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Barnes' multiple Bernoulli and generalized Barnes' multiple Frobenius-Euler mixed-type polynomials

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Abstract
In this paper, by considering Barnes' multiple Bernoulli polynomials as well as generalized Barnes' multiple Frobenius-Euler polynomials, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.
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1 Introduction

In this paper, we consider the polynomials

$$BH_n(x) = BH_n(x|a;b;\lambda;\mu) = BH_n(x|a_1,\dots,a_r;b_1,\dots,b_s;\lambda_1,\dots,\lambda_s;\mu_1,\dots,\mu_s)$$

called Barnes' multiple Bernoulli and generalized Barnes' multiple Frobenius-Euler mixed-type polynomials, whose generating function is given by

$$\prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)^{\mu_j}e^{xt} = \sum_{n=0}^{\infty}BH_n(x)\frac{t^n}{n!}, \tag{1}$$

where $a_1,\dots,a_r,b_1,\dots,b_s,\lambda_1,\dots,\lambda_s,\mu_1,\dots,\mu_s \in \mathbb{C}$ with $a_1,\dots,a_r,b_1,\dots,b_s \neq 0$, $\lambda_1,\dots,\lambda_s \neq 1$. When $x = 0$,

$$BH_n = BH_n(0) = BH_n(0;a_1,\dots,a_r;b_1,\dots,b_s;\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$$

are called Barnes' multiple Bernoulli and generalized Barnes' multiple Frobenius-Euler mixed-type numbers.

Recall that Barnes' multiple Bernoulli polynomials, denoted by $B_n(x|a_1,\dots,a_r)$, are given by the generating function as

$$\prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)e^{xt} = \sum_{n=0}^{\infty}B_n(x|a_1,\dots,a_r)\frac{t^n}{n!}, \tag{2}$$

where $a_1, \ldots, a_r \neq 0$ [1, 2]. In addition, the generalized Barnes' multiple Frobenius-Euler polynomials, denoted by $H_n(x|b;\lambda;\mu) = H_n(x|b_1,\ldots,b_s;\lambda_1,\ldots,\lambda_s;\mu_1,\ldots,\mu_s)$, are given by the generating function as

$$\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} e^{xt} = \sum_{n=0}^{\infty} H_n(x|b;\lambda;\mu)\frac{t^n}{n!} \tag{3}$$

(see *e.g.* [3–7]). If $\mu_1 = \cdots = \mu_s = 1$, then

$$H_n(x|b_1,\ldots,b_s;\lambda_1,\ldots,\lambda_s) = H_n(x|b_1,\ldots,b_s;\lambda_1,\ldots,\lambda_s;1,\ldots,1)$$

are called Barnes-type Frobenius-Euler polynomials. If further $\lambda_1 = \cdots = \lambda_s = \lambda$ and $b_1 = \cdots = b_s = 1$, then $H_n^{(s)}(x|\lambda) = H_n(x|1,\ldots,1;\lambda,\ldots,\lambda;1,\ldots,1)$ are called Frobenius-Euler polynomials of order s (see *e.g.* [8, 9]). If $\lambda_1 = \cdots = \lambda_s = -1$, then $E_n(x|b_1,\ldots,b_s;\mu_1,\ldots,\mu_s) = H_n(x|b_1,\ldots,b_s;1,\ldots,1;\mu_1,\ldots,\mu_s)$ are called generalized Barnes-type Euler polynomials. These polynomials arise naturally in connection with the study of Barnes-type Peters polynomials. Peters polynomials were mentioned in [10, p.128] and were investigated in *e.g.* [11].

In this paper, by considering Barnes' multiple Bernoulli polynomials as well as generalized Barnes' multiple Frobenius-Euler polynomials, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be a complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{4}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad (n \geq 0). \tag{5}$$

In particular,

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{6}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ *umbral algebra*, and *umbral calculus* is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a *delta series*; if $O(f(t)) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k | s_n(x)\rangle = n!\delta_{n,k}$ for $n, k \geq 0$ [10, Theorem 2.3.1]. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\bigl\langle f(t)g(t)|p(x)\bigr\rangle = \bigl\langle f(t)|g(t)p(x)\bigr\rangle = \bigl\langle g(t)|f(t)p(x)\bigr\rangle, \tag{7}$$

and

$$f(t) = \sum_{k=0}^{\infty} \bigl\langle f(t)|x^k\bigr\rangle \frac{t^k}{k!}, \qquad p(x) = \sum_{k=0}^{\infty} \bigl\langle t^k|p(x)\bigr\rangle \frac{x^k}{k!}, \tag{8}$$

[10, Theorem 2.2.5]. Thus, by (8), we get

$$t^k p(x) = p(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x+y). \tag{9}$$

Sheffer sequences are characterized in the generating function [10, Theorem 2.3.4].

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim (g(t), f(t))$, we have the following equations [10, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t)s_n(x) = n s_{n-1}(x) \quad (n \geq 0), \tag{10}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \bigl\langle g\bigl(\bar{f}(t)\bigr)^{-1} \bar{f}(t)^j | x^n \bigr\rangle x^j, \tag{11}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{12}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then the transfer formula [10, Corollary 3.8.2] is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have [10, p.132]

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle. \tag{13}$$

3 Main results

From definitions (2), (3) and (1), $B_n(x|a_1, \ldots, a_r)$, $H_n(x|b_1, \ldots, b_s; \lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_s)$ and $BH_n(x|a_1, \ldots, a_r; b_1, \ldots, b_s; \lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_s)$ are the Appell sequences for

$$\prod_{i=1}^{r} \left(\frac{e^{a_i t} - 1}{t} \right), \qquad \prod_{j=1}^{s} \left(\frac{e^{b_j t} - \lambda_j}{1 - \lambda_j} \right)^{\mu_j} \quad \text{and} \quad \prod_{i=1}^{r} \left(\frac{e^{a_i t} - 1}{t} \right) \prod_{j=1}^{s} \left(\frac{e^{b_j t} - \lambda_j}{1 - \lambda_j} \right)^{\mu_j},$$

respectively. So,

$$B_n(x|a_1, \ldots, a_r) \sim \left(\sum_{i=1}^{r} \left(\frac{e^{a_i t} - 1}{t} \right), t \right), \tag{14}$$

$$H_n(x|b_1, \ldots, b_s; \lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_s) \sim \left(\prod_{j=1}^{s} \left(\frac{e^{b_j t} - \lambda_j}{1 - \lambda_j} \right)^{\mu_j}, t \right), \tag{15}$$

$$\begin{aligned} &BH_n(x|a_1, \ldots, a_r; b_1, \ldots, b_s; \lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_s) \\ &\quad \sim \left(\prod_{i=1}^{r} \left(\frac{e^{a_i t} - 1}{t} \right) \prod_{j=1}^{s} \left(\frac{e^{b_j t} - \lambda_j}{1 - \lambda_j} \right)^{\mu_j}, t \right). \end{aligned} \tag{16}$$

In particular,

$$\begin{aligned} tB_n(x|a) &= \frac{d}{dx} B_n(x|a) = nB_{n-1}(x|a), \\ tH_n(x|b; \lambda; \mu) &= \frac{d}{dx} H_n(x|b; \lambda; \mu) = nH_{n-1}(x|b; \lambda; \mu), \\ tBH_n(x|a; b; \lambda; \mu) &= \frac{d}{dx} BH_n(x|a; b; \lambda; \mu) = nBH_{n-1}(x|a; b; \lambda; \mu), \end{aligned}$$

where $a = (a_1, \ldots, a_r)$, $b = (b_1, \ldots, b_s)$, $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_s)$.

3.1 Explicit expressions

Let $(n)_j = n(n-1) \cdots (n-j+1)$ $(j \geq 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n, m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n, m) x^m.$$

Notice that

$$(x)_n \sim \left(1, e^t - 1\right). \tag{17}$$

Theorem 1 *We have*

$$BH_n(x|a;b;\lambda;\mu) = \sum_{j=0}^{n} \binom{n}{j} BH_{n-j}x^j \tag{18}$$

$$= \sum_{l=0}^{n} \binom{n}{l} B_{n-l}(a)H_l(x|b;\lambda;\mu) \tag{19}$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(b;\lambda;\mu)B_l(x|a). \tag{20}$$

Proof By (11) with (16), we get

$$\begin{aligned}
&\left\langle g\left(\bar{f}(t)\right)^{-1}\bar{f}(t)^j | x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} t^j \middle| x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} \middle| t^j x^n \right\rangle \\
&= (n)_j\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} \middle| x^{n-j} \right\rangle \\
&= (n)_j\left\langle \sum_{i=0}^{\infty} BH_i \frac{t^i}{i!} \middle| x^{n-j} \right\rangle \\
&= (n)_j BH_{n-j}.
\end{aligned}$$

Thus, we obtain identity (18).

Next,

$$\begin{aligned}
BH_n(y|a;b;\lambda;\mu) &= \left\langle \sum_{i=0}^{\infty} BH_i(y|a;b;\lambda;\mu)\frac{t^i}{i!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} e^{yt} \middle| x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right) \middle| \prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} e^{yt} x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right) \middle| \sum_{l=0}^{\infty} H_l(y|b;\lambda;\mu)\frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l} H_l(y|b;\lambda;\mu)\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right) \middle| x^{n-l} \right\rangle
\end{aligned}$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_l(y|b;\lambda;\mu) \left\langle \sum_{i=0}^{\infty} B_i(a) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle$$
$$= \sum_{l=0}^{n} \binom{n}{l} H_l(y|b;\lambda;\mu) B_{n-l}(a).$$

Thus, we obtain (19).

Finally, we obtain that

$$\begin{aligned}
BH_n(y|a;b;\lambda;\mu) &= \left\langle \sum_{i=0}^{\infty} BH_i(y|a;b;\lambda;\mu) \frac{t^i}{i!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} e^{yt} \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} \middle| \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) e^{yt} x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} \middle| \sum_{l=0}^{\infty} B_l(y|a) \frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} B_l(y|a) \left\langle \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} B_l(y|a) \left\langle \sum_{i=0}^{\infty} H_i(b;\lambda;\mu) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} B_l(y|a) H_{n-l}(b;\lambda;\mu).
\end{aligned}$$

Thus, we get identity (20). □

3.2 The Sheffer identity

Theorem 2

$$BH_n(x+y|a;b;\lambda;\mu) = \sum_{j=0}^{n} \binom{n}{j} BH_j(x|a;b;\lambda;\mu) y^{n-j}. \tag{21}$$

Proof By (16) with

$$\begin{aligned}
p_n(x) &= \prod_{i=1}^{r} \left(\frac{e^{a_i t} - 1}{t} \right) \prod_{j=1}^{s} \left(\frac{e^{b_j t} - \lambda_j}{1-\lambda_j} \right)^{\mu_j} BH_n(x|a;b;\lambda;\mu) \\
&= x^n \sim (1, t),
\end{aligned}$$

using (12), we have (21). □

3.3 Recurrence

Theorem 3

$$\begin{aligned}&\left(1-\frac{r}{n+1}\right)BH_{n+1}(x|a;b;\lambda;\mu)\\&\quad=xBH_n(x|a;b;\lambda;\mu)-\frac{1}{n+1}\sum_{i=1}^{r}a_iBH_{n+1}(x+a_i|a,a_i;b;\lambda;\mu)\\&\qquad-\sum_{j=1}^{s}\frac{\mu_jb_j}{1-\lambda_j}BH_n(x+b_j|a;b;\lambda;\mu+e_j).\end{aligned}\tag{22}$$

Remark When $n=r-1$, as the left-hand side of (22) is equal to 0, we have

$$\begin{aligned}xBH_{r-1}(x|a;b;\lambda;\mu)&=\frac{1}{r}\sum_{i=1}^{r}a_iBH_r(x+a_i|a,a_i;b;\lambda;\mu)\\&\quad+\sum_{j=1}^{s}\frac{\mu_jb_j}{1-\lambda_j}BH_{r-1}(x+b_j|a;b;\lambda;\mu+e_j).\end{aligned}$$

Proof By applying

$$s_{n+1}(x)=\left(x-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x)\tag{23}$$

[10, Corollary 3.7.2] with (16), we get

$$BH_{n+1}(x|a;b;\lambda;\mu)=xBH_n(x|a;b;\lambda;\mu)-\frac{g'(t)}{g(t)}BH_n(x|a;b;\lambda;\mu).$$

Since

$$\begin{aligned}\frac{g'(t)}{g(t)}&=\left(\ln g(t)\right)'=\left(\sum_{i=1}^{r}\ln\left(e^{a_it}-1\right)-r\ln t+\sum_{j=1}^{s}\mu_j\ln\left(e^{b_jt}-\lambda_j\right)-\sum_{j=1}^{s}\mu_j\ln(1-\lambda_j)\right)'\\&=\sum_{i=1}^{r}\frac{a_ie^{a_it}}{e^{a_it}-1}-\frac{r}{t}+\sum_{j=1}^{s}\frac{\mu_jb_je^{b_jt}}{e^{b_jt}-\lambda_j}=\frac{1}{t}\sum_{i=1}^{r}\left(\frac{a_ite^{a_it}}{e^{a_it}-1}-1\right)+\sum_{j=1}^{s}\frac{\mu_jb_je^{b_jt}}{e^{b_jt}-\lambda_j}\end{aligned}$$

