Candan *Advances in Difference Equations* 2014, 2014:163
http://www.advancesindifferenceequations.com/content/2014/1/163

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Domain of the double sequential band matrix in the spaces of convergent and null sequences

Murat Candan*

*Correspondence:
murat.candan@inonu.edu.tr
Department of Mathematics, Faculty of Arts and Sciences, İnönü University, The University Campus, Malatya, 44280, Turkey

Abstract
What stands out in this article is the sequence spaces of a new brand $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$, derived by using a double sequential band matrix $B(\tilde{r},\tilde{s})$ which generalizes the previous work of Sönmez and Başar (Abstr. Appl. Anal. 2012:435076, 2012), where $(r_n)_{n=0}^{\infty}$ and $(s_n)_{n=0}^{\infty}$ are given convergent sequences of positive real numbers. The aforementioned spaces are in fact the BK-spaces of non-absolute type. Moreover, they are norm isomorphic to the spaces c_0 and c, respectively. Then, some inclusion relations are derived to determine the α-, β- and γ-duals of these spaces. Next, their Schauder bases are constructed. In conclusion, some matrix classes from the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ to the spaces ℓ_p, c_0 and c are characterized. When compared with the corresponding results in the literature, it is seen that the results of the present study are more general and more inclusive.
MSC: 46A45; 40C05

Keywords: sequence spaces; Schauder basis; α-, β- and γ-duals; matrix transformations

1 Fundamental facts

There are many ways to construct new sequence spaces from old ones. In recent years the construction of a new sequence space by means of the domain of triangle matrix has been used by some of the researchers in many scientific articles. Purely for the development of this approach, the very readable book of Başar [1] is recommended especially for interesting historical developments. Let us start here with a definition of just what a sequence is. There is a variety of ways to define a sequence, each of which is an equivalent way of defining the same thing. Instead, we prefer the following definition. A sequence can easily be described as an ordered list of numbers. Although these lists may or may not include infinite number of terms, here we will exclusively deal with those consisting of infinite number of terms. A sequence can be described as a function having a domain $\{k_0, k_0+1, k_0+2, \ldots\}$ assuming values in $\mathbb{R}$ or $\mathbb{C}$, here k_0 is any given integer, mostly $k_0 = 0$ or 1. Usually, subscript notation is used and (x_n) is written instead of $x(n)$. A sequence (x_n) converges to limit a if each neighborhood of a contains almost all terms of the sequence. In this case we say that (x_n) converges to a as n goes to ∞. We denote by c, the set of all convergent sequences in $\mathbb{K}$, where $\mathbb{K}$ denotes either of fields $\mathbb{R}$ and $\mathbb{C}$. A sequence (x_n) in $\mathbb{K}$ is called a null sequence if it converges to zero. We denote the set of all null sequences in $\mathbb{K}$ by c_0.

A sequence is bounded if the set of its terms is bounded. The set of all bounded sequences is denoted by ℓ_∞. Any vector subspace of $\omega = \omega(\mathbb{K}) = \mathbb{K}^{\mathbb{N}}$ is known as a sequence space. It is clear that the sets c, c_0 and ℓ_∞ are the subspaces of the ω. Therefore, c, c_0 and ℓ_∞, equipped with a vector space structure, form a sequence space. Also by bs, cs, ℓ_1 and ℓ_p we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively. As is well known, we call a sequence space X with a linear topology a K-space if and only if each of the maps $p_n : X \to \mathbb{R}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$. A K-space X is called an FK-space if and only if X is a complete linear metric space. In other words, we can say that an FK-space is a complete total paranormed space. Note here that some discussion of FK-spaces is given in [2]. An FK-space whose topology is normable is called a BK-space or a Banach coordinate space, so a BK-space is a normed FK-space. The space ℓ_p $(1 \leq p < \infty)$ is a BK-space with $\|x\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}$ and c_0, c and ℓ_∞ are BK-spaces with $\|x\|_\infty = \sup_k |x_k|$. An FK-space X is said to have the AK property if $\phi \subset X$ and $\{e^{(k)}\}$ is a basis for X, where e^k is a sequence whose only non-zero term is a 1 in kth place for each $k \in \mathbb{N}$ and $\phi = \operatorname{span}\{e^k\}$, the set of all finitely non-zero sequences. If ