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Barnes-type Narumi polynomials

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Abstract
In this paper, we study the Barnes-type Narumi polynomials with umbral calculus viewpoint. From our study, we derive various identities of the Barnes-type Narumi polynomials.
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Keywords: Barnes-type Narumi polynomial; umbral calculus

1 Introduction

As is well known, the Narumi polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^{\alpha}(1+t)^x=\sum_{n=0}^{\infty}N_n^{(\alpha)}(x)\frac{t^n}{n!}\quad\text{(see [1])}.\tag{1}$$

Let $r\in\mathbb{Z}_{>0}$. We consider the polynomials $N_n(x|a_1,\dots,a_r)$ and $\hat{N}_n(x|a_1,\dots,a_r)$, respectively, called the Barnes-type Narumi polynomials of the first kind and those of the second kind and respectively given by

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^x=\sum_{n=0}^{\infty}N_n(x|a_1,\dots,a_r)\frac{t^n}{n!}\tag{2}$$

and

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)(1+t)^{a_j}}\right)(1+t)^x=\sum_{n=0}^{\infty}\hat{N}_n(x|a_1,\dots,a_r)\frac{t^n}{n!},\tag{3}$$

where $a_1,a_2,\dots,a_r\neq 0$.

When $x=0$,

$$N_n(a_1,\dots,a_r)=N_n(0|a_1,\dots,a_r)$$

and

$$\hat{N}_n(a_1,\dots,a_r)=\hat{N}_n(0|a_1,\dots,a_r)$$

are respectively called the Barnes-type Narumi numbers of the first kind and those of the second kind.

Note that

$$N_n(x|\underbrace{1,\ldots,1}_{r}) = N_n^{(r)}(x),$$

$$\hat{N}_n(x|\underbrace{1,\ldots,1}_{r}) = \hat{N}_n^{(r)}(x)$$

and

$$\hat{N}_n(x|\underbrace{1,\ldots,1}_{r}) = N_n^{(r)}(x-r).$$

In the previous paper [2], $N_n^{(\alpha)}(x)$ was denoted by $N_n^{(-\alpha)}$ and called the Narumi polynomial of order α.

The Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{(see [3–6])}. \tag{4}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. In [7], it is known that the Cauchy numbers are given by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} C_n\frac{t^n}{n!}. \tag{5}$$

It is well known that the Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{\infty} S_1(n,l)x^l \quad (n \geq 0) \text{ (see [1, 2, 7–11])}. \tag{6}$$

From (6), we have

$$\bigl(\log(1+t)\bigr)^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!} \quad (n \geq 0). \tag{7}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on $p(x)$ which satisfies $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, and $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant. The linear functional $\langle f(t)|\cdot\rangle$ on $\mathbb{P}$ is defined by $\langle f(t)|x^n\rangle = a_n$ $(n \geq 0)$, where $f(t) \in \mathcal{F}$. Thus, we note that

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{8}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [12–18]).

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra. The order $o(f(t))$ of a power series $f(t) \neq 0$ is the smallest integer for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a delta series; if $o(f(t)) = 0$, then $g(t)$ is called an invertible series. Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$ for $n, k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{9}$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \qquad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!}. \tag{10}$$

From (10), we can derive the following equation (11):

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \qquad e^{yt}p(x) = p(x+y) \quad \text{(see [1])}. \tag{11}$$

Let $s_n(x) \sim (g(t), f(t))$. Then the following will be used:

$$\frac{ds_n(x)}{dx} = \sum_{l=0}^{n-1} \binom{n}{l} \langle \bar{f}(t)|x^{n-l}\rangle s_l(x), \tag{12}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$,

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!} \quad \text{for all } x \in \mathbb{C}, \tag{13}$$

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 1), \qquad s_n(x) = \sum_{j=0}^{n} \frac{\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j|x^n\rangle}{j!} x^j, \tag{14}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x)p_{n-j}(y), \quad \text{where } p_n(x) = g(t)s_n(x), \tag{15}$$

$$\langle f(t)|xp(x)\rangle = \langle \partial_t f(t)|p(x)\rangle, \quad \text{where } \partial_t f(t) = \frac{df(t)}{dt} \tag{16}$$

and

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} s_n(x) \quad (n \geq 0) \text{ (see [1, 19])}. \tag{17}$$

Let us assume that $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$. Then we have

$$s_n(x) = \sum_{m=0}^{n} c_{n,m} r_m(x) \quad (n \geq 0), \tag{18}$$

where

$$c_{n,m} = \frac{1}{m!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l\bigl(\bar{f}(t)\bigr)^m \middle| x^n \right\rangle \quad \text{(see [1, 5])}. \tag{19}$$

From (2), (3) and (13), we note that

$$N_n(x|a_1, \ldots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{t}{e^{a_j t} - 1} \right), e^t - 1 \right) \tag{20}$$

and

$$\hat{N}_n(x|a_1, \ldots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{t e^{a_j t}}{e^{a_j t} - 1} \right), e^t - 1 \right). \tag{21}$$

In this paper, we study the Barnes-type Narumi polynomials with umbral calculus viewpoint. From our study, we derive various identities of the Barnes-type Narumi polynomials.

2 Barnes-type Narumi polynomials

From (21), we note that

$$\prod_{j=1}^{r} \left(\frac{t}{e^{a_j t} - 1} \right) N_n(x|a_1, \ldots, a_r) \sim \bigl(1, e^t - 1\bigr) \tag{22}$$

and

$$(x)_n \sim \bigl(1, e^t - 1\bigr). \tag{23}$$

Thus, by (22) and (23), we get

$$\begin{aligned} N_n(x|a_1, \ldots, a_r) &= \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t} \right) (x)_n \\ &= \sum_{m=0}^{n} S_1(n, m) \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t} \right) x^m. \end{aligned} \tag{24}$$

Note that

$$\begin{aligned} \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t} \right) &= \left(\sum_{l_1=0}^{\infty} \frac{a_1^{l_1+1}}{(l_1+1)!} t^{l_1} \right) \times \cdots \times \left(\sum_{l_r=0}^{\infty} \frac{a_r^{l_r+1}}{(l_r+1)!} t^{l_r} \right) \\ &= \sum_{l_1, \ldots, l_r=0}^{\infty} \sum_{l_1 + \cdots + l_r = i} \frac{a_1^{l_1+1} \cdots a_r^{l_r+1}}{(l_1+1)! \cdots (l_r+1)!} t^i. \end{aligned} \tag{25}$$

From (24) and (25), we have

$$\begin{aligned} N_n(x|a_1,\ldots,a_r) &= \sum_{m=0}^{\infty} S_1(n,m) \sum_{i=0}^{m} \sum_{l_1+\cdots+l_r=i} \frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!} t^i x^m \\ &= \sum_{m=0}^{n} S_1(n,m) \sum_{i=0}^{m} \frac{i!}{(i+r)!} \sum_{l_1+\cdots+l_r=i} \binom{i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} \\ &\quad \times a_1^{l_1+1}\cdots a_r^{l_r+1} x^{m-i} \\ &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \right. \\ &\quad \left. \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1}\cdots a_r^{l_r+1} \right\} x^i. \end{aligned} \tag{26}$$

By (21), we see that

$$\prod_{j=1}^{r} \left(\frac{te^{a_j t}}{e^{a_j t}-1} \right) \hat{N}_n(x|a_1,\ldots,a_r) \sim \left(1, e^t - 1\right), \tag{27}$$

and we recall (23).

Thus, we have

$$\begin{aligned} \hat{N}_n(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r} \left(\frac{e^{a_j t}-1}{te^{a_j t}} \right) (x)_n = e^{-\sum_{j=1}^{r} a_j t} \prod_{j=1}^{r} \left(\frac{e^{a_j t}-1}{t} \right) (x)_n \\ &= e^{-\sum_{j=1}^{r} a_j t} N_n(x|a_1,\ldots,a_r) \\ &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \right. \\ &\quad \left. \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1}\cdots a_r^{l_r+1} \right\} e^{-\sum_{j=1}^{r} a_j t} x^i \\ &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \right. \\ &\quad \left. \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1}\cdots a_r^{l_r+1} \right\} \left(x - \sum_{j=1}^{r} a_j \right)^i \end{aligned} \tag{28}$$

