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Interval-valued functional integro-differential equations

Ngo Van Hoa[1], Nguyen Dinh Phu[2], Tran Thanh Tung[3*] and Le Thanh Quang[2]

*Correspondence:
tttung@ttn.edu.vn;
thanhtung_bmt@yahoo.com
[3]Faculty of Natural Science and Technology, Tay Nguyen University, 567 Le Duan Road, Buon Ma Thuot City, Daklak Province, Vietnam
Full list of author information is available at the end of the article

Abstract
This paper is devoted to studying the local and global existence and uniqueness results for interval-valued functional integro-differential equations (IFIDEs). In the paper, for the local existence and uniqueness, the method of successive approximations is used and for the global existence and uniqueness, the contraction principle is a good tool in investigating. Some examples are given to illustrate the results.

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1 Introduction

Functional differential equations (or, as they are called, delay differential equations) play an important role in an increasing number of system models in biology, engineering, physics and other sciences. There exists an extensive amount of literature dealing with functional differential equations and their applications; the reader is referred to the monographs [1–6] and the references therein.

The set-valued differential and integral equations are an important part of the theory of set-valued analysis. They have an important value in theory and application in control theory; and they were studied in 1969 by De Blasi and Iervolino [7]. Recently, set-valued differential equations have been studied by many authors due to their application in many areas. For many results in the theory of set-valued differential and integral equations, the readers can be referred to the following books and papers [8–23] and the references therein. The interval-valued analysis and interval-valued differential equations (IDEs) are the particular cases of the set-valued analysis and set differential equations, respectively. In many cases, when modeling real-world phenomena, information about the behavior of a dynamic system is uncertain, and we have to consider these uncertainties to gain more models. The interval-valued differential and integro-differential equations can be used to model dynamic systems subject to uncertainties. Recently, many works have been done by several authors in the theory of interval-valued differential equations (see, *e.g.*, [24–26]). These equations can be studied with a framework of the Hukuhara derivative [27]. However, it causes that the solutions have increasing length of their values. Stefanini and Bede [26] proposed to consider the so-called strongly generalized derivative of interval-valued functions. The interval-valued differential equations with this derivative can have solutions with decreasing length of their values. This approach was the starting point for the topic of interval-valued differential equations (see [24, 25]). Besides that, some very important extensions of the interval-valued differential equations are the set differential equations (see [6, 8, 11–14, 16, 18, 20, 23, 28–31]).

The connection between the fuzzy analysis and the interval analysis is very well known (Moore and Lodwick [32]). Interval analysis and fuzzy analysis were introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. Based on the results in [33], there are some very important extensions, and the development related to the subject of the present paper is in the field of fuzzy sets, *i.e.*, fuzzy calculus and fuzzy differential equations under generalized Hukuhara derivative. Recently, several works, *e.g.*, [5, 9, 10, 30, 34–45], have been done on fuzzy differential equations and fuzzy integro-differential equations, the fuzzy stochastic differential equations [46–51], fractional fuzzy differential equations [30, 52–55], and some methods for solving fuzzy differential equations [56, 57].

In the papers [24–26], one can find the studies on interval-valued differential equations under generalized Hukuhara differentiability, *i.e.*, equations of the form

$$D_H^g X(t) = F\bigl(t, X(t)\bigr), \qquad X(t_0) = X_0 \in K_C(\mathbb{R}), \quad t \in [t_0, t_0 + p], \tag{1.1}$$

where D_H^g denotes two kinds of derivatives, namely the classical Hukuhara derivative and the second-type Hukuhara derivative (generalized Hukuhara differentiability). The existence and uniqueness of a Cauchy problem is then obtained under an assumption that the coefficients satisfy a condition with the Lipschitz constant (see [26]). The proof is based on the application of the Banach fixed point theorem. In [25], under the generalized Lipschitz condition, Malinowski obtained the existence and uniqueness of solutions to both kinds of IDEs. In this paper, we study two kinds of solutions to IFIDEs. The different types of solutions to IFIDEs are generated by the usage of two different concepts of interval-valued derivative. Furthermore, in [5], Lupulescu established the local and global existence and uniqueness results for fuzzy functional differential equations. Malinowski [6] studied the existence and uniqueness result of solution to the delay set-valued differential equations under the condition that the right-hand side of an equation is Lipschitzian in the functional variable. Inspired and motivated by the results of Stefanini and Bede [26], Malinowski [24, 25] and Lupulescu [5], we consider the interval-valued functional integro-differential equations under generalized Hukuhara derivative. The paper is organized as follows. As preliminaries, we recall some basic concepts and notations about interval analysis and interval-valued differential equations. In Section 3, we present the local and global existence and uniqueness theorem of solution of IFIDEs under generalized Hukuhara derivatives. In the last section, we give some examples as simple illustrations of the theory of interval-valued functional integro-differential equations.

2 Preliminaries

Let us denote by $K_C(\mathbb{R})$ the set of any nonempty compact intervals of the real line $\mathbb{R}$. The addition and scalar multiplication in $K_C(\mathbb{R})$ are defined as usual, *i.e.*, for $A, B \in K_C(\mathbb{R})$,

$A = [\underline{A}, \overline{A}]$, $B = [\underline{B}, \overline{B}]$, where $\underline{A} \le \overline{A}$, $\underline{B} \le \overline{B}$, and $\lambda \ge 0$, then we have

$$A + B = [\underline{A} + \underline{B}, \overline{A} + \overline{B}], \qquad \lambda A = [\lambda \underline{A}, \lambda \overline{A}] \qquad \bigl(-\lambda A = [-\lambda \overline{A}, -\lambda \underline{A}]\bigr).$$

Furthermore, let $A \in K_C(\mathbb{R})$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \in \mathbb{R}$ and $\lambda_3\lambda_4 \ge 0$, then we have $\lambda_1(\lambda_2 A) = (\lambda_1\lambda_2)A$ and $(\lambda_3 + \lambda_4)A = \lambda_3 A + \lambda_4 A$. Let $A, B \in K_C(\mathbb{R})$ as above, the Hausdorff metric H in $K_C(\mathbb{R})$ is defined as follows:

$$H[A, B] = \max\bigl\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\bigr\}. \tag{2.1}$$

It is known that $(K_C(\mathbb{R}), H)$ is a complete, separable and locally compact metric space. We define the magnitude and the length of $A \in K_C(\mathbb{R})$ by

$$H[A, 0] = \|A\| = \max\bigl\{|\underline{A}|, |\overline{A}|\bigr\}, \qquad \operatorname{len}(A) = \overline{A} - \underline{A},$$

respectively, where 0 is the zero element of $K_C(\mathbb{R})$ which is regarded as one point.

The Hausdorff metric (2.1) satisfies the following properties:

$$H[A + C, B + C] = H[A, B] \quad \text{and} \quad H[A, B] = H[B, A],$$
$$H[A + B, C + D] \le H[A, C] + H[B, D],$$
$$H[\lambda A, \lambda B] = |\lambda| H[A, B]$$

for all $A, B, C, D \in K_C(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Let $A, B \in K_C(\mathbb{R})$. If there exists an interval $C \in K_C(\mathbb{R})$ such that $A = B + C$, then we call C the Hukuhara difference of A and B. We denote the interval C by $A \ominus B$. Note that $A \ominus B \ne A + (-)B$. It is known that $A \ominus B$ exists in the case $\operatorname{len}(A) \ge \operatorname{len}(B)$. Besides that, we can see the following properties for $A, B, C, D \in K_C(\mathbb{R})$ (see [24]):

- If $A \ominus B$, $A \ominus C$ exist, then $H[A \ominus B, A \ominus C] = H[B, C]$;
- If $A \ominus B$, $C \ominus D$ exist, then $H[A \ominus B, C \ominus D] = H[A + D, B + C]$;
- If $A \ominus B$, $A \ominus (B + C)$ exist, then there exist $(A \ominus B) \ominus C$ and $(A \ominus B) \ominus C = A \ominus (B + C)$;
- If $A \ominus B$, $A \ominus C$, $C \ominus B$ exist, then there exist $(A \ominus B) \ominus (A \ominus C)$ and $(A \ominus B) \ominus (A \ominus C) = C \ominus B$.

