} \middle| \tau \right) = \begin{cases} R_{4,4}(m, n; \tau) \vartheta_3(mnz | m^2 n \tau), & mn \text{ is even,} \\ R_{4,4}(m, n; \tau) \vartheta_4(mnz | m^2 n \tau), & mn \text{ is odd.} \end{cases} \quad (6.12)$$

Here

$$R_{4,4}(m, n; \tau) = mn \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2}. \quad (6.13)$$

Taking $n = 1$ in (6.12), we deduce

$$\sum_{k=0}^{m-1} \vartheta_4\left(z + \frac{k\pi}{m} \middle| \tau\right) = \begin{cases} m\vartheta_3(mz|m^2\tau), & m \text{ is even,} \\ m\vartheta_4(mz|m^2\tau), & m \text{ is odd,} \end{cases} \quad (6.14)$$

which is just Boon's result (see [8, p.3440, Eq. (7)]), with $z \mapsto z + \frac{\pi}{2}$ and the transformation equation (1.7) and (1.12).

Taking $n = 2$ in (6.12) and noting that

$$R_{4,4}^{(1)}(m; \tau) = 2m \sum_{\substack{r_1+r_2=0 \\ r_1, r_2=-\infty}}^{\infty} q^{r_1^2+r_2^2} = 2m\vartheta_3(0|2\tau), \quad (6.15)$$

we have the following.

Corollary 6.6 *Suppose that m is any positive integer; we have*

$$\sum_{k=0}^{2m-1} \vartheta_4^2\left(z + \frac{k\pi}{2m} \middle| \tau\right) = 2m\vartheta_3(0|2\tau)\vartheta_3(2mz|2m^2\tau). \quad (6.16)$$

The above formula is just Boon's result [8, p.3440, Eq. (10)], with $z \mapsto z + \frac{\pi}{2}$ and the transformation equation (1.7) and (1.12).

Taking $n = 3$ in (6.12) and noting that

$$R_{4,4}(m; \tau) = 3m \sum_{\substack{r_1+r_2+r_3=0 \\ r_1, r_2, r_3=-\infty}}^{\infty} q^{r_1^2+r_2^2+r_3^2} = 3ma(2\tau), \quad (6.17)$$

we have the following.

Corollary 6.7 *Suppose that m is a positive integer.*

$$\sum_{k=0}^{3m-1} \vartheta_4^3\left(z + \frac{k\pi}{3m} \middle| \tau\right) = \begin{cases} 3ma(2\tau)\vartheta_3(3mz|3m^2\tau), & m \text{ is even,} \\ 3ma(2\tau)\vartheta_4(3mz|3m^2\tau), & m \text{ is odd.} \end{cases} \quad (6.18)$$

We set $m = 1$,

$$\vartheta_4^3(z|\tau) + \vartheta_4^3\left(z + \frac{\pi}{3} \middle| \tau\right) + \vartheta_4^3\left(z - \frac{\pi}{3} \middle| \tau\right) = 3a(2\tau)\vartheta_4(3z|3\tau). \quad (6.19)$$

Taking $m = 1$ in (6.12), we have

$$\sum_{k=0}^{n-1} \vartheta_4^n\left(z + \frac{k\pi}{n} \middle| \tau\right) = \begin{cases} R_{4,4}(n; \tau)\vartheta_3(nz|n\tau), & n \text{ is even,} \\ R_{4,4}(n; \tau)\vartheta_4(nz|n\tau), & n \text{ is odd.} \end{cases} \quad (6.20)$$

Here

$$R_{4,4}(n; \tau) = n \sum_{\substack{r_1+\dots+r_n=0 \\ r_1, \dots, r_n=-\infty}}^{\infty} q^{r_1^2+\dots+r_n^2}. \quad (6.21)$$

The above formula is just Ramanujan's circular summation formula (see [6, p.632, Theorem 3.1]), with $z \mapsto z + \frac{\pi}{2}$ and the transformation equation (1.7) and (1.12).