or

$$\begin{aligned} \hat{N}_n(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r} \left(\frac{e^{a_j t}-1}{te^{a_j t}} \right) (x)_n = \prod_{j=1}^{r} \left(\frac{e^{-a_j t}-1}{-t} \right) (x)_n \\ &= \sum_{m=0}^{n} S_1(n,m) \sum_{i=0}^{m} (-1)^i \frac{i!}{(i+r)!} \\ &\quad \times \sum_{l_1+\cdots+l_r=i} \binom{i+r}{i_1+1,\ldots,i_r+1} \binom{m}{i} a_1^{l_1+1}\cdots a_r^{l_r+1} x^{m-i} \end{aligned}$$

$$= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} (-1)^{m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1} \cdots a_r^{l_r+1} \right\} x^i. \tag{29}$$

Therefore, by (26), (28) and (29), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1} \cdots a_r^{l_r+1} \right\} x^i$$

and

$$\begin{aligned} \hat{N}_n(x|a_1,\ldots,a_r) &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1} \cdots a_r^{l_r+1} \right\} \left(x - \sum_{j=1}^{r} a_j \right)^i \\ &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} (-1)^{m-i} S_1(n,m) \times \frac{(m-i)!}{(m-i+r)!} \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{j} a_1^{l_1+1} \cdots a_r^{l_r+1} \right\} x^i. \end{aligned}$$

From (14), we can derive the following equation (30):

$$N_n(x|a_1,\ldots,a_r) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \left(\log(1+t)\right)^j \middle| x^n \right\rangle x^j, \tag{30}$$

where

$$\begin{aligned} &\left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \left(\log(1+t)\right)^j \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \middle| \left(\log(1+t)\right)^j x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \middle| j! \sum_{l=j}^{\infty} S_1(l,j) \frac{t^l}{l!} x^n \right\rangle \\ &= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) N_{n-l}(a_1,\ldots,a_r). \end{aligned} \tag{31}$$

Thus, by (30) and (31), we obtain the following theorem.

Theorem 2 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{j=0}^{n} \left\{ \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) N_{n-l}(a_1,\ldots,a_r) \right\} x^j.$$

By the same methods as in (28), (29) and (30), we get

$$\begin{aligned} &\hat{N}_n(x|a_1,\ldots,a_r) \\ &= \sum_{j=0}^{n} \left\{ \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) N_{n-l}(a_1,\ldots,a_r) \right\} \left(x - \sum_{i=1}^{r} a_i \right)^j \\ &= \sum_{j=0}^{n} \left\{ \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \hat{N}_{n-l}(a_1,\ldots,a_r) \right\} x^j. \end{aligned} \tag{32}$$

By (8), we get

$$\begin{aligned} N_n(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} N_i(y|a_1,\ldots,a_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) (1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \middle| \sum_{m=0}^{\infty} (y)_m \frac{t^m}{m!} x^m \right\rangle \\ &= \sum_{m=0}^{n} (y)_m \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \middle| x^{n-m} \right\rangle \\ &= \sum_{m=0}^{n} (y)_m \binom{n}{m} N_{n-m}(a_1,\ldots,a_r) \end{aligned} \tag{33}$$

and

$$\begin{aligned} \hat{N}_n(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} \hat{N}_i(y|a_1,\ldots,a_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)(1+t)^{a_j}} \right) (1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)(1+t)^{a_j}} \right) \middle| \sum_{m=0}^{\infty} (y)_m \frac{t^m}{m!} x^n \right\rangle \\ &= \sum_{m=0}^{n} (y)_m \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)(1+t)^{a_j}} \right) \middle| x^{n-m} \right\rangle \\ &= \sum_{m=0}^{n} \binom{n}{m} (y)_m \hat{N}_{n-m}(a_1,\ldots,a_r). \end{aligned} \tag{34}$$

Therefore, by (33) and (34), we obtain the following theorem.