and

$$\sum_{i=1}^{r}\left(\frac{a_ite^{a_it}}{e^{a_it}-1}-1\right)=\frac{1}{2}\left(\sum_{i=1}^{r}a_i\right)t+\cdots$$

is a series with order at least one, we have

$$\begin{aligned}&\frac{g'(t)}{g(t)}BH_n(x|a;b;\lambda;\mu)\\&\quad=\left(\frac{1}{t}\sum_{i=1}^{r}\left(\frac{a_ite^{a_it}}{e^{a_it}-1}-1\right)+\sum_{j=1}^{s}\frac{\mu_jb_je^{b_jt}}{1-\lambda_j}\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)BH_n(x|a;b;\lambda;\mu)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t}\sum_{i=1}^{r}\left(\frac{a_i t e^{a_i t}}{e^{a_i t}-1}-1\right) BH_n(x|a;b;\lambda;\mu) \\
&\quad + \sum_{j=1}^{s}\frac{\mu_j b_j e^{a_j t}}{1-\lambda_j}\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} x^n \\
&= \frac{1}{n+1}\left(\sum_{i=1}^{r}\frac{a_i t e^{a_i t}}{e^{a_i t}-1}-r\right) BH_{n+1}(x|a;b;\lambda;\mu) + \sum_{j=1}^{s}\frac{\mu_j b_j}{1-\lambda_j} BH_n(x+b_j|a;b;\lambda;\mu+e_j) \\
&= \frac{1}{n+1}\sum_{i=1}^{r} a_i e^{a_i t}\frac{t}{e^{a_i t}-1}\prod_{\nu=1}^{r}\left(\frac{t}{e^{a_\nu t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} x^{n+1} \\
&\quad - \frac{r}{n+1} BH_{n+1}(x|a;b;\lambda;\mu) + \sum_{j=1}^{s}\frac{\mu_j b_j}{1-\lambda_j} BH_n(x+b_j|a;b;\lambda;\mu+e_j) \\
&= \frac{1}{n+1}\sum_{i=1}^{r} a_i BH_{n+1}(x+a_i|a,a_i;b;\lambda;\mu) - \frac{r}{n+1} BH_{n+1}(x|a;b;\lambda;\mu) \\
&\quad + \sum_{j=1}^{s}\frac{\mu_j b_j}{1-\lambda_j} BH_n(x+b_j|a;b;\lambda;\mu+e_j).
\end{aligned}
$$

Here $(a,a_i) = (a_1,\ldots,a_r,a_i)$ and $e_i = (\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{r-i})$ $(i = 1,2,\ldots,r)$. Therefore, we obtain

$$
\begin{aligned}
&\left(1-\frac{r}{n+1}\right) BH_{n+1}(x|a;b;\lambda;\mu) \\
&\quad = xBH_n(x|a;b;\lambda;\mu) - \frac{1}{n+1}\sum_{i=1}^{r} a_i BH_{n+1}(x+a_i|a,a_i;b;\lambda;\mu) \\
&\quad\quad - \sum_{j=1}^{s}\frac{\mu_j b_j}{1-\lambda_j} BH_n(x+b_j|a;b;\lambda;\mu+e_j),
\end{aligned}
$$

which is (22). □

3.4 More relations

Theorem 4 *For $n \geq 1$, we have*

$$
\begin{aligned}
BH_n(x|a;b;\lambda;\mu) &= xBH_{n-1}(x|a;b;\lambda;\mu) - \sum_{j=1}^{s}\frac{\mu_j b_j}{1-\lambda_j} BH_{n-1}(x+b_j|a;b;\lambda;\mu+e_j) \\
&\quad + \sum_{m=1}^{n}\frac{(-1)^{m-1}\binom{n-1}{m-1}B_m}{m} BH_{n-m}(x|a;b;\lambda;\mu)\sum_{i=1}^{r} a_i^m. \qquad (24)
\end{aligned}
$$

Proof For $n \geq 1$, we have

$$
\begin{aligned}
BH_n(y|a;b;\lambda;\mu) &= \left\langle \sum_{l=0}^{\infty} BH_l(y|a;b;\lambda;\mu)\frac{t^l}{l!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} e^{yt} \middle| x^n \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \left\langle \partial_t \left(\prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} e^{yt} \right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} e^{yt} \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \left(\partial_t \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} \right) e^{yt} \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} \left(\partial_t e^{yt} \right) \middle| x^{n-1} \right\rangle.
\end{aligned}
$$

