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A note on poly-Bernoulli numbers and polynomials of the second kind

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Abstract

In this paper, we consider the poly-Bernoulli numbers and polynomials of the second kind and presents new and explicit formulas for calculating the poly-Bernoulli numbers of the second kind and the Stirling numbers of the second kind.

Keywords: Bernoulli polynomials of the second kind; poly-Bernoulli numbers and polynomials; Stirling number of the second kind

1 Introduction

As is well known, the Bernoulli polynomials of the second kind are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \quad \text{(see [1–3])}. \tag{1}$$

When $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind. The first few Bernoulli numbers b_n of the second kind are $b_0 = 1$, $b_1 = 1/2$, $b_2 = -1/12$, $b_3 = 1/24$, $b_4 = -19/720$, $b_5 = 3/160, \ldots$.

From (1), we have

$$b_n(x) = \sum_{l=0}^{n} \binom{n}{l} b_l(x)_{n-l}, \tag{2}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \geqq 0)$. The Stirling number of the second kind is defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \quad (n \geqq 0). \tag{3}$$

The ordinary Bernoulli polynomials are given by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{(see [1–18])}. \tag{4}$$

When $x = 0$, $B_n = B_n(0)$ are called Bernoulli numbers.

It is well known that the classical poly-logarithmic function $\mathrm{Li}_k(x)$ is given by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (k \in \mathbb{Z}) \text{ (see [8–10])}. \tag{5}$$

For $k = 1$, $\mathrm{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$. The Stirling number of the first kind is defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \quad (n \geq 0) \text{ (see [16])}. \tag{6}$$

In this paper, we consider the poly-Bernoulli numbers and polynomials of the second kind and presents new and explicit formulas for calculating the poly-Bernoulli number and polynomial and the Stirling number of the second kind.

2 Poly-Bernoulli numbers and polynomials of the second kind

For $k \in \mathbb{Z}$, we consider the poly-Bernoulli polynomials $b_n^{(k)}(x)$ of the second kind:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $b_n^{(k)} = b_n^{(k)}(0)$ are called the poly-Bernoulli numbers of the second kind.

Indeed, for $k = 1$, we have

$$\frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^x = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{8}$$

By (7) and (8), we get

$$b_n^{(1)}(x) = b_n(x) \quad (n \geq 0). \tag{9}$$

It is well known that

$$\frac{t(1+t)^{x-1}}{\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)}(x)\frac{t^n}{n!}, \tag{10}$$

where $B_n^{(\alpha)}(x)$ are the Bernoulli polynomials of order α which are given by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \text{(see [1–18])}.$$

By (1) and (10), we get

$$b_n(x) = B_n^{(n)}(x+1) \quad (n \geq 0).$$

Now, we observe that

$$\begin{aligned}
&\frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^x \\
&\quad= \sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!} \\
&\quad= \frac{1}{\log(1+t)} \underbrace{\int_0^t \frac{1}{e^x-1}\int_0^t \frac{1}{e^x-1}\cdots\frac{1}{e^x-1}}_{k-1\ \text{times}}\int_0^t \frac{x}{e^x-1}\,dx\cdots dx(1+t)^x. \qquad (11)
\end{aligned}$$

Thus, by (11), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n^{(2)}(x)\frac{t^n}{n!} &= \frac{(1+t)^x}{\log(1+t)}\int_0^t \frac{x}{e^x-1}\,dx \\
&= \frac{(1+t)^x}{\log(1+t)}\sum_{l=0}^{\infty}\frac{B_l}{l!}\int_0^t x^l\,dx \\
&= \left(\frac{t}{\log(1+t)}\right)(1+t)^x\sum_{l=0}^{\infty}\frac{B_l}{(l+1)}\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\binom{n}{l}\frac{B_l b_{n-l}(x)}{l+1}\right\}\frac{t^n}{n!}. \qquad (12)
\end{aligned}$$

Therefore, by (12), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$ we have*

$$b_n^{(2)}(x) = \sum_{l=0}^{n}\binom{n}{l}\frac{B_l b_{n-l}(x)}{l+1}.$$

From (11), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^x \\
&= \frac{t}{\log(1+t)}\frac{\mathrm{Li}_k(1-e^{-t})}{t}(1+t)^x. \qquad (13)
\end{aligned}$$

We observe that

$$\begin{aligned}
\frac{1}{t}\mathrm{Li}_k\left(1-e^{-t}\right) &= \frac{1}{t}\sum_{n=1}^{\infty}\frac{1}{n^k}\left(1-e^{-t}\right)^n \\
&= \frac{1}{t}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^k}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{(-t)^l}{l!} \\
&= \frac{1}{t}\sum_{l=1}^{\infty}\sum_{n=1}^{l}\frac{(-1)^{n+l}}{n^k}n!S_2(l,n)\frac{t^l}{l!} \\
&= \sum_{l=0}^{\infty}\sum_{n=1}^{l+1}\frac{(-1)^{n+l+1}}{n^k}n!\frac{S_2(l+1,n)}{l+1}\frac{t^l}{l!}. \qquad (14)
\end{aligned}$$

Thus, by (10) and (14), we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty} b_m(x)\frac{t^m}{m!}\right)\left\{\sum_{l=0}^{\infty}\left(\sum_{p=1}^{l+1}\frac{(-1)^{p+l+1}}{p^k}p!\frac{S_2(l+1,p)}{l+1}\right)\frac{t^l}{l!}\right\}\\ &= \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\binom{n}{l}\left(\sum_{p=1}^{l+1}\frac{(-1)^{p+l+1}p!}{p^k}\frac{S_2(l+1,p)}{l+1}\right)b_{n-l}(x)\right\}\frac{t^n}{n!}.\end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$b_n^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}\left(\sum_{p=1}^{l+1}\frac{(-1)^{p+l+1}}{p^k}p!\frac{S_2(l+1,p)}{l+1}\right)b_{n-l}(x).$$