Theorem 3 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} \binom{n}{m} N_{n-m}(a_1,\ldots,a_r)(x)_m$$

and

$$\hat{N}_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} \binom{n}{m} \hat{N}_{n-m}(a_1,\ldots,a_r)(x)_m.$$

From (15), we note that

$$N_n(x+y|a_1,\ldots,a_r) = \sum_{j=0}^{n} \binom{n}{j} N_j(x|a_1,\ldots,a_r)(y)_{n-j} \tag{35}$$

and

$$\hat{N}_n(x+y|a_1,\ldots,a_r) = \sum_{j=0}^{n} \binom{n}{j} \hat{N}_j(x|a_1,\ldots,a_r)(y)_{n-j}. \tag{36}$$

By (14), we get

$$\left(e^t - 1\right) N_n(x|a_1,\ldots,a_r) = nN_{n-1}(x|a_1,\ldots,a_r) \tag{37}$$

and

$$\begin{aligned}\left(e^t - 1\right) N_n(x|a_1,\ldots,a_r) &= e^t N_n(x|a_1,\ldots,a_r) - N_n(x|a_1,\ldots,a_r) \\ &= N_n(x+1|a_1,\ldots,a_r) - N(x|a_1,\ldots,a_r).\end{aligned} \tag{38}$$

From (37) and (38), we have

$$N_n(x+1|a_1,\ldots,a_r) - N_n(x|a_1,\ldots,a_r) = nN_{n-1}(x|a_1,\ldots,a_r). \tag{39}$$

By the same method as (39), we get

$$\hat{N}_n(x+1|a_1,\ldots,a_r) - \hat{N}_n(x|a_1,\ldots,a_r) = n\hat{N}_{n-1}(x|a_1,\ldots,a_r). \tag{40}$$

Recall that $N_n(x|a_1,\ldots,a_r) \sim (\prod_{j=1}^{r}(\frac{t}{e^{a_j t}-1}), e^t - 1)$.
From (17), we can derive the following equation (41):

$$N_{n+1}(x|a_1,\ldots,a_r) = xN_n(x-1|a_1,\ldots,a_r) - e^{-t}\frac{g'(t)}{g(t)} N_n(x|a_1,\ldots,a_r). \tag{41}$$

Now, we observe that

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \left(\log g(t)\right)' = \left(r\log t - \sum_{j=1}^{r} \log\left(e^{a_j t} - 1\right)\right)' = \frac{r}{t} - \sum_{j=1}^{r} \frac{a_j e^{a_j t}}{e^{a_j t} - 1} \\ &= \frac{\sum_{j=1}^{r} \prod_{i \neq j}^{r}(e^{a_i t} - 1)\{e^{a_j t} - 1 - t a_j e^{a_j t}\}}{t\prod_{j=1}^{r}(e^{a_j t} - 1)},\end{aligned} \tag{42}$$

where

$$r-\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}=\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)\{e^{a_jt}-1-a_jte^{a_jt}\}}{\prod_{j=1}^{r}(e^{a_jt}-1)}$$
$$=\frac{-\frac{1}{2}(\sum_{j=1}^{r}a_1a_2\cdots a_{j-1}a_j^2a_{j+1}\cdots a_r)t^{r+1}+\cdots}{(a_1a_2\cdots a_r)t^r}$$
$$=-\frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots \tag{43}$$

has at least the order 1.

By (42) and (43), we get

$$\begin{aligned}
&\frac{g'(t)}{g(t)}N_n(x|a_1,\ldots,a_r)\\
&=\frac{r-\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}}{t}\left(\sum_{i=0}^{n}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}x^i\right)\\
&=\sum_{i=0}^{n}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\frac{r-\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}}{t}x^i\\
&=\sum_{i=0}^{n}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\left(r-\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}\right)\frac{x^{i+1}}{i+1}\\
&=\sum_{i=0}^{n}\frac{1}{i+1}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\left(rx^{i+1}-\sum_{j=1}^{r}\sum_{m=0}^{\infty}B_m\frac{(-a_jt)^m}{m!}x^{i+1}\right)\\
&=-\sum_{i=0}^{n}\frac{1}{i+1}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\\
&\quad\times\sum_{j=1}^{r}\sum_{m=1}^{i+1}(-1)^m\binom{i+1}{m}B_ma_j^mx^{i+1-m}\\
&=-\sum_{i=0}^{n}\frac{1}{i+1}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\\
&\quad\times\sum_{j=1}^{r}\sum_{m=0}^{i}(-1)^{i+1-m}\binom{i+1}{m}a_j^{i+1-m}B_{i+1-m}x^m.
\end{aligned}\tag{44}$$

Therefore, by (41) and (44), we obtain the following theorem.