The third term is

$$
y \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} e^{yt} \middle| x^{n-1} \right\rangle = yBH_{n-1}(y|a;b;\lambda;\mu).
$$

Since

$$
\begin{aligned}
\partial_t \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) &= \sum_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right)' \prod_{\nu \neq i} \frac{t}{e^{a_\nu t} - 1} \\
&= \frac{t^{r-1}}{\prod_{\nu=1}^{r} (e^{a_\nu t} - 1)} \left(r - \sum_{i=1}^{r} \frac{a_i t e^{a_i t}}{e^{a_i t} - 1} \right) \\
&= \frac{t^{r-1}}{\prod_{\nu=1}^{r} (e^{a_\nu t} - 1)} \left(r - \sum_{i=1}^{r} \frac{-a_i t}{e^{-a_i t} - 1} \right) \\
&= \frac{t^{r-1}}{\prod_{\nu=1}^{r} (e^{a_\nu t} - 1)} \left(r - \sum_{i=1}^{r} \sum_{m=0}^{\infty} \frac{(-a_i)^m B_m t^m}{m!} \right) \\
&= \frac{t^{r-1}}{\prod_{\nu=1}^{r} (e^{a_\nu t} - 1)} \left(r - \sum_{m=0}^{\infty} \sum_{i=1}^{r} (-a_i)^m \frac{B_m t^m}{m!} \right) \\
&= \frac{t^{r}}{\prod_{\nu=1}^{r} (e^{a_\nu t} - 1)} \sum_{m=1}^{\infty} \sum_{i=1}^{r} a_i^m \frac{(-1)^{m-1} B_m}{m!} t^{m-1},
\end{aligned}
$$

the first term is

$$
\begin{aligned}
&\left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} e^{yt} \middle| \sum_{m=1}^{\infty} \sum_{i=1}^{r} a_i^m \frac{(-1)^{m-1} B_m}{m!} t^{m-1} x^{n-1} \right\rangle \\
&\quad = \sum_{m=1}^{n} \frac{(-1)^{m-1} \binom{n-1}{m-1} B_m}{m} \sum_{i=1}^{r} a_i^m \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} e^{yt} \middle| x^{n-m} \right\rangle \\
&\quad = \sum_{m=1}^{n} \frac{(-1)^{m-1} \binom{n-1}{m-1} B_m}{m} BH_{n-m}(y|a;b;\lambda;\mu) \sum_{i=1}^{r} a_i^m.
\end{aligned}
$$

Since

$$\partial_t \prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} = \sum_{j=1}^{s}\left(\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j}\right)' \prod_{\nu\neq j}\left(\frac{1-\lambda_\nu}{e^{b_\nu t}-\lambda_\nu}\right)^{\mu_\nu}$$
$$= -\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} \sum_{\nu=1}^{s} \frac{\mu_\nu b_\nu e^{b_\nu t}}{1-\lambda_\nu} \frac{1-\lambda_\nu}{e^{b_\nu t}-\lambda_\nu},$$

the second term is

$$-\sum_{\nu=1}^{s}\frac{\mu_\nu b_\nu}{1-\lambda_\nu}\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\frac{1-\lambda_\nu}{e^{b_\nu t}-\lambda_\nu}\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} e^{(y+b_\nu)t}\middle| x^{n-1}\right\rangle$$
$$= -\sum_{\nu=1}^{s}\frac{\mu_\nu b_\nu}{1-\lambda_\nu} BH_{n-1}(y+b_\nu|a;b;\lambda;\mu+e_\nu).$$

Therefore, we obtain

$$BH_n(x|a;b;\lambda;\mu) = xBH_{n-1}(x|a;b;\lambda;\mu) - \sum_{j=1}^{s}\frac{\mu_j b_j}{1-\lambda_j} BH_{n-1}(x+b_j|a;b;\lambda;\mu+e_j)$$
$$+ \sum_{m=1}^{n}\frac{(-1)^{m-1}\binom{n-1}{m-1}B_m}{m} BH_{n-m}(x|a;b;\lambda;\mu)\sum_{i=1}^{r} a_i^m,$$

which is identity (24). □

3.5 A relation including Bernoulli numbers

Theorem 5 *For $n-1 \geq m \geq 1$, we have*

$$\begin{aligned}
&(n-m)BH_{n-m}(a;b;\lambda;\mu) \\
&\quad = -(n-m)\sum_{j=1}^{s}\frac{\mu_j b_j}{1-\lambda_j} BH_{n-m-1}(b_i|a;b;\lambda;\mu+e_j) \\
&\qquad + \sum_{l=1}^{n-m}(-1)^{l-1}\binom{n-m}{l} B_l BH_{n-m-l}(a;b;\lambda;\mu)\sum_{i=1}^{r} a_i^l.
\end{aligned} \tag{25}$$