We set $m = 1$ in (6.10),

$$\sum_{k=0}^{n-1} \vartheta_3^n \left(z + \frac{\pi}{2n} + \frac{k\pi}{n} \middle| \tau \right) = R_{4,4}(n; \tau) \vartheta_4(nz|n\tau), \quad (6.22)$$

where

$$R_{4,4}(n; \tau) = n \sum_{\substack{r_1 + \dots + r_n = 0 \\ r_1, \dots, r_n = -\infty}}^{\infty} q^{r_1^2 + \dots + r_n^2}. \quad (6.23)$$

We set $n = 2$ in (6.10),

$$\vartheta_3^2 \left(z + \frac{\pi}{4} \middle| \tau \right) + \vartheta_4^2 \left(z + \frac{\pi}{4} \middle| \tau \right) = 2\vartheta_3(0|2\tau)\vartheta_4(2z|2\tau). \quad (6.24)$$

We set $n = 3$ in (6.10),

$$\vartheta_3^3 \left(z + \frac{\pi}{6} \middle| \tau \right) + \vartheta_3^3 \left(z - \frac{\pi}{6} \middle| \tau \right) + \vartheta_4^3(z|\tau) = 3a(2\tau)\vartheta_4(3z|3\tau). \quad (6.25)$$

Comparing (6.19) and (6.25), we find that

$$\vartheta_3^3 \left(z + \frac{\pi}{6} \middle| \tau \right) + \vartheta_3^3 \left(z - \frac{\pi}{6} \middle| \tau \right) = \vartheta_4^3 \left(z + \frac{\pi}{3} \middle| \tau \right) + \vartheta_4^3 \left(z - \frac{\pi}{3} \middle| \tau \right). \quad (6.26)$$

Taking $n = 3$ in (6.3) and noting that $y_1 + y_2 + y_3 = 0$, we obtain

$$R_{4,4}(y_1, y_2, ; \tau) = 3ma(y_1 - y_3, y_2 - y_3|2\tau).$$

Hence, we have

$$\begin{aligned} & \sum_{k=0}^{3m-1} \vartheta_4 \left(z + y_1 + \frac{k\pi}{3m} \middle| \tau \right) \vartheta_4 \left(z + y_2 + \frac{k\pi}{3m} \middle| \tau \right) \vartheta_4 \left(z + y_3 + \frac{k\pi}{3m} \middle| \tau \right) \\ &= \begin{cases} 3ma(y_1 - y_3, y_2 - y_3|2\tau)\vartheta_3(3mz|3m^2\tau), & m \text{ is even,} \\ 3ma(y_1 - y_3, y_2 - y_3|2\tau)\vartheta_4(3mz|3m^2\tau), & m \text{ is odd,} \end{cases} \end{aligned} \quad (6.27)$$

where $a(y_1, y_2|2\tau)$ is defined by (6.2).

Taking $m = 2$ in the first equation of (6.27), we get

$$\begin{aligned} & \vartheta_3(z + y_1|\tau)\vartheta_3(z + y_2|\tau)\vartheta_3(z + y_3|\tau) + \vartheta_4(z + y_1|\tau)\vartheta_4(z + y_2|\tau)\vartheta_4(z + y_3|\tau) \\ &+ \vartheta_4 \left(z + y_1 + \frac{\pi}{6} \middle| \tau \right) \vartheta_4 \left(z + y_2 + \frac{\pi}{6} \middle| \tau \right) \vartheta_4 \left(z + y_3 + \frac{\pi}{6} \middle| \tau \right) \\ &+ \vartheta_4 \left(z + y_1 - \frac{\pi}{6} \middle| \tau \right) \vartheta_4 \left(z + y_2 - \frac{\pi}{6} \middle| \tau \right) \vartheta_4 \left(z + y_3 - \frac{\pi}{6} \middle| \tau \right) \\ &+ \vartheta_4 \left(z + y_1 + \frac{\pi}{3} \middle| \tau \right) \vartheta_4 \left(z + y_2 + \frac{\pi}{3} \middle| \tau \right) \vartheta_4 \left(z + y_3 + \frac{\pi}{3} \middle| \tau \right) \end{aligned}$$

$$\begin{aligned}
 & + \vartheta_4\left(z + y_1 - \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_2 - \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_3 - \frac{\pi}{3} \middle| \tau\right) \\
 & = 6a(y_1 - y_3, y_2 - y_3 | 2\tau) \vartheta_3(6z | 12\tau).
 \end{aligned} \tag{6.28}$$

Setting $y_1 = y_2 = y_3 = 0$ in (6.28), we get

$$\begin{aligned}
 & \vartheta_3^3(z | \tau) + \vartheta_4^3(z | \tau) + \vartheta_4^3\left(z + \frac{\pi}{6} \middle| \tau\right) + \vartheta_4^3\left(z - \frac{\pi}{6} \middle| \tau\right) \\
 & + \vartheta_4^3\left(z + \frac{\pi}{3} \middle| \tau\right) + \vartheta_4^3\left(z - \frac{\pi}{3} \middle| \tau\right) \\
 & = 6a(2\tau) \vartheta_3(6z | 12\tau).
 \end{aligned} \tag{6.29}$$