By (7), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(b_n^{(k)}(x+1)-b_n^{(k)}(x)\right)\frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^{x+1} - \frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^{x}\\ &= \frac{t\,\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^{x}\\ &= \left(\frac{t}{\log(1+t)}(1+t)^{x}\right)\mathrm{Li}_k\left(1-e^{-t}\right)\\ &= \left(\sum_{l=0}^{\infty}\frac{b_l(x)}{l!}t^l\right)\left\{\sum_{p=1}^{\infty}\left(\sum_{m=1}^{p}\frac{(-1)^{m+p}m!}{m^k}S_2(p,m)\right)\right\}\frac{t^p}{p!} \qquad (16)\\ &= \sum_{n=1}^{\infty}\left(\sum_{p=1}^{n}\sum_{m=1}^{p}\frac{(-1)^{m+p}}{m^k}m!S_2(p,m)\frac{b_{n-p}(x)n!}{(n-p)!p!}\right)\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty}\left\{\sum_{p=1}^{n}\sum_{m=1}^{p}\binom{n}{p}\frac{(-1)^{m+p}m!}{m^k}S_2(p,m)b_{n-p}(x)\right\}\frac{t^n}{n!}. \qquad (17)\end{aligned}$$

Therefore, by (16), we obtain the following theorem.

Theorem 2.3 *For $n \geq 1$, we have*

$$b_n^{(k)}(x+1) - b_n^{(k)}(x) = \sum_{p=1}^{n}\sum_{m=1}^{p}\binom{n}{p}\frac{(-1)^{m+p}m!}{m^k}S_2(p,m)b_{n-p}(x). \tag{18}$$

From (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_n^{(k)}(x+y)\frac{t^n}{n!} &= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}\right)^k(1+t)^{x+y}\\ &= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}\right)^k(1+t)^{x}(1+t)^{y}\\ &= \left(\sum_{l=0}^{\infty} b_l^{(k)}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(y)_m\frac{t^m}{m!}\right)\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(y)_l b_{n-l}^{(k)}(x)\frac{n!}{(n-l)!l!}\right)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} b_{n-l}^{(k)}(x)(y)_l\right)\frac{t^n}{n!}. \tag{19}$$

Therefore, by (17), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we have*

$$b_n^{(k)}(x+y) = \sum_{l=0}^{n}\binom{n}{l} b_{n-l}^{(k)}(x)(y)_l.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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References

1. Kim, DS, Kim, T, Lee, S-H: Poly-Cauchy numbers and polynomials with umbral calculus viewpoint. Int. J. Math. Anal. **7**, 2235-2253 (2013)
2. Prabhakar, TR, Gupta, S: Bernoulli polynomials of the second kind and general order. Indian J. Pure Appl. Math. **11**, 1361-1368 (1980)
3. Roman, S, Rota, GC: The umbral calculus. Adv. Math. **27**(2), 95-188 (1978)
4. Choi, J, Kim, DS, Kim, T, Kim, YH: Some arithmetic identities on Bernoulli and Euler numbers arising from the p-adic integrals on Z_p. Adv. Stud. Contemp. Math. **22**, 239-247 (2012)
5. Ding, D, Yang, J: Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials. Adv. Stud. Contemp. Math. **20**, 7-21 (2010)
6. Gaboury, S, Tremblay, R, Fugère, B-J: Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials. Proc. Jangjeon Math. Soc. **17**, 115-123 (2014)
7. King, D, Lee, SJ, Park, L-W, Rim, S-H: On the twisted weak weight q-Bernoulli polynomials and numbers. Proc. Jangjeon Math. Soc. **16**, 195-201 (2013)
8. Kim, DS, Kim, T, Lee, S-H: A note on poly-Bernoulli polynomials arising from umbral calculus. Adv. Stud. Theor. Phys. **7**(15), 731-744 (2013)
9. Kim, DS, Kim, T: Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials. Adv. Differ. Equ. **2013**, 251 (2013)
10. Kim, DS, Kim, T, Lee, S-H, Rim, S-H: Umbral calculus and Euler polynomials. Ars Comb. **112**, 293-306 (2013)
11. Kim, DS, Kim, T: Higher-order Cauchy of first kind and poly-Cauchy of the first kind mixed type polynomials. Adv. Stud. Contemp. Math. **23**(4), 621-636 (2013)
12. Kim, T: q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients. Russ. J. Math. Phys. **15**, 51-57 (2008)
13. Kim, Y-H, Hwang, K-W: Symmetry of power sum and twisted Bernoulli polynomials. Adv. Stud. Contemp. Math. **18**, 127-133 (2009)
14. Ozden, H, Cangul, IN, Simsek, Y: Remarks on q-Bernoulli numbers associated with Daehee numbers. Adv. Stud. Contemp. Math. **18**, 41-48 (2009)
15. Park, J-W: New approach to q-Bernoulli polynomials with weight or weak weight. Adv. Stud. Contemp. Math. **24**(1), 39-44 (2014)
16. Roman, S: The Umbral Calculus. Pure and Applied Mathematics, vol. 111. Academic Press, New York (1984). ISBN:0-12-594380-6
17. Simsek, Y: Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions. Adv. Stud. Contemp. Math. **16**, 251-278 (2008)
18. Srivastava, HM, Kim, T, Simsek, Y: q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series. Russ. J. Math. Phys. **12**, 241-278 (2005)

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