Theorem 4 *For $n\geq 0$, we have*

$$\begin{aligned}
&N_{n+1}(x|a_1,\ldots,a_r)\\
&=xN_n(x-1|a_1,\ldots,a_r)+\sum_{m=0}^{n}\left\{\sum_{i=m}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\
&\quad\left.\times B_{i+1-m}(-a_j)^{i+1-m}N_{n-l}(a_1,\ldots,a_r)\right\}(x-1)^m.
\end{aligned}$$

By the same method as the proof of Theorem 4, we get

$$\begin{aligned}\hat{N}_{n+1}(x|a_1,\ldots,a_r) &= \left(x-\sum_{j=1}^{r}a_j\right)\hat{N}_n(x-1|a_1,\ldots,a_r)\\&\quad+\sum_{m=0}^{n}\left\{\sum_{i=m}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\&\quad\left.\times B_{i+1-m}(-a_j)^{i+1-m}\hat{N}_{n-l}(a_1,\ldots,a_r)\right\}(x-1)^m.\end{aligned}\tag{45}$$

From (12) and (20), we can derive the following equation (46):

$$\left\langle\overline{f}(t)|x^{n-l}\right\rangle=\left\langle\log(1+t)|x^{n-l}\right\rangle=\left\langle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}t^m\middle|x^{n-l}\right\rangle=(-1)^{n-l-1}(n-l-1)!.\tag{46}$$

Thus, by (46), we get

$$\begin{aligned}\frac{d}{dx}N_n(x|a_1,\ldots,a_r)&=\sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!N_l(x|a_1,\ldots,a_r)\\&=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}N_l(x|a_1,\ldots,a_r).\end{aligned}\tag{47}$$

By the same method as (47), we get

$$\frac{d}{dx}\hat{N}_n(x|a_1,\ldots,a_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\hat{N}_l(x|a_1,\ldots,a_r).\tag{48}$$

From (8), we note that, for $n\geq 1$,

$$\begin{aligned}&N_n(y|a_1,\ldots,a_r)\\&\quad=\left\langle\sum_{i=0}^{\infty}N_i(y|a_1,\ldots,a_r)\frac{t^i}{i!}\middle|x^n\right\rangle\\&\quad=\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^y\middle|x^n\right\rangle\\&\quad=\left\langle\partial_t\left(\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^y\right)\middle|x^{n-1}\right\rangle\\&\quad=\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\left(\partial_t(1+t)^y\right)\middle|x^{n-1}\right\rangle\\&\qquad+\left\langle\left(\partial_t\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)(1+t)^y\middle|x^{n-1}\right\rangle\\&\quad=yN_{n-1}(y-1|a_1,\ldots,a_r)+\left\langle\left(\partial_t\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)(1+t)^y\middle|x^{n-1}\right\rangle.\end{aligned}\tag{49}$$

Now, we observe that

$$\begin{aligned}
&\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right) \\
&= \sum_{j=1}^{r}\prod_{i\neq j}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)\frac{a_j(1+t)^{a_j-1}\log(1+t)-((1+t)^{a_j}-1)\frac{1}{1+t}}{(\log(1+t))^2} \\
&= \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)\sum_{j=1}^{r}\left\{\frac{a_j(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{1}{\log(1+t)}\right\} \\
&= \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)\frac{\sum_{j=1}^{r}\{\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\}}{t},
\end{aligned} \tag{50}$$

where

$$\sum_{j=1}^{r}\left\{\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\right\} = \frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots \tag{51}$$

is a series with order greater than or equal to 1.