Proof We shall compute

$$\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} t^m \middle| x^n\right\rangle$$

in two different ways. On the one hand, it is equal to

$$\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} \middle| t^m x^n\right\rangle$$
$$= (n)_m \left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} \middle| x^{n-m}\right\rangle$$
$$= (n)_m BH_{n-m}(a;b;\lambda;\mu).$$

On the other hand, it is equal to

$$\begin{aligned}
&\left\langle \partial_t \left(\prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} t^m \right) \middle| x^{n-1} \right\rangle \\
&\quad = \left\langle \left(\partial_t \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} t^m \middle| x^{n-1} \right\rangle \\
&\qquad + \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \left(\partial_t \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} \right) t^m \middle| x^{n-1} \right\rangle \\
&\qquad + \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} \left(\partial_t t^m \right) \middle| x^{n-1} \right\rangle. \qquad (26)
\end{aligned}$$

The third term of (26) is equal to

$$\begin{aligned}
&m \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} t^{m-1} \middle| x^{n-1} \right\rangle \\
&\quad = m(n-1)_{m-1} BH_{n-m}(a;b;\lambda;\mu).
\end{aligned}$$

The second term of (26) is equal to

$$\begin{aligned}
&-\sum_{l=1}^{s} \frac{\mu_l b_l}{1-\lambda_l} \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \frac{1-\lambda_l}{e^{b_l t} - \lambda_l} \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} e^{b_l t} \middle| t^m x^{n-1} \right\rangle \\
&\quad = -(n-1)_m \sum_{l=1}^{s} \frac{\mu_l b_l}{1-\lambda_l} \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \frac{1-\lambda_l}{e^{b_l t} - \lambda_l} \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} e^{b_l t} \middle| x^{n-m-1} \right\rangle \\
&\quad = -(n-1)_m \sum_{l=1}^{s} \frac{\mu_l b_l}{1-\lambda_l} BH_{n-m-1}(b_l | a;b;\lambda;\mu + e_l).
\end{aligned}$$

Since

$$(n-1)_{l-1}(n-l)_m = (n-1)_{l+m-1} = (n-1)_{m-1}(n-m)_l,$$

the first term of (26) is equal to

$$\begin{aligned}
&\left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} t^m \middle| \sum_{l=1}^{\infty} \left(\sum_{\nu=1}^{r} a_\nu^l \right) \frac{(-1)^{l-1} B_l}{l!} t^{l-1} x^{n-1} \right\rangle \\
&\quad = \sum_{l=1}^{n-m} \left(\sum_{\nu=1}^{r} a_\nu^l \right) \frac{(-1)^{l-1} B_l}{l!} (n-1)_{l-1} \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t} - \lambda_j} \right)^{\mu_j} t^m \middle| x^{n-l} \right\rangle \\
&\quad = \sum_{l=1}^{n-m} \left(\sum_{\nu=1}^{r} a_\nu^l \right) \frac{(-1)^{l-1} B_l}{l!} (n-1)_{l-1}(n-l)_m BH_{n-m-l}(a;b;\lambda;\mu) \\
&\quad = (n-1)_{m-1} \sum_{l=1}^{n-m} (-1)^{l-1} \binom{n-m}{l} B_l BH_{n-m-l}(a;b;\lambda;\mu) \sum_{\nu=1}^{r} a_\nu^l.
\end{aligned}$$

Therefore, we get, for $n-1 \geq m \geq 1$,

$$\begin{aligned}
&(n)_m BH_{n-m}(a;b;\lambda;\mu) \\
&\quad = m(n-1)_{m-1} BH_{n-m}(a;b;\lambda;\mu) \\
&\qquad - (n-1)_m \sum_{j=1}^{s} \frac{\mu_j b_j}{1-\lambda_j} BH_{n-m-1}(b_j|a;b;\lambda;\mu+e_j) \\
&\qquad + (n-1)_{m-1} \sum_{l=1}^{n-m} (-1)^{l-1} \binom{n-m}{l} B_l BH_{n-m-l}(a;b;\lambda;\mu) \sum_{i=1}^{r} a_i^l.
\end{aligned}$$