Definition 2.1 We say that the interval-valued mapping $X : [a, b] \subset \mathbb{R}^+ \to K_C(\mathbb{R})$ is continuous at the point $t \in [a, b]$ if for every $\varepsilon > 0$ there exists $\delta = \delta(t, \varepsilon) > 0$ such that

$$H\bigl[X(t), X(s)\bigr] \le \varepsilon$$

for all $s \in [a, b]$ with $|t - s| < \delta$.

The strongly generalized differentiability was introduced in [26] and studied in [6, 24, 25, 31, 41–43].

Definition 2.2 Let $X : (a, b) \to K_C(\mathbb{R})$ and $t \in (a, b)$. We say that X is strongly generalized differentiable at t if there exists $D_H^g X(t) \in K_C(\mathbb{R})$ such that

(i) for all $h > 0$ sufficiently small, $\exists X(t+h) \ominus X(t)$, $\exists X(t) \ominus X(t-h)$ and

$$\lim_{h\searrow 0} H\left[\frac{X(t+h)\ominus X(t)}{h}, D_H^g X(t)\right] = 0,$$
$$\lim_{h\searrow 0} H\left[\frac{X(t)\ominus X(t-h)}{h}, D_H^g X(t)\right] = 0,$$

or

(ii) for all $h > 0$ sufficiently small, $\exists X(t) \ominus X(t+h)$, $\exists X(t-h) \ominus X(t)$ and

$$\lim_{h\searrow 0} H\left[\frac{X(t)\ominus X(t+h)}{-h}, D_H^g X(t)\right] = 0,$$
$$\lim_{h\searrow 0} H\left[\frac{X(t-h)\ominus X(t)}{-h}, D_H^g X(t)\right] = 0,$$

or

(iii) for all $h > 0$ sufficiently small, $\exists X(t+h) \ominus X(t)$, $\exists X(t-h) \ominus X(t)$ and

$$\lim_{h\searrow 0} H\left[\frac{X(t+h)\ominus X(t)}{h}, D_H^g X(t)\right] = 0,$$
$$\lim_{h\searrow 0} H\left[\frac{X(t-h)\ominus X(t)}{-h}, D_H^g X(t)\right] = 0,$$

or

(iv) for all $h > 0$ sufficiently small, $\exists X(t) \ominus X(t+h)$, $\exists X(t) \ominus X(t-h)$ and the limits

$$\lim_{h\searrow 0} H\left[\frac{X(t)\ominus X(t+h)}{-h}, D_H^g X(t)\right] = 0,$$
$$\lim_{h\searrow 0} H\left[\frac{X(t)\ominus X(t-h)}{h}, D_H^g X(t)\right] = 0$$

(h at denominators means $\frac{1}{h}$).

In this definition, case (i) ((i)-differentiability for short) corresponds to the classical H-derivative, so this differentiability concept is a generalization of the Hukuhara derivative. In this paper we consider only the two first of Definition 2.2. In the other cases, the derivative is trivial because it is reduced to a crisp element (more precisely, $D_H^g X(t) \in \mathbb{R}$). Further, we say that X is (i)-differentiable or (ii)-differentiable on $[a,b]$, if it is differentiable in the sense (i) or (ii) of Definition 2.2, respectively.

Theorem 2.1 *Let $X : (a,b) \to K_C(\mathbb{R})$ be* (i)*-differentiable or* (ii)*-differentiable on (a,b), and assume that the derivative $D_H^g X$ is integrable over (a,b). We have*

(a) *if X is* (i)*-differentiable on (a,b), then $\int_a^b D_H^g X(t)\,dt = X(b) \ominus X(a)$;*

(b) *if X is* (ii)*-differentiable on (a,b), then $\int_a^b D_H^g X(t)\,dt = (-1)(X(a) \ominus X(b))$.*

Provided that, the above Hukuhara differences exist.

Lemma 2.1 (see [24–26]) *Assume that $F : [t_0, t_0 + p] \times K_C(\mathbb{R}) \to K_C(\mathbb{R})$ is continuous. The interval-valued differential equation* (1.1) *is equivalent to one of the following integral equations*:

$$X(t) = X(t_0) + \int_{t_0}^{t} F\bigl(s, X(s)\bigr)\,ds, \quad \forall t \in [t_0, t_0 + p],$$

if X is (i)*-differentiable, and*

$$X(t) = X(t_0) \ominus (-1) \int_{t_0}^{t} F\bigl(s, X(s)\bigr)\, ds, \quad \forall t \in [t_0, t_0 + p]$$

if X is (ii)*-differentiable, provided that the H-difference exists.*

The following well-known result is useful in the next section.

Lemma 2.2 *Let $a(t)$, $b(t)$ and $c(t)$ be real-valued nonnegative continuous functions defined on $\mathbb{R}_+$, $d \geq 0$ is a constant for which the inequality*

$$a(t) \leq d + \int_0^t \biggl[b(s)a(s) + b(s) \int_0^s c(r)a(r)\, dr \biggr] ds \tag{2.2}$$

holds for all $t \in \mathbb{R}_+$. Then

$$a(t) \leq d \biggl[1 + \int_0^t b(s) \exp\biggl(\int_0^s \bigl(b(r) + c(r)\bigr)\, dr \biggr) ds \biggr].$$

Corollary 2.1 (see [24–26]) *Let $X : [t_0, t_0 + p] \to K_C(\mathbb{R})$ be given. Denote $X(t) = [\underline{X}(t), \overline{X}(t)]$ for $t \in [t_0, t_0 + p]$, where $\underline{X}, \overline{X} : [t_0, t_0 + p] \to \mathbb{R}$.*

(i) *If the mapping X is* (i)*-differentiable* (*i.e., classical Hukuhara differentiable*) *at $t \in [t_0, t_0 + p]$, then the real-valued functions $\underline{X}$, $\overline{X}$ are differentiable at t and $D_H^g X(t) = [\underline{X}'(t), \overline{X}'(t)]$.*
(ii) *If the mapping X is* (ii)*-differentiable at $t \in [t_0, t_0 + p]$, then the real-valued functions $\underline{X}$, $\overline{X}$ are differentiable at t and $D_H^g X(t) = [\overline{X}'(t), \underline{X}'(t)]$.*

3 Main results

For a positive number σ, we denote by $C_\sigma = C([-\sigma, 0], K_C(\mathbb{R}))$ the space of continuous mappings from $[-\sigma, 0]$ to $K_C(\mathbb{R})$. Define a metric H_σ in C_σ by

$$H_\sigma[X, Y] = \sup_{t \in [-\sigma, 0]} H\bigl[X(t), Y(t)\bigr].$$

Let $p > 0$. Denote $I = [t_0, t_0 + p]$, $J = [t_0 - \sigma, t_0] \cup I = [t_0 - \sigma, t_0 + p]$. For any $t \in I$, denote X_t by the element of C_σ defined by $X_t(s) = X(t + s)$ for $s \in [-\sigma, 0]$.

Let us consider the interval-valued functional integro-differential equations (IFIDEs) with the generalized Hukuhara derivative under the form

$$\begin{cases} D_H^g X(t) = F(t, X_t) + \int_{t_0}^t G(t, s, X_s)\, ds, & t \geq t_0, \\ X(t) = \varphi(t - t_0) = \varphi_0, & t_0 \geq t \geq t_0 - \sigma, \end{cases} \tag{3.1}$$

where $F : I \times C_\sigma \to K_C(\mathbb{R})$, $G : I \times I \times C_\sigma \to K_C(\mathbb{R})$, $\varphi \in C_\sigma$ and the symbol D_H^g denotes the generalized Hukuhara derivative from Definition 2.2. By a solution to equation (3.1) we mean an interval-valued mapping $X \in C(J, K_C(\mathbb{R}))$ that satisfies $X(t) = \varphi(t - t_0)$ for $t \in [t_0 - \sigma, t_0]$, X is differentiable on $[t_0, t_0 + p]$ and $D_H^g X(t) = F(t, X_t) + \int_{t_0}^t G(t, s, X_s)\, ds$ for $t \in I$. We note that the solution in this sense is considered just one-side differentiable at $t = t_0$ (specifically, right-differentiable at $t = t_0$).

Lemma 3.1 *Assume that $F \in C(I \times C_\sigma, K_C(\mathbb{R}))$, $G \in C(I \times I \times C_\sigma, K_C(\mathbb{R}))$ and $X \in C(J, K_C(\mathbb{R}))$. Then the interval-valued mapping $t \to F(t, X_t) + \int_{t_0}^t G(t, s, X_s)\,ds$ belongs to $C(I, K_C(\mathbb{R}))$.*

Remark 3.1 Under assumptions of the lemma above, the mapping $t \to F(t, X_t) + \int_{t_0}^t G(t, s, X_s)\,ds$ is integrable over the interval I.

Remark 3.2 If $F : I \times C_\sigma \to K_C(\mathbb{R})$, $G : I \times I \times C_\sigma \to K_C(\mathbb{R})$ are continuous and $X \in C(J, K_C(\mathbb{R}))$, then the mapping $t \to F(t, X_t) + \int_{t_0}^t G(t, s, X_s)\,ds$ is bounded on I. Also, the function $t \to F(t, 0) + \int_{t_0}^t G(t, s, 0)\,ds$ is bounded on I.

Lemma 3.2 *Assume that $F : I \times C_\sigma \to K_C(\mathbb{R})$, $G : I \times I \times C_\sigma \to K_C(\mathbb{R})$ are continuous. An interval-valued mapping $X : J \to K_C(\mathbb{R})$ is called a local solution to problem* (3.1) *on J if and only if X is a continuous interval-valued mapping and it satisfies one of the following interval-valued integral equations*:

$$(\mathrm{S1})\quad \begin{cases} X(t) = \varphi(t - t_0) & \textit{for } t \in [t_0 - \sigma, t_0], \\ X(t) = \varphi(0) + \int_{t_0}^t (F(s, X_s) + \int_{t_0}^t G(t, s, X_s)\,ds)\,ds, & t \in I, \end{cases} \tag{3.2}$$

if X is (i)*-differentiable,*

$$(\mathrm{S2})\quad \begin{cases} X(t) = \varphi(t - t_0) & \textit{for } t \in [t_0 - \sigma, t_0], \\ X(t) = \varphi(0) \ominus (-1) \int_{t_0}^t (F(s, X_s) + \int_{t_0}^t G(t, s, X_s)\,ds)\,ds, & t \in I, \end{cases} \tag{3.3}$$

if X is (ii)*-differentiable. We remark that in* (3.3), *the following statement is hidden*: *there exists the Hukuhara difference $\varphi(0) \ominus (-1) \int_{t_0}^t (F(s, X_s) + \int_{t_0}^t G(t, s, X_s)\,ds)\,ds$.*

Proof We prove the case of (ii)-differentiability, the proof of the other case being similar. Assume that $X : [t_0, t_0 + p] \to K_C(\mathbb{R})$ is a solution to problem (3.1). Hence X is (ii)-differentiable on $[t_0, t_0 + p]$ and $D_H^g X$ is integrable as a continuous function. Applying Theorem 2.1, we obtain that

$$X(t_0) = X(t) + (-1) \int_{t_0}^t D_H^g X(s)\,ds$$

for every $t \in [t_0, t_0 p]$. Since $X(t_0) = \varphi(0)$ and $D_H^g X(s) = F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\,d\tau$ for $s \in [t_0, t_0 + p]$, we easily obtain

$$\begin{cases} X(t) = \varphi(t - t_0) & \text{for } t \in [t_0 - \sigma, t_0], \\ X(t) = \varphi(0) \ominus (-1) \int_{t_0}^t (F(s, X_s) + \int_{t_0}^t G(t, s, X_s)\,ds)\,ds, & t \in I. \end{cases}$$

To show that the opposite implication is true, let us assume that $X : [t_0, t_0 + p] \to K_C(\mathbb{R})$ is a continuous interval-valued mapping and it satisfies equation (3.3). Equation (3.3) allows us to claim that $\varphi(0) = X(t_0)$ and that there exists the Hukuhara difference

$$\varphi(0) \ominus (-1) \int_{t_0}^t \left(F(s, X_s) + \int_{t_0}^t G(t, s, X_s)\,ds \right) ds \quad \text{for every } t \in [t_0, t_0 + p].$$

Now, let $t \in [t_0, t_0 + p)$ and h be such that $t + h \in [t_0, t_0 + p]$. We observe that

$$X(t) \ominus X(t+h) = (-1) \int_t^{t+h} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds. \tag{3.4}$$

Indeed, we have by direct computation

$$\begin{aligned}
&X(t+h) + (-1) \int_t^{t+h} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds \\
&\quad = \varphi(0) \ominus (-1) \int_{t_0}^{t+h} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds \\
&\qquad + (-1) \int_t^{t+h} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds \\
&\quad = \varphi(0) \ominus (-1) \int_{t_0}^{t+h} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds \\
&\qquad + (-1) \int_{t_0}^{t+h} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds \\
&\qquad \ominus (-1) \int_{t_0}^{t} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds \\
&\quad = \varphi(0) \ominus (-1) \int_{t_0}^{t} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds \\
&\quad = X(t).
\end{aligned}$$

Similarly to (3.4), we can obtain

$$X(t-h) \ominus X(t) = (-1) \int_{t-h}^{t} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds \tag{3.5}$$

for $t \in (t_0, t_0 + p]$. Multiplying (3.4)-(3.5) by $\frac{1}{-h}$ and passing to limit with $h \searrow 0$, we have by Definition 2.2 that X is (ii)-differentiable, and consequently

$$D_H^g X(t) = F(t, X_t) + \int_{t_0}^t G(t, s, X_s)\, ds \quad \text{for } t \in [t_0, t_0 + p].$$

Indeed, we have, for every $t \in [t_0, t_0 + p]$,

$$\lim_{h \to 0^+} \frac{X(t) \ominus X(t+h)}{-h} = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds$$

and

$$\lim_{h \to 0^+} \frac{X(t-h) \ominus X(t)}{-h} = \lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^{t} \left(F(s, X_s) + \int_{t_0}^s G(s, \tau, X_\tau)\, d\tau \right) ds.$$

Since F, G are continuous, for $h \to 0^+$, we obtain

$$\lim_{h \to 0^+} \frac{X(t) \ominus X(t+h)}{-h} = F(t, X_t) + \int_{t_0}^t G(t, s, X_s)\, ds.$$

Proceeding as above, we can obtain

$$\lim_{h\to 0^+} \frac{X(t-h)\ominus X(t)}{-h} = F(t,X_t) + \int_{t_0}^{t} G(t,s,X_s)\,ds.$$

The proof is complete. □

Definition 3.1 Let $X : J \to K_C(\mathbb{R})$ be an interval-valued function which is (i)-differentiable. If X and its derivative satisfy problem (3.1), we say that X is (i)-solution of problem (3.1). (i)-solution $X : J \to K_C(\mathbb{R})$ is unique if it holds $H[X(t), Y(t)] = 0$ for any $Y : J \to K_C(\mathbb{R})$ which is (i)-solution of (3.1).

Definition 3.2 Let $X : J \to K_C(\mathbb{R})$ be an interval-valued function which is (ii)-differentiable. If X and its derivative satisfy problem (3.1), we say that X is (ii)-solution of problem (3.1). (ii)-solution $X : J \to K_C(\mathbb{R})$ is unique if it holds $H[X(t), Y(t)] = 0$ for any $X : J \to K_C(\mathbb{R})$ which is (ii)-solution of (3.1).

Theorem 3.1 *Let $\varphi(t-t_0) \in C_\sigma$ and suppose that $F \in C(I \times C_\sigma, K_C(\mathbb{R}))$, $G \in C(I \times I \times C_\sigma, K_C(\mathbb{R}))$ satisfy the conditions: there exists a constant $L > 0$ such that*

$$\max\bigl\{H\bigl[F(t,X),F(t,Y)\bigr],H\bigl[G(t,s,X),G(t,s,Y)\bigr]\bigr\} \le LH_\sigma[X,Y]$$

for every $t \in [t_0, t_0+p]$, $(t,s) \in [t_0,t_0+p] \times [t_0,t_0+p]$ and $X, Y \in C_\sigma$. Moreover, there exists $M > 0$ such that $\max\{H[F(t,X),0],H[G(t,s,X),0]\} \le M$. *Then the following successive approximations given by*

$$\begin{aligned}
&\widehat{X}^0(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-\sigma, t_0],\\ \varphi(0), & t \in I,\end{cases}\\
&\widehat{X}^{n+1}(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-\sigma, t_0],\\ \varphi(0) + \int_{t_0}^{t} (F(s,\widehat{X}^n_s) + \int_{t_0}^{s} G(s,\tau,\widehat{X}^n_\tau)\,d\tau)\,ds\end{cases}
\end{aligned} \tag{3.6}$$

for the case of (i)-*differentiability, and*

$$\begin{aligned}
&\widetilde{X}^0(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-\sigma, t_0],\\ \varphi(0), & t \in [t_0, t_0+d],\end{cases}\\
&\widetilde{X}^{n+1}(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-\sigma, t_0],\\ \varphi(0) \ominus (-1) \int_{t_0}^{t} (F(s,\widetilde{X}^n_s) + \int_{t_0}^{s} G(s,\tau,\widetilde{X}^n_\tau)\,d\tau)\,ds\end{cases}
\end{aligned} \tag{3.7}$$

for the case of (ii)-*differentiability* (*where* $0 < d \le p$ *such that equation* (3.7) *is well defined, i.e., the foregoing Hukuhara differences do exist*), *converge uniformly to two unique solutions* $\widehat{X}(t)$ *and* $\widetilde{X}(t)$ *of* (3.1), *respectively, on* $[a, a+r]$ *where* $r = \min\{p, d\}$.

Proof We prove that for the case of (ii)-differentiability, the proof of the other case is similar. From assumptions of the theorem, we have

$$\begin{aligned}
&H\bigl[\widetilde{X}^1(t),\widetilde{X}^0(t)\bigr]\\
&\quad = H\biggl[\varphi(0) \ominus (-1)\int_{t_0}^{t} \biggl(F\bigl(s,\widetilde{X}^0_s\bigr) + \int_{t_0}^{s} G\bigl(s,\tau,\widetilde{X}^0_\tau\bigr)\,d\tau\biggr)\,ds, \varphi(0)\biggr]
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_0}^{t} \left(H\big[F\big(s,\widetilde{X}_s^0\big),0\big] + \int_{t_0}^{s} H\big[G\big(s,\tau,\widetilde{X}_\tau^0\big),0\big]\,d\tau \right) ds \\
&\leq M(t-t_0) + M\frac{(t-t_0)^2}{2!}
\end{aligned}$$

for $t \in [t_0, t_0 + r]$. Further, for every $n \geq 2$ and $t \in [t_0, t_0 + r]$, we get

$$\begin{aligned}
&H\big[\widetilde{X}^{n+1}(t),\widetilde{X}^n(t)\big] \\
&\quad = H\bigg[\int_{t_0}^{t} \left(F\big(s,\widetilde{X}_s^n\big) + \int_{t_0}^{s} G\big(s,\tau,\widetilde{X}_\tau^n\big)\,d\tau \right) ds, \\
&\qquad \int_{t_0}^{t} \left(F\big(s,\widetilde{X}_s^{n-1}\big) + \int_{t_0}^{s} G\big(s,\tau,\widetilde{X}_\tau^{n-1}\big)\,d\tau \right) ds\bigg] \\
&\quad \leq L\int_{t_0}^{t} \left(H_\sigma\big[\widetilde{X}_s^n,\widetilde{X}_s^{n-1}\big] + \int_{t_0}^{s} H_\sigma\big[\widetilde{X}_\tau^n,\widetilde{X}_\tau^{n-1}\big]\,d\tau \right) ds \\
&\quad \leq L\int_{t_0}^{t} \bigg(\sup_{\theta\in[-\sigma,0]} H\big[\widetilde{X}^n(s+\theta),\widetilde{X}^{n-1}(s+\theta)\big] \\
&\qquad + \int_{t_0}^{s} \sup_{\theta\in[-\sigma,0]} H\big[\widetilde{X}^n(\tau+\theta),\widetilde{X}^{n-1}(\tau+\theta)\big]\,d\tau \bigg) ds \\
&\quad = L\int_{t_0}^{t} \left(\sup_{r\in[s-\sigma,s]} H\big[\widetilde{X}^n(r),\widetilde{X}^{n-1}(r)\big] + \int_{t_0}^{s} \sup_{\upsilon\in[\tau-\sigma,\tau]} H\big[\widetilde{X}^n(\upsilon),\widetilde{X}^{n-1}(\upsilon)\big]\,d\upsilon \right) dr.
\end{aligned}$$

In particular, from (3.7) it follows that

$$H\big[\widetilde{X}^2(t),\widetilde{X}^1(t)\big] \leq LM\left(\frac{(t-t_0)^2}{2!} + 2\frac{(t-t_0)^3}{3!} + \frac{(t-t_0)^4}{4!} \right).$$

Therefore, by mathematical induction, for every $n \in \mathbb{N}$ and $t \in [t_0, t_0 + r]$,

$$\begin{aligned}
&H\big[\widetilde{X}^{n+1}(t),\widetilde{X}^n(t)\big] \\
&\quad \leq ML^n\left(\frac{(t-t_0)^{n+1}}{(n+1)!} +^n \lambda_1\frac{(t-t_0)^{n+2}}{(n+2)!} + \cdots +^n \lambda_n\frac{(t-t_0)^{2n+1}}{(2n+1)!} + \frac{(t-t_0)^{2n+2}}{(2n+2)!} \right).
\end{aligned} \tag{3.8}$$

In inequality (3.8), $\lambda_1,\ldots,\lambda_n$ are balancing constants. We observe that for every $n \in \{0,1,2,\ldots\}$, the function $\widetilde{X}^n(\cdot) : [t_0 - \sigma, t_0 + r] \to K_C(\mathbb{R})$ is continuous. Indeed, since $\varphi \in C_\sigma$, $\widetilde{X}^0(t)$ is continuous on $t \in [t_0 - \sigma, t_0]$. We see that

$$\begin{aligned}
H\big[\widetilde{X}^1(t+h),\widetilde{X}^1(t)\big] &= H\bigg[\varphi(0) \ominus (-1)\int_{t_0}^{t+h} \left(F\big(s,\widetilde{X}_s^0\big) + \int_{t_0}^{s} G\big(s,\tau,\widetilde{X}_\tau^0\big)\,d\tau \right) ds, \\
&\qquad \varphi(0) \ominus (-1)\int_{t_0}^{t} \left(F\big(s,\widetilde{X}_s^0\big) + \int_{t_0}^{s} G\big(s,\tau,\widetilde{X}_\tau^0\big)\,d\tau \right) ds\bigg] \\
&\leq \int_{t}^{t+h} \left(H\big[F\big(s,\widetilde{X}_s^0\big),0\big] + \int_{t_0}^{s} H\big[G\big(s,\tau,\widetilde{X}_\tau^0\big),0\big]\,d\tau \right) ds \\
&\leq Mh + \frac{Mh^2}{2!} \to 0 \quad \text{as } h \to 0^+.
\end{aligned}$$

Thus, by mathematical induction, for every $n \geq 2$, we deduce that

$$H\bigl[\widetilde{X}^n(t+h),\widetilde{X}^n(t)\bigr] \to 0$$

as $h \to 0^+$. A similar inequality is obtained for $H[\widetilde{X}^n(t-h),\widetilde{X}^n(t)] \to 0$ as $h \to 0^+$. In the sequel, we shall show that for $\{\widetilde{X}^n(t)\}$ the Cauchy convergence condition is satisfied uniformly in t, and as a consequence $\{\widetilde{X}^n(\cdot)\}$ is uniformly convergent. For $n > m > 0$, from (3.8) we obtain

$$\begin{aligned}
&\sup_{t\in I} H\bigl[\widetilde{X}^n(t),\widetilde{X}^m(t)\bigr] \\
&\quad = \sup_{t\in J} H\bigl[\widetilde{X}^n(t),\widetilde{X}^m(t)\bigr] \leq \sum_{k=m}^{n-1} \sup_{t\in J} H\bigl[\widetilde{X}^{k+1}(t),\widetilde{X}^k(t)\bigr] \\
&\quad \leq M \sum_{k=m}^{n-1} \biggl(\frac{(t-t_0)^{k+1}}{(k+1)!} +^k \lambda_1 \frac{(t-t_0)^{k+2}}{(k+2)!} + \cdots +^k \lambda_k \frac{(t-t_0)^{2k+1}}{(2k+1)!} + \frac{(t-t_0)^{2k+2}}{(2k+2)!} \biggr).
\end{aligned}$$