By (50) and (51), we get

$$\begin{aligned}
&\left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)(1+t)^y \middle| x^{n-1}\right\rangle \\
&= \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)\frac{(1+t)^y}{1+t} \middle| \frac{\sum_{j=1}^{r}\{\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\}}{t}x^{n-1}\right\rangle \\
&= \frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y-1} \middle| \sum_{j=1}^{r}\left\{\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\right\}x^n\right\rangle \\
&= \frac{1}{n}\left\{\sum_{j=1}^{r}a_j\left\langle \frac{\log(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y+a_j-1} \middle| \frac{t}{\log(1+t)}x^n\right\rangle\right. \\
&\quad \left. - r\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y-1} \middle| \frac{t}{\log(1+t)}x^n\right\rangle\right\} \\
&= \frac{1}{n}\sum_{j=1}^{r}a_j\sum_{l=0}^{n}\binom{n}{l}C_l\left\langle \frac{\log(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y+a_j-1} \middle| x^{n-l}\right\rangle \\
&\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}C_l\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y-1} \middle| x^{n-l}\right\rangle \\
&= \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_j C_l N_{n-l}(y+a_j-1|a_1,\ldots,\hat{a}_j,\ldots,a_r) \\
&\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}C_l N_{n-l}(y-1|a_1,\ldots,a_r),
\end{aligned} \tag{52}$$

where $\hat{a}_j$ means that a_j is omitted.

Therefore, by (49) and (52), we obtain the following theorem.

Theorem 5 *For $n \geq 1$, we have*

$$N_n(x|a_1,\dots,a_r) = xN_{n-1}(x-1|a_1,\dots,a_r) + \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jC_lN_{n-l}(x+a_j-1|a_1,\dots,\hat{a}_j,\dots,a_r) - \frac{r}{n}\sum_{l=0}^{r}\binom{n}{l}C_lN_{n-l}(x-1|a_1,\dots,a_r),$$

where C_n are the Cauchy numbers with the generating function given by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty}C_n\frac{t^n}{n!}.$$

By the same method as the proof of Theorem 5, we get

$$\hat{N}_n(x|a_1,\dots,a_r) = \left(x - \sum_{j=1}^{r}a_j\right)\hat{N}_{n-1}(x-1|a_1,\dots,a_r) - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}C_l\hat{N}_{n-l}(x-1|a_1,\dots,a_r) - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jC_l\hat{N}_{n-l}(x-1|a_1,\dots,\hat{a}_j,\dots,a_r). \tag{53}$$

Now we compute the following formula (54) in two different ways:

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\left(\log(1+t)\right)^m \middle| x^n\right\rangle. \tag{54}$$

On the one hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\left(\log(1+t)\right)^m \middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right) \middle| m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right) \middle| x^{n-l}\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)N_{n-l}(a_1,\dots,a_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)N_l(a_1,\dots,a_r).
\end{aligned} \tag{55}$$

On the other hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\left(\log(1+t)\right)^m \middle| x^n \right\rangle \\
&= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\left(\log(1+t)\right)^m\right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\left(\partial_t\left(\left(\log(1+t)\right)^m\right)\right) \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)\left(\log(1+t)\right)^m \middle| x^{n-1} \right\rangle. \qquad (56)
\end{aligned}$$

Note that

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\left(\partial_t\left(\left(\log(1+t)\right)^m\right)\right) \middle| x^{n-1} \right\rangle \\
&= m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\frac{1}{1+t} \middle| \log(1+t)^{m-1}x^{n-1} \right\rangle \\
&= m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^{-1} \middle| (m-1)!\sum_{l=m-1}^{\infty} S_1(l,m-1)\frac{t^l}{l!}x^{n-1} \right\rangle \\
&= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^{-1} \middle| x^{n-1-l} \right\rangle \\
&= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)N_{n-1-l}(-1|a_1,\ldots,a_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)N_l(-1|a_1,\ldots,a_r) \qquad (57)
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)\left(\log(1+t)\right)^m \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right) \middle| m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}x^{n-1} \right\rangle \\
&= m!\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m) \\
&\quad \times \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{-1} \middle| \frac{\sum_{j=1}^{r}\{\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\}}{t}x^{n-1-l} \right\rangle \\
&= m!\sum_{l=m}^{n-1}\binom{n-1}{l}\frac{S_1(l,m)}{n-l}
\end{aligned}$$

$$\times \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{-1} \middle| \sum_{j=1}^{r}\left\{\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\right\}x^{n-l}\right\rangle$$

$$= \frac{m!}{n}\sum_{l=m}^{n-1}\binom{n}{l}S_1(l,m)$$

$$\times \left\{\sum_{j=1}^{r} a_j\left\langle \frac{\log(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{a_j-1} \middle| \frac{t}{\log(1+t)}x^{n-l}\right\rangle\right.$$