Dividing both sides by $(n-1)_{m-1}$, we obtain, for $n-1 \geq m \geq 1$,

$$\begin{aligned}
&(n-m) BH_{n-m}(a;b;\lambda;\mu) \\
&\quad = -(n-m) \sum_{j=1}^{s} \frac{\mu_j b_j}{1-\lambda_j} BH_{n-m-1}(b_j|a;b;\lambda;\mu+e_j) \\
&\qquad + \sum_{l=1}^{n-m} (-1)^{l-1} \binom{n-m}{l} B_l BH_{n-m-l}(a;b;\lambda;\mu) \sum_{i=1}^{r} a_i^l.
\end{aligned}$$

Thus, we get (25). □

3.6 A relation with Stirling numbers

The Stirling numbers of the second kind $S_2(n,m)$ are defined by

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!}.$$

Then

$$\phi_n(x) := \sum_{m=0}^{n} S_2(n,m) x^m \sim \bigl(1, \ln(1+t)\bigr). \tag{27}$$

Theorem 6

$$BH_n(x|a;b;\lambda;\mu) = \sum_{m=0}^{n} \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) BH_{n-l} \phi_m(x). \tag{28}$$

Proof For (16) and (27), assume that $BH_n(x|a;b;\lambda;\mu) = \sum_{m=0}^{n} C_{n,m} \phi_m(x)$. By (13), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{i=1}^{r} (\frac{e^{a_i t}-1}{t}) \prod_{j=1}^{s} (\frac{e^{b_j t}-\lambda_j}{1-\lambda_j})^{\mu_j}} \bigl(\ln(1+t)\bigr)^m \middle| x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t}-1}\right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} \middle| \frac{1}{m!} \bigl(\ln(1+t)\bigr)^m x^n \right\rangle \\
&= \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t}-1}\right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j}\right)^{\mu_j} \middle| \sum_{l=m}^{\infty} S_1(l,m) \frac{t^l}{l!} x^n \right\rangle
\end{aligned}$$

$$= \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t}-1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j} \right)^{\mu_j} \middle| x^{n-l} \right\rangle$$
$$= \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) BH_{n-l}.$$

Thus, we get identity (28). □

3.7 A relation with falling factorials

Theorem 7

$$BH_n(x|a;b;\lambda;\mu) = \sum_{m=0}^{n} \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) BH_{n-l}(x)_m. \tag{29}$$

Proof For (16) and (17), assume that $BH_n(x|a;b;\lambda;\mu) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (13), we have

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{1}{\prod_{i=1}^{r}(\frac{e^{a_i t}-1}{t}) \prod_{j=1}^{s}(\frac{e^{b_j t}-\lambda_j}{1-\lambda_j})^{\mu_j}} (e^t-1)^m \middle| x^n \right\rangle$$
$$= \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t}-1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j} \right)^{\mu_j} \middle| \frac{1}{m!} (e^t-1)^m x^n \right\rangle$$
$$= \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t}-1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j} \right)^{\mu_j} \middle| \sum_{l=m}^{\infty} S_2(l,m) \frac{t^l}{l!} x^n \right\rangle$$
$$= \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) \left\langle \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t}-1} \right) \prod_{j=1}^{s} \left(\frac{1-\lambda_j}{e^{b_j t}-\lambda_j} \right)^{\mu_j} \middle| x^{n-l} \right\rangle$$
$$= \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) BH_{n-l}.$$

Thus, we get identity (29). □

3.8 A relation with higher-order Frobenius-Euler polynomials

Theorem 8

$$BH_n(x|a;b;\lambda;\mu) = \frac{1}{(1-\alpha)^p} \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{l=0}^{p} \sum_{\nu=0}^{n-m} \binom{p}{l} \binom{n-m}{\nu} (-\alpha)^{p-l} l^{\nu} BH_{n-m-\nu} \right) H_m^{(p)}(x|\alpha). \tag{30}$$