Taking $m = 1$ in the second equation of (6.27), we get

$$\begin{aligned}
 & \vartheta_4(z + y_1 | \tau) \vartheta_4(z + y_2 | \tau) \vartheta_4(z + y_3 | \tau) \\
 & + \vartheta_4\left(z + y_1 + \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_2 + \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_3 + \frac{\pi}{3} \middle| \tau\right) \\
 & + \vartheta_4\left(z + y_1 - \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_2 - \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_3 - \frac{\pi}{3} \middle| \tau\right) \\
 & = 3a(y_1 - y_3, y_2 - y_3 | 2\tau) \vartheta_4(3z | 3\tau).
 \end{aligned} \tag{6.30}$$

Setting $y_1 = y_2 = y_3 = 0$ in (6.28), we get again

$$\vartheta_3^3\left(z + \frac{\pi}{6} \middle| \tau\right) + \vartheta_3^3\left(z - \frac{\pi}{6} \middle| \tau\right) + \vartheta_4^3(z | \tau) = 3a(2\tau) \vartheta_4(3z | 3\tau). \tag{6.31}$$

Taking $n = 3$ in (6.4) and noting that $y_1 + y_2 + y_3 = \frac{\pi}{2m}$, we obtain

$$R_{4,4}(y_1, y_2, ; \tau) = 3ma(y_1 - y_3, y_2 - y_3 | 2\tau).$$

Hence, we have

$$\begin{aligned}
 & \sum_{k=0}^{3m-1} \vartheta_4\left(z + y_1 + \frac{k\pi}{3m} \middle| \tau\right) \vartheta_4\left(z + y_2 + \frac{k\pi}{3m} \middle| \tau\right) \vartheta_4\left(z + y_3 + \frac{k\pi}{3m} \middle| \tau\right) \\
 & = 3ma(y_1 - y_3, y_2 - y_3 | 2\tau) \vartheta_4(3mz | 3m^2\tau), \quad m \text{ is even.}
 \end{aligned} \tag{6.32}$$

Taking $m = 2$ in (6.32) and noting that $y_1 + y_2 + y_3 = \frac{\pi}{4}$, we get

$$\begin{aligned}
 & \vartheta_3(z + y_1 | \tau) \vartheta_3(z + y_2 | \tau) \vartheta_3(z + y_3 | \tau) + \vartheta_4(z + y_1 | \tau) \vartheta_4(z + y_2 | \tau) \vartheta_4(z + y_3 | \tau) \\
 & + \vartheta_4\left(z + y_1 + \frac{\pi}{6} \middle| \tau\right) \vartheta_4\left(z + y_2 + \frac{\pi}{6} \middle| \tau\right) \vartheta_4\left(z + y_3 + \frac{\pi}{6} \middle| \tau\right) \\
 & + \vartheta_4\left(z + y_1 - \frac{\pi}{6} \middle| \tau\right) \vartheta_4\left(z + y_2 - \frac{\pi}{6} \middle| \tau\right) \vartheta_4\left(z + y_3 - \frac{\pi}{6} \middle| \tau\right) \\
 & + \vartheta_4\left(z + y_1 + \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_2 + \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_3 + \frac{\pi}{3} \middle| \tau\right)
 \end{aligned}$$

$$\begin{aligned}
 & + \vartheta_4\left(z + y_1 - \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_2 - \frac{\pi}{3} \middle| \tau\right) \vartheta_4\left(z + y_3 - \frac{\pi}{3} \middle| \tau\right) \\
 & = 6a(y_1 - y_3, y_2 - y_3 | 2\tau) \vartheta_4(6z | 12\tau).
 \end{aligned} \tag{6.33}$$

Setting $y_1 = y_2 = y_3 = \frac{\pi}{12}$ in (6.33), we get the following new identity for the theta function $\vartheta_4(z | \tau)$:

$$\begin{aligned}
 & \vartheta_3^3\left(z + \frac{\pi}{12} \middle| \tau\right) + \vartheta_4^3\left(z + \frac{\pi}{12} \middle| \tau\right) + \vartheta_4^3\left(z - \frac{\pi}{12} \middle| \tau\right) + \vartheta_4^3\left(z + \frac{\pi}{4} \middle| \tau\right) \\
 & + \vartheta_4^3\left(z - \frac{\pi}{4} \middle| \tau\right) + \vartheta_4^3\left(z + \frac{5\pi}{12} \middle| \tau\right) \\
 & = 6a(2\tau) \vartheta_4(6z | 12\tau).
 \end{aligned} \tag{6.34}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in writing this paper, and read and approved the manuscript.

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