$$\left. - r\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{-1} \middle| \frac{t}{\log(1+t)}x^{n-l}\right\rangle\right\}$$

$$= \frac{m!}{n}\sum_{l=m}^{n-1}\binom{n}{l}S_1(l,m)\left\{\sum_{j=1}^{r} a_j\sum_{k=0}^{n-l}C_k\binom{n-l}{k}N_{n-l-k}(a_j-1|a_1,\ldots,\hat{a}_j,\ldots,a_r)\right.$$

$$\left. - r\sum_{k=0}^{n-l}C_k\binom{n-l}{k}N_{n-l-k}(-1|a_1,\ldots,a_r)\right\}. \tag{58}$$

Therefore, by (55), (56), (57) and (58), we obtain the following theorem.

Theorem 6 *For $n-1 \geq m \geq 1$, we have*

$$\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)N_l(a_1,\ldots,a_r)$$

$$= \sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)N_l(-1|a_1,\ldots,a_r)$$

$$+ \frac{1}{n}\sum_{l=m}^{n-1}\sum_{k=0}^{n-l}\sum_{j=1}^{r}\binom{n}{l}\binom{n-l}{k}a_j C_{n-l-k}S_1(l,m)N_k(a_j-1|a_1,\ldots,\hat{a}_j,\ldots,a_r)$$

$$- \frac{r}{n}\sum_{l=m}^{n-1}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}C_{n-l-k}S_1(l,m)N_k(-1|a_1,\ldots,a_r).$$

By the same method as the proof of Theorem 6, we get

$$\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\hat{N}_l(a_1,\ldots,a_r)$$

$$= \sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\hat{N}_l(-1|a_1,\ldots,a_r)$$

$$+ \frac{1}{n}\sum_{j=1}^{r}\sum_{l=m}^{n-1}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}a_j C_{n-l-k}S_1(l,m)\hat{N}_k(-1|a_1,\ldots,\hat{a}_j,\ldots,a_r)$$

$$- \frac{r}{n}\sum_{l=m}^{n-1}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}C_{n-l-k}S_1(l,m)\hat{N}_k(-1|a_1,\ldots,a_r)$$

$$- \sum_{j=1}^{r}a_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\hat{N}_k(-1|a_1,\ldots,a_r), \tag{59}$$

where $n-1 \geq m \geq 1$.

Let us consider the following two Sheffer sequences:

$$N_n(x|a_1,\ldots,a_r) \sim \left(\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right), e^t - 1\right) \tag{60}$$

and (23).

We let

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} C_{n,m}(x)_m. \tag{61}$$

From (18) and (19), we note that

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)t^m \middle| x^n \right\rangle \\ &= \binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right) \middle| x^{n-m} \right\rangle \\ &= \binom{n}{m} N_{n-m}(a_1,\ldots,a_r). \end{aligned} \tag{62}$$

Therefore, by (61) and (62), we obtain the following theorem.

Theorem 7 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n}\binom{n}{m} N_{n-m}(a_1,\ldots,a_r)(x)_m.$$

By the same method as the proof of Theorem 7, we get

$$\hat{N}_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n}\binom{n}{m} \hat{N}_{n-m}(a_1,\ldots,a_r)(x)_m. \tag{63}$$

For

$$N_n(x|a_1,\ldots,a_r) \sim \left(\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right), e^t - 1\right)$$

and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right), \quad \lambda \in \mathbb{C} \text{ with } \lambda \neq 1,$$

let us assume that

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\lambda), \tag{64}$$

where $H_m^{(s)}(x|\lambda)$ are the Frobenius-Euler polynomials of order s defined by the generating function as

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x|\lambda)\frac{t^n}{n!}.$$

From (18) and (19), we note that

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m(1-\lambda+t)^s \middle| x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| \sum_{j=0}^{\min\{s,n\}}\binom{s}{j}(1-\lambda)^{s-j}t^j x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\lambda)^{s-j}(n)_j \\
&\quad \times \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1_t)}\right)\bigl(\log(1+t)\bigr)^m \middle| x^{n-j}\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\lambda)^{s-j}(n)_j m! \\
&\quad \times \sum_{l=0}^{n-j-m}\binom{n-j}{l}S_1(n-j-l,m)N_l(a_1,\ldots,a_r) \\
&= \sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l} \\
&\quad \times (n)_j(1-\lambda)^{-j}S_1(n-j-l,m)N_l(a_1,\ldots,a_r). \qquad (65)
\end{aligned}$$

Therefore, by (64) and (65), we obtain the following theorem.