The convergence of this series implies that for any $\varepsilon > 0$ we find $n_0 \in \mathbb{N}$ large enough such that for $n, m > n_0$,

$$H\bigl[\widetilde{X}^n(t),\widetilde{X}^m(t)\bigr] \leq \varepsilon. \tag{3.9}$$

Since $(K_C(\mathbb{R}), H)$ is a complete metric space and (3.9) holds, the sequence $\{\widetilde{X}^n(\cdot)\}$ is uniformly convergent to a mapping $\widetilde{X} \in C([t_0-\sigma, t_0+r], K_C(\mathbb{R}))$. We shall show that $\widetilde{X}$ is (ii)-solution to (3.1). Since $\widetilde{X}^n(t) = \varphi(t-t_0)$ for every $n = 0,1,2,\ldots$ and every $t \in [t_0-\sigma, t_0]$, we easily have $\widetilde{X}(t) = \varphi(t-t_0)$. For $s \in [t_0, t_0+r]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&H\biggl[\int_{t_0}^t F\bigl(s,\widetilde{X}_s^n\bigr)\,ds, \int_{t_0}^t F(s,\widetilde{X}_s)\,ds\biggr] \leq L \int_{t_0}^t \sup_{\theta\in[s-\sigma,s]} H\bigl[\widetilde{X}^n(\theta),\widetilde{X}(\theta)\bigr]\,d\theta \to 0, \\
&H\biggl[\int_{t_0}^t \int_{t_0}^s G\bigl(s,\tau,\widetilde{X}_\tau^n\bigr)\,d\tau\,ds, \int_{t_0}^t \int_{t_0}^s G(s,\tau,\widetilde{X}_\tau)\,d\tau\,ds\biggr] \\
&\quad \leq L \int_{t_0}^t \int_{t_0}^s \sup_{\theta\in[\tau-\sigma,\tau]} H\bigl[\widetilde{X}^n(\upsilon),\widetilde{X}(\upsilon)\bigr]\,d\upsilon\,ds \to 0
\end{aligned}$$

as $n \to \infty$ for any $t \in [t_0, t_0+r]$. Consequently, we have

$$\begin{aligned}
&H\biggl[\varphi(0),\widetilde{X}(t) + (-1)\int_{t_0}^t \biggl(F(s,\widetilde{X}_s) + \int_{t_0}^t G(t,s,\widetilde{X}_s)\,ds\biggr)\,ds\biggr] \\
&\quad \leq H\bigl[\widetilde{X}^n(t),\widetilde{X}(t)\bigr] + \int_{t_0}^t \biggl(H\bigl[F\bigl(s,\widetilde{X}_s^{n-1}\bigr),F(s,\widetilde{X}_s)\bigr] \\
&\qquad + \int_{t_0}^s H\bigl[G\bigl(s,\tau,\widetilde{X}_\tau^{n-1}\bigr),G(s,\tau,\widetilde{X}_\tau)\bigr]\,d\tau\biggr)\,ds.
\end{aligned}$$

We infer that

$$H\biggl[\varphi(0),\widetilde{X}(t) + (-1)\int_{t_0}^t \biggl(F(s,\widetilde{X}_s) + \int_{t_0}^t G(t,s,\widetilde{X}_s)\,ds\biggr)\,ds\biggr] = 0$$

for every $t \in [t_0, t_0 + r]$. Therefore, $\widetilde{X}$ is the (ii)-solution of (3.1), due to Lemma 3.2 it follows that $\widetilde{X}$ is the (ii)-solution of (3.1). For the uniqueness of the (ii)-solution $\widetilde{X}$, let us assume that $\widetilde{X}, \widetilde{Y} \in C([t_0 - \sigma, t_0 + r], K_C(\mathbb{R}))$ are two solutions of (3.3). By definition of the solution, $\widetilde{X}(t) = \widetilde{Y}(t)$ if $t \in [t_0 - \sigma, t_0]$. Note that for $t \in [t_0, t_0 + r]$,

$$\begin{aligned} & H\bigl[\widetilde{X}(t), \widetilde{Y}(t)\bigr] \\ & \quad \leq L \int_{t_0}^{t} \left(\sup_{\theta \in [s-\sigma, s]} H\bigl[\widetilde{X}(\theta), \widetilde{Y}(\theta)\bigr] + \int_{t_0}^{s} \sup_{\upsilon \in [\tau-\sigma, \tau]} H\bigl[\widetilde{X}(\upsilon), \widetilde{Y}(\upsilon)\bigr] \, d\tau \right) ds. \end{aligned}$$

If we put $a(s) = \sup_{r \in [s-\sigma, s]} H[\widetilde{X}(r), \widetilde{Y}(r)]$, $s \in [t_0, t] \subset [t_0, t_0 + r]$, then we obtain

$$a(t) \leq L \int_{t_0}^{t} \left(a(s) + \int_{t_0}^{s} a(\tau) \, d\tau \right) ds,$$

and by Lemma 2.2 we obtain that $a(t) = 0$ on $[t_0, t_0 + r]$. This proves the uniqueness of the (ii)-solution for (3.1) □

Remark 3.3 The existence and uniqueness results for solutions of problem (3.1) can be obtained by using the contraction principle.

Now, we present the studies and results concerning the global existence and uniqueness of two solutions for (3.1), each one corresponding to a different type of differentiability, by using the contraction principle, which was studied in [5] for fuzzy functional differential equations. In the following, for a given $k > 0$, we consider the set S_k of all continuous interval-valued functions $X \in C([t_0 - \sigma, \infty), K_C(\mathbb{R}))$ such that $X(t) = \varphi(t - t_0)$ on $[t_0 - \sigma, t_0]$ and $\sup_{t \geq t_0 - \sigma} \{H[X(t), 0] \exp(-kt)\} < \infty$. On S_k we can define the following metric:

$$H_k[X, Y] = \sup_{t \geq t_0 - \sigma} \bigl\{ H\bigl[X(t), Y(t)\bigr] \exp(-kt) \bigr\}, \tag{3.10}$$

where $k > 0$ is chosen suitably later. It is easy to prove that the space (S_k, H_k) of continuous interval-valued functions $X : [t_0, \infty) \to K_C(\mathbb{R})$ is a complete metric space with distance (3.10).

Theorem 3.2 *Assume that*

(i) $F \in C([t_0, \infty) \times C_\sigma, K_C(\mathbb{R}))$, $G \in C([t_0, \infty) \times [t_0, \infty) \times C_\sigma, K_C(\mathbb{R}))$ *and there exists a constant* $L > 0$ *such that*

$$\max\bigl\{ H\bigl[F(t, X), F(t, Y)\bigr], H\bigl[G(t, s, X), G(t, s, Y)\bigr] \bigr\} \leq L H_\sigma[X, Y]$$

for all $X, Y \in C_\sigma$ *and* $t, s \geq t_0$;

(ii) *there exist* $M > 0$ *and* $b > 0$ *such that*

$$\max\bigl\{ H\bigl[F(t, 0), 0\bigr], H\bigl[G(t, s, 0), 0\bigr] \bigr\} \leq M \exp(bt)$$

for all $t \geq t_0$, *where* $b < k$.

Then

(a) *the interval-valued functional integro-differential equation* (3.1) *has* (i)*-solution on* $[t_0,\infty)$;

(b) *the interval-valued functional integro-differential equation* (3.1) *has* (ii)*-solution on* $[t_0,\infty)$ *if the following condition holds*:

$$\int_{t_0}^{t}\left(\operatorname{len}\bigl(F(s,X_s)\bigr)+\int_{t_0}^{t}\operatorname{len}\bigl(G(t,s,X_s)\bigr)\,ds\right)ds\le\operatorname{len}\bigl(\varphi(0)\bigr),\quad t\ge t_0. \tag{3.11}$$

Proof Since the way of the proof is similar for both cases, we only consider the case of (ii)-differentiability for X. Note that the space (S_k,H_k) under inequality (3.11) depends on the positive constant k, the functions F, G and the initial condition φ. In (S_k,H_k), the continuity of F, G guarantees that S_k under inequality (3.11) is a closed set in $C([t_0,\infty),K_C(\mathbb{R}))$, so that S_k under inequality (3.11) is a complete metric space considering the distance H_k. We consider the complete metric space (S_k,H_k) and define an operator

$$\begin{aligned}&\mathbb{T}:S_k\to S_k,\\&X\to\mathbb{T}X\end{aligned}$$

given by

$$(\mathbb{T}X)(t)=\begin{cases}\varphi(t-t_0) & \text{if } t\in[t_0-\sigma,t_0],\\ \varphi(0)\ominus(-1)\int_{t_0}^{t}(F(s,X_s)+\int_{t_0}^{s}G(s,\tau,X_\tau)\,d\tau)\,ds & \text{if } t\ge t_0.\end{cases}$$

We can choose a big enough value for k such that $\mathbb{T}$ is a contraction, so the Banach fixed point theorem provides the existence of a unique fixed point for $\mathbb{T}$, that is, a unique solution for (3.1).

First, we shall prove that $\mathbb{T}(S_k)\subseteq S_k$, *i.e.*, the operator T is well defined, with assumption $k>b$. Indeed, let $X\in S_k$. For each $t\ge t_0$, we get

$$\begin{aligned}
&H_k\bigl[(\mathbb{T}X)(t),0\bigr]\\
&\quad=\sup_{t\ge t_0}\left\{H\left[\varphi(0)\ominus(-1)\int_{t_0}^{t}\left(F(s,X_s)+\int_{t_0}^{s}G(s,\tau,X_\tau)\,d\tau\right)ds,0\right]\exp(-kt)\right\}\\
&\quad\le\sup_{t\ge t_0}\left\{\left(H\bigl[\varphi(0),0\bigr]+\int_{t_0}^{t}\bigl\{H\bigl[F(s,X_s),F(s,0)\bigr]+H\bigl[F(s,0),0\bigr]\bigr\}\,ds\right.\right.\\
&\qquad\left.\left.+\int_{t_0}^{t}\left(\int_{t_0}^{s}\bigl\{H\bigl[G(s,\tau,X_\tau),G(s,\tau,0)\bigr]+H\bigl[G(s,\tau,0),0\bigr]\bigr\}\,d\tau\right)ds\right)\exp(-kt)\right\}\\
&\quad\le\sup_{t\ge t_0}\left\{\left(H\bigl[\varphi(0),0\bigr]+L\int_{t_0}^{t}H_\sigma[X_s,0]\,ds+\frac{M}{b}\exp(bt)\right.\right.\\
&\qquad\left.\left.+L\int_{t_0}^{t}\left(\int_{t_0}^{s}H_\sigma[X_\tau,0]\,d\tau\right)ds+\frac{M}{b^2}\exp(bt)\right)\exp(-kt)\right\}\\
&\quad\le\sup_{t\ge t_0}\left\{\left(H\bigl[\varphi(0),0\bigr]+L\int_{t_0}^{t}\sup_{\theta\in[-\sigma,0]}H\bigl[X(s+\theta),0\bigr]\,ds+\frac{M}{b}\exp(bt)\right.\right.\\
&\qquad\left.\left.+L\int_{t_0}^{t}\left(\int_{t_0}^{s}\sup_{\theta\in[-\sigma,0]}H\bigl[X(\tau+\theta),0\bigr]\,d\tau\right)ds+\frac{M}{b^2}\exp(bt)\right)\exp(-kt)\right\}.
\end{aligned}$$

Since $X \in S_k$, there exists $\rho > 0$ such that $H[X(t), 0] < \rho \exp(kt)$ for all $t \geq t_0 - \sigma$. Therefore, for all $t \geq t_0$, we obtain

$$\begin{aligned} &H_k\big[(\mathbb{T}X)(t), 0\big] \\ &\quad \leq \sup_{t \geq t_0}\left\{\left(H\big[\varphi(0), 0\big] + \left(1 + \frac{1}{k}\right)\frac{\rho L}{k}\exp(kt) + \left(1 + \frac{1}{b}\right)\frac{M}{b}\exp(bt)\right)\exp(-kt)\right\} \\ &\quad \leq H\big[\varphi(0), 0\big] + \left(1 + \frac{1}{b}\right)\frac{1}{b}(M + \rho L) \leq K + \left(1 + \frac{1}{b}\right)\frac{1}{b}(M + \rho L) < \infty. \end{aligned}$$

We infer that $\mathbb{T}X \in S_k$.

Next, we shall prove that $\mathbb{T}$ is a contraction by metric H_k. Let $X, Y \in S_k$. Then, for $\theta \in [-\sigma, 0]$, $H[(\mathbb{T}X)(t_0 + \theta), (\mathbb{T}Y)(t_0 + \theta)] = 0$. For each $t \geq t_0$, we have

$$\begin{aligned} &H_k\big[(\mathbb{T}X)(t), (\mathbb{T}Y)(t)\big] \\ &\quad = \sup_{t \geq t_0}\big\{H\big[(\mathbb{T}X)(t), (\mathbb{T}Y)(t)\big]\exp(-kt)\big\} \\ &\quad = \sup_{t \geq t_0}\left\{H\left[\varphi(0) \ominus (-1)\int_{t_0}^{t}\left(F(s, X_s) + \int_{t_0}^{s} G(s, \tau, X_\tau)\,d\tau\right)ds,\right.\right. \\ &\qquad \left.\left.\varphi(0) \ominus (-1)\int_{t_0}^{t}\left(F(s, Y_s) + \int_{t_0}^{s} G(s, \tau, Y_\tau)\,d\tau\right)ds\right]\exp(-kt)\right\} \\ &\quad \leq \sup_{t \geq t_0}\left\{\left(L\int_{t_0}^{t}\left(H_\sigma[X_s, Y_s] + \int_{t_0}^{s} H_\sigma[X_\tau, Y_\tau]\,d\tau\right)ds\right)\exp(-kt)\right\} \\ &\quad = \sup_{t \geq t_0}\left\{\left(L\int_{t_0}^{t}\sup_{\theta \in [-\sigma, 0]} H\big[X(s + \theta), Y(s + \theta)\big]\,ds\right.\right. \\ &\qquad \left.\left. + L\int_{t_0}^{t}\left(\int_{t_0}^{s}\sup_{\theta \in [-\sigma, 0]} H\big[X(\tau + \theta), Y(\tau + \theta)\big]\right)ds\right)\exp(-kt)\right\} \\ &\quad = \sup_{t \geq t_0}\left\{\left(L\int_{t_0}^{t}\sup_{r \in [s - \sigma, s]} H\big[X(r), Y(r)\big]\,dr\right.\right. \\ &\qquad \left.\left. + L\int_{t_0}^{t}\left(\int_{t_0}^{s}\sup_{\upsilon \in [\tau - \sigma, \tau]} H\big[X(\upsilon), Y(\upsilon)\big]\,d\upsilon\right)ds\right)\exp(-kt)\right\} \\ &\quad \leq LH_k[X, Y]\sup_{t \geq t_0}\left(\int_{t_0}^{t}\left(\exp\big(k(r - t)\big) + \int_{t_0}^{s}\exp\big(k(\upsilon - t)\big)\,d\upsilon\right)dr\right) \\ &\quad \leq \frac{(1 + k)LH_k[x, y]}{k^2}. \end{aligned}$$

Choosing $k > b$ and $\frac{(1+k)L}{k^2} < 1$, it follows that the operator $\mathbb{T}$ on S_k is a contraction. Using the Banach fixed point theorem provides the existence of a unique fixed point for $\mathbb{T}$, and the unique fixed point of $\mathbb{T}$ is in the space S_k, that is, a unique solution for (3.1) in the case of (ii)-differentiability. □

4 Illustrations

In this part, some simple examples are given to illustrate the theory of IFIDEs. We shall consider IFIDEs (3.1) with (i) and (ii) derivatives, respectively. Let us start the illustrations with considering the following interval-valued functional integro-differential equation:

$$\begin{cases} D_H^g X(t) = F(t, X_t) + \int_{t_0}^t k(t,s) X_s \, ds, & t \in J, \\ X(t) = \varphi(t - t_0), & t \in [-\sigma, t_0], \end{cases} \tag{4.1}$$

where $F : I \times C_\sigma \to K_C(\mathbb{R})$, $k(t,s) : I \times I \to \mathbb{R}$. Let $X(t) = [\underline{X}(t), \overline{X}(t)]$. By using Corollary 2.1, we have the following two cases.

If we consider the derivative of $X(t)$ by using (i)-differentiability, then from Corollary 2.1, we have $D_H^g X(t) = [\underline{X}'(t), \overline{X}'(t)]$ for $t \geq t_0$. Therefore, (4.1) is translated into the following delay integro-differential system:

$$\begin{cases} \underline{X}'(t) = \underline{F}(t, \underline{X}_t, \overline{X}_t) + \int_{t_0}^t \underline{k(t,s) X_s} \, ds, & t \geq t_0, \\ \overline{X}'(t) = \overline{F}(t, \underline{X}_t, \overline{X}_t) + \int_{t_0}^t \overline{k(t,s) X_s} \, ds, & t \geq t_0, \\ \underline{X}(t) = \underline{\varphi}(t - t_0), & \sigma \leq t \leq t_0, \\ \overline{X}(t) = \overline{\varphi}(t - t_0), & \sigma \leq t \leq t_0. \end{cases} \tag{4.2}$$

If we consider the derivative of $X(t)$ by using (ii)-differentiability, then from Corollary 2.1, we have $D_H^g X(t) = [\overline{X}'(t), \underline{X}'(t)]$ for $t \geq t_0$. Therefore, (4.1) is translated into the following delay integro-differential system:

$$\begin{cases} \overline{X}'(t) = \underline{F}(t, \underline{X}_t, \overline{X}_t) + \int_{t_0}^t \underline{k(t,s) X_s} \, ds, & t \geq t_0, \\ \underline{X}'(t) = \overline{F}(t, \underline{X}_t, \overline{X}_t) + \int_{t_0}^t \overline{k(t,s) X_s} \, ds, & t \geq t_0, \\ \underline{X}(t) = \underline{\varphi}(t - t_0), & \sigma \leq t \leq t_0, \\ \overline{X}(t) = \overline{\varphi}(t - t_0), & \sigma \leq t \leq t_0, \end{cases} \tag{4.3}$$

where

$$\underline{k(t,s) X_s} = \begin{cases} k(t,s) \underline{X}_s, & k(t,s) \geq 0, \\ k(t,s) \overline{X}_s, & k(t,s) < 0, \end{cases}$$

$$\overline{k(t,s) X_s} = \begin{cases} k(t,s) \overline{X}_s, & k(t,s) \geq 0, \\ k(t,s) \underline{X}_s, & k(t,s) < 0. \end{cases}$$

Remark 4.1 If we ensure that the solutions $(\underline{X}(t), \overline{X}(t))$ of systems (4.2) and (4.3) respectively are valid sets of interval-valued functions and if the derivatives $(\underline{X}'(t), \overline{X}'(t))$ are valid sets of interval-valued functions with two kinds of differentiability respectively, then we can construct the solution of interval-valued functional differential equation (4.1).

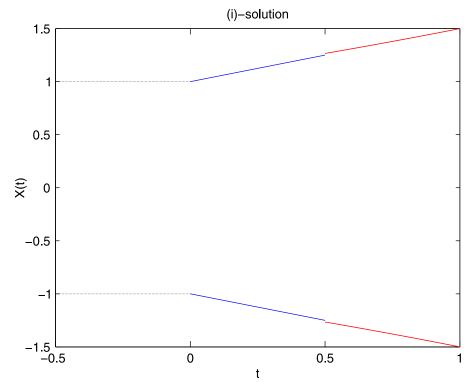
Next, we shall consider two examples being a simple illustration for the theory of interval-valued functional integro-differential equations.

Example 4.1 Let us consider the linear interval-valued functional integro-differential equation (with $k(t,s) \equiv 0$) under two kinds of Hukuhara derivatives

$$\begin{cases} D_H^g X(t) = -\lambda X(t - \frac{1}{2}), \\ X(t) = \varphi(t), & t \in [-\frac{1}{2}, 0], \end{cases} \tag{4.4}$$

where $\varphi(t) = [-1, 1]$, $\lambda > 0$. In this example we shall solve (4.4) on $[0, 1]$.

Figure 1 (i)-solution to (4.4) (λ = 0.5).



Case 1. Considering (i)-differentiability, problem (4.4) is translated into the following delay system:

$$\begin{cases} \underline{X}'(t) = -\lambda\overline{X}(t-\frac{1}{2}), & t \geq 0, \\ \overline{X}'(t) = -\lambda\underline{X}(t-\frac{1}{2}), & t \geq 0, \\ \underline{X}(t) = -1, & -\frac{1}{2} \leq t \leq 0, \\ \overline{X}(t) = 1, & -\frac{1}{2} \leq t \leq 0. \end{cases} \tag{4.5}$$

Solving delay system (4.5) by using the method of steps, we obtain a unique (i)-solution to (4.4) defined on $[0,1]$ and it is of the form

$$X(t) = \begin{cases} [-1(1+\lambda t), (1+\lambda t)] & \text{for } t \in [0, \frac{1}{2}], \\ [-1(1+\lambda t + \frac{\lambda^2(t-1)^2}{2}), 1+\lambda t + \frac{\lambda^2(t-1)^2}{2}] & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

The (i)-solution is illustrated in Figure 1.

Case 2. Considering (ii)-differentiability, problem (4.4) is translated into the following delay system:

$$\begin{cases} \overline{X}'(t) = -\lambda\overline{X}(t-\frac{1}{2}), & t \geq 0, \\ \underline{X}'(t) = -\lambda\underline{X}(t-\frac{1}{2}), & t \geq 0, \\ \underline{X}(t) = -1, & -\frac{1}{2} \leq t \leq 0, \\ \overline{X}(t) = 1, & -\frac{1}{2} \leq t \leq 0. \end{cases} \tag{4.6}$$

We obtain a unique (ii)-solution to (4.4) defined on $[0,1]$ and it is of the form

$$X(t) = \begin{cases} [\lambda t - 1, 1 - \lambda t] & \text{for } t \in [0, \frac{1}{2}], \\ [-1(1-\lambda t + \frac{\lambda^2(t-1)^2}{2}), 1-\lambda t + \frac{\lambda^2(t-1)^2}{2}] & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

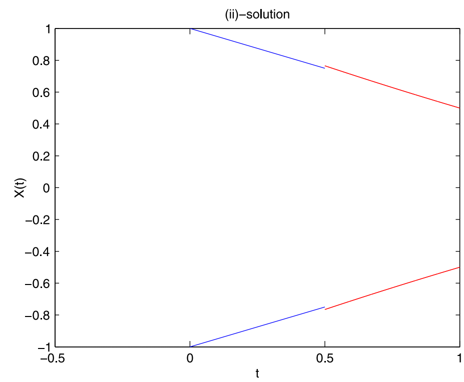
The (ii)-solution is illustrated in Figure 2.

Example 4.2 Let us consider the linear interval-valued functional integro-differential equation under two kinds of Hukuhara derivatives

$$\begin{cases} D_H^g X(t) = X(t-\frac{1}{2}) + \alpha \int_{t_0}^t e^{(s-t)} X(s-\frac{1}{2})\,ds, \\ X(t) = \varphi(t), \quad t \in [-\frac{1}{2}, 0], \end{cases} \tag{4.7}$$

where $\varphi(t) = [1, 2-t]$, $\alpha \in \mathbb{R}\backslash\{0\}$. In this example we shall solve (4.7) on $[0,1/2]$.

Figure 2 (ii)-solution to (4.4) (λ = 0.5).



Case 1. ($\alpha > 0$) From (4.2), we get

$$\begin{cases} \underline{X}'(t) = \underline{X}(t-\frac{1}{2}) + \alpha \int_0^t e^{(s-t)}\underline{X}(s-\frac{1}{2})\,ds, & t \geq 0, \\ \overline{X}'(t) = \overline{X}(t-\frac{1}{2}) + \alpha \int_0^t e^{(s-t)}\overline{X}(s-\frac{1}{2})\,ds, & t \geq 0, \\ \underline{X}(t) = 1, & \frac{-1}{2} \leq t \leq 0, \\ \overline{X}(t) = 2-t, & \frac{-1}{2} \leq t \leq 0. \end{cases} \tag{4.8}$$

Following the method of steps, we obtain the (i)-solution to (4.7) defined on $[0,1/2]$ and it is of the form

$$X(t) = \left[1 + \frac{t}{2} + \frac{\alpha}{2}\left(1 - e^{-t}\right), 2 + 2t - \frac{t^2}{2} + \alpha\left(3 - t - 3e^{-t}\right)\right], \quad t \in [0,1/2].$$

From (4.3) we obtain

$$\begin{cases} \overline{X}'(t) = \underline{X}(t-\frac{1}{2}) + \alpha \int_0^t e^{(s-t)}\underline{X}(s-\frac{1}{2})\,ds, & t \geq 0, \\ \underline{X}'(t) = \overline{X}(t-\frac{1}{2}) + \alpha \int_0^t e^{(s-t)}\overline{X}(s-\frac{1}{2})\,ds, & t \geq 0, \\ \underline{X}(t) = 1, & \frac{-1}{2} \leq t \leq 0, \\ \overline{X}(t) = 2-t, & \frac{-1}{2} \leq t \leq 0. \end{cases} \tag{4.9}$$

The (ii)-solution to (4.7) defined on $[0,1/2]$ is of the form

$$X(t) = \left[1 + 2t - \frac{t^2}{2} + \alpha\left(3 - t - 3e^{-t}\right), 2 + \frac{\alpha}{2}\left(1 - e^{-t}\right)\right], \quad t \in [0,1/2].$$

In Figures 3 and 4, (i)-solution and (ii)-solution curves of (4.7) are given.

Case 2. ($\alpha < 0$) From (4.2) we get

$$\begin{cases} \underline{X}'(t) = \underline{X}(t-\frac{1}{2}) + \alpha \int_0^t e^{(s-t)}\overline{X}(s-\frac{1}{2})\,ds, & t \geq 0, \\ \overline{X}'(t) = \overline{X}(t-\frac{1}{2}) + \alpha \int_0^t e^{(s-t)}\underline{X}(s-\frac{1}{2})\,ds, & t \geq 0, \\ \underline{X}(t) = 1, & \frac{-1}{2} \leq t \leq 0, \\ \overline{X}(t) = 2-t, & \frac{-1}{2} \leq t \leq 0. \end{cases} \tag{4.10}$$

By solving delay integro-differential system (4.10), we obtain (i)-solution

$$X(t) = \left[1 + \frac{t}{2} + \alpha\left(3 - t - 3e^{-t}\right), 2 + 2t - \frac{t^2}{2} + \frac{\alpha}{2}\left(1 - e^{-t}\right)\right], \quad t \in [0,1/2].$$

The (i)-solution of (4.7) on $[-1/2,1/2]$ is illustrated in Figure 5.

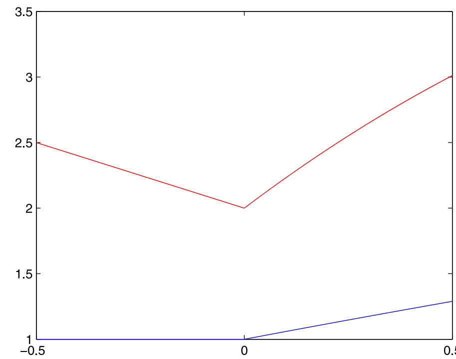


Figure 3 Graphs of $\underline{X}(t), \overline{X}(t)$ for $t \in [\frac{-1}{2}, \frac{1}{2}]$, $\alpha = 0.1$.

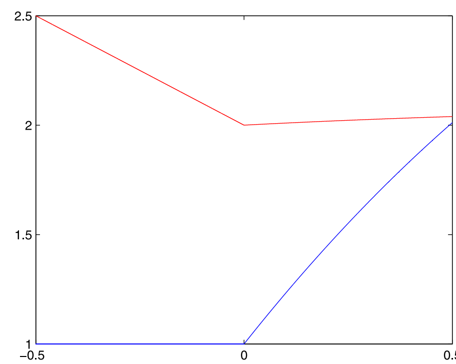


Figure 4 Graphs of $\underline{X}(t), \overline{X}(t)$ for $t \in [\frac{-1}{2}, \frac{1}{2}]$, $\alpha = 0.1$.

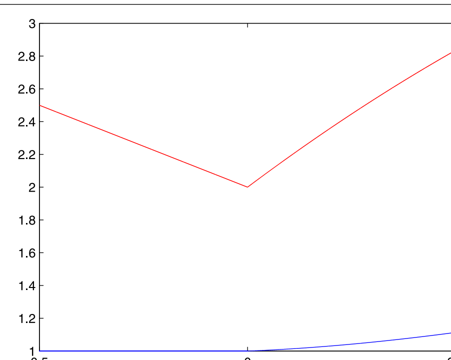


Figure 5 Graphs of $\underline{X}(t), \overline{X}(t)$ for $t \in [\frac{-1}{2}, \frac{1}{2}]$, $\alpha = -0.1$.

From (4.3) we obtain

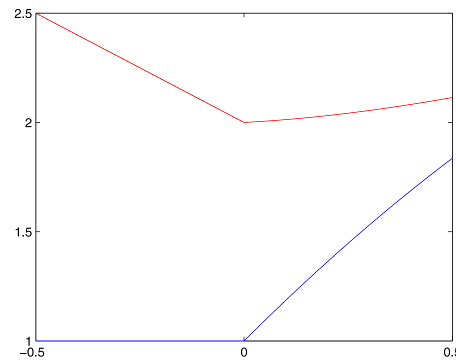
$$\begin{cases} \overline{X}'(t) = \underline{X}(t - \frac{1}{2}) + \alpha \int_0^t e^{(s-t)} \overline{X}(s - \frac{1}{2})\, ds, & t \geq 0, \\ \underline{X}'(t) = \overline{X}(t - \frac{1}{2}) + \alpha \int_0^t e^{(s-t)} \underline{X}(s - \frac{1}{2})\, ds, & t \geq 0, \\ \underline{X}(t) = 1, \quad \frac{-1}{2} \leq t \leq 0, \\ \overline{X}(t) = 2 - t, \quad \frac{-1}{2} \leq t \leq 0. \end{cases} \tag{4.11}$$

By solving delay integro-differential systems (4.11), we obtain (ii)-solution

$$X(t) = \left[1 + 2t - \frac{t^2}{2} + \frac{\alpha}{2}(1 - e^{-t}), 2 + \frac{t}{2} + \alpha\left(3 - t - 3e^{-t}\right)\right], \quad t \in [0, 1/2].$$

The (ii)-solution of (4.7) on $[-1/2, 1/2]$ is illustrated in Figure 6.

Figure 6 Graphs of $\underline{X}(t)$, $\overline{X}(t)$ for $t \in [\frac{-1}{2}, \frac{1}{2}]$, $\alpha = -0.1$.



5 Conclusion

In this study, we have established the local and global existence and uniqueness results of two solutions for interval-valued functional integro-differential equations. For the local existence and uniqueness, we use the method of successive approximations under the Lipschitz condition, and for global existence and uniqueness, we use the contraction principle under suitable conditions. In our further work, we would like to use these results to study the local and global existence and uniqueness results of solutions for interval-valued functional integro-differential equations under Caputo-type interval-valued fractional derivatives.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final version of the manuscript.

Author details
[1]Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam. [2]Faculty of Mathematics and Computer Science, University of Science, VNU, Ho Chi Minh City, Vietnam. [3]Faculty of Natural Science and Technology, Tay Nguyen University, 567 Le Duan Road, Buon Ma Thuot City, Daklak Province, Vietnam.

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