Proof For (16) and

$$H_n^{(p)}(x|\alpha) \sim \left(\left(\frac{e^t-\alpha}{1-\alpha} \right)^p, t \right), \tag{31}$$

assume that $BH_n(x|a;b;\lambda;\mu) = \sum_{m=0}^{n} C_{n,m}H_m^{(s)}(x|\alpha)$. By (13), similarly to the proof of (25), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{(\frac{e^t-\alpha}{1-\alpha})^p}{\prod_{i=1}^{r}(\frac{e^{a_it}-1}{t})\prod_{j=1}^{s}(\frac{e^{b_jt}-\lambda_j}{1-\lambda_j})^{\mu_j}} t^m \middle| x^n \right\rangle \\
&= \frac{\binom{n}{m}}{(1-\alpha)^p}\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)^{\mu_j} \middle| (e^t-\alpha)^p x^{n-m} \right\rangle \\
&= \frac{\binom{n}{m}}{(1-\alpha)^p}\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)^{\mu_j} \middle| \sum_{l=0}^{p}\binom{p}{l}(-\alpha)^{p-l}e^{lt}x^{n-m} \right\rangle \\
&= \frac{\binom{n}{m}}{(1-\alpha)^p}\sum_{l=0}^{p}\binom{p}{l}(-\alpha)^{p-l}\left\langle e^{lt} \middle| \prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)^{\mu_j} x^{n-m} \right\rangle \\
&= \frac{\binom{n}{m}}{(1-\alpha)^p}\sum_{l=0}^{p}\binom{p}{l}(-\alpha)^{p-l}\left\langle e^{lt} \middle| \sum_{\nu=0}^{\infty} BH_\nu \frac{t^\nu}{\nu!} x^{n-m} \right\rangle \\
&= \frac{\binom{n}{m}}{(1-\alpha)^p}\sum_{l=0}^{p}\binom{p}{l}(-\alpha)^{p-l}\sum_{\nu=0}^{n-m}\binom{n-m}{\nu}BH_\nu \langle e^{lt}|x^{n-m-\nu}\rangle \\
&= \frac{\binom{n}{m}}{(1-\alpha)^p}\sum_{l=0}^{p}\binom{p}{l}(-\alpha)^{p-l}\sum_{\nu=0}^{n-m}\binom{n-m}{\nu}BH_\nu l^{n-m-\nu} \\
&= \frac{\binom{n}{m}}{(1-\alpha)^p}\sum_{l=0}^{p}\binom{p}{l}(-\alpha)^{p-l}\sum_{\nu=0}^{n-m}\binom{n-m}{\nu}BH_{n-m-\nu} l^{\nu}.
\end{aligned}$$

Thus, we get identity (30). □

3.9 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(p)}(x)$ of order p are defined by

$$\left(\frac{t}{e^t-1}\right)^p e^{xt} = \sum_{n=0}^{\infty}\frac{\mathfrak{B}_n^{(p)}(x)}{n!}t^n$$

(see *e.g.* [10, Section 2.2]).

Theorem 9

$$BH_n(x|a;b;\lambda;\mu) = \sum_{m=0}^{n}\binom{n}{m}\left(\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{n-m-l+p}{p}}S_2(n-m-l+p,p)BH_l\right)\mathfrak{B}_m^{(p)}(x). \tag{32}$$

Proof For (16) and

$$\mathfrak{B}_n^{(p)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^p, t\right), \tag{33}$$

assume that $BH_n(x|a;b;\lambda;\mu) = \sum_{m=0}^{n} C_{n,m}\mathfrak{B}_m^{(p)}(x)$. By (13), similarly to the proof of (25), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{(\frac{e^t-1}{t})^p}{\prod_{i=1}^{r}(\frac{e^{a_it}-1}{t})\prod_{j=1}^{s}(\frac{e^{b_jt}-\lambda_j}{1-\lambda_j})^{\mu_j}} t^m \middle| x^n \right\rangle \\
&= \binom{n}{m}\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)^{\mu_j} \middle| \left(\frac{e^t-1}{t}\right)^p x^{n-m} \right\rangle \\
&= \binom{n}{m}\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)^{\mu_j} \middle| p!\sum_{l=0}^{\infty} S_2(l+p,p)\frac{t^l}{(l+p)!}x^{n-m} \right\rangle \\
&= \binom{n}{m}p!\sum_{l=0}^{n-m}\frac{(n-m)_l}{(l+p)!}S_2(l+p,p)\left\langle \prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)\prod_{j=1}^{s}\left(\frac{1-\lambda_j}{e^{b_jt}-\lambda_j}\right)^{\mu_j} \middle| x^{n-m-l} \right\rangle \\
&= \binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+p}{p}}S_2(l+p,p)BH_{n-m-l} \\
&= \binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{n-m-l+p}{p}}S_2(n-m-l+p,p)BH_l.
\end{aligned}$$

Thus, we get identity (32). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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