Theorem 8 *For $n \geq 0$, we have*

$$\begin{aligned}
&N_n(x|a_1,\ldots,a_r) \\
&\quad = \sum_{m=0}^{n}\left\{\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j(1-\lambda)^{-j}\right. \\
&\qquad \left. \times S_1(n-j-l,m)N_l(a_1,\ldots,a_r)\right\}H_m^{(s)}(x|\lambda).
\end{aligned}$$

By the same method as the proof of Theorem 8, we get

$$\begin{aligned}
&\hat{N}_n(x|a_1,\ldots,a_r) \\
&\quad = \sum_{m=0}^{n}\left\{\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j\right. \\
&\qquad \left. \times (1-\lambda)^{-j}S_1(n-j-l,m)\hat{N}_l(a_1,\ldots,a_r)\right\}H_m^{(s)}(x|\lambda). \qquad (66)
\end{aligned}$$

Now, we consider the following two Sheffer sequences:

$$N_n(x|a_1,\ldots,a_r) \sim \left(\prod_{j=1}^{r}\left(\frac{t}{e^{a_jt}-1}\right), e^t-1\right) \tag{67}$$

and

$$B_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, e^t-1\right),$$

where $B_n^{(s)}(x)$ are the Bernoulli polynomials of order s given by the generating function as

$$\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty} B_n^{(s)}(x)\frac{t^n}{n!}.$$

Let us assume that

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} C_{n,m}B_m^{(s)}(x). \tag{68}$$

By (18) and (19), we get

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m\left(\frac{t}{\log(1+t)}\right)^s \middle| x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| \left(\frac{t}{\log(1+t)}\right)^s x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| \sum_{k=0}^{\infty}\mathbb{C}_k^{(s)}\frac{t^k}{k!}x^n\right\rangle \\
&= \frac{1}{m!}\sum_{k=0}^{n-m}\binom{n}{k}\mathbb{C}_k^{(s)}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| x^{n-k}\right\rangle \\
&= \frac{1}{m!}\sum_{k=0}^{n-m}\binom{n}{k}\mathbb{C}_k^{(s)} m! \sum_{l=0}^{n-m-k}\binom{n-k}{l}S_1(n-l-k,m)N_l(a_1,\ldots,a_r) \\
&= \sum_{k=0}^{n-m}\sum_{l=0}^{n-m-k}\binom{n}{k}\binom{n-k}{l}\mathbb{C}_k^{(s)}S_1(n-k-l,m)N_l(a_1,\ldots,a_r),
\end{aligned} \tag{69}$$

where $\mathbb{C}_k^{(s)}$ are the Cauchy numbers of the first kind of order s defined by the generating function as

$$\left(\frac{t}{\log(1+t)}\right)^s = \sum_{n=0}^{\infty}\mathbb{C}_n^{(s)}\frac{t^n}{n!}.$$

Therefore, by (68) and (69), we obtain the following theorem.

Theorem 9 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n}\left\{\sum_{k=0}^{n-m}\sum_{l=0}^{n-m-k}\binom{n}{k}\binom{n-k}{l}\mathbb{C}_k^{(s)} \times S_1(n-k-l,m)N_l(a_1,\ldots,a_r)\right\}B_m^{(s)}(x).$$

By the same method as the proof of Theorem 9, we get

$$\hat{N}_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n}\left\{\sum_{k=0}^{n-m}\sum_{l=0}^{n-m-k}\binom{n}{k}\binom{n-k}{l}\mathbb{C}_k^{(s)} \times S_1(n-k-l,m)\hat{N}_l(a_1,\ldots,a_r)\right\}B_m^{(s)}(x).$$

Recently, several authors have studied umbral calculus (see [1–5, 7–18, 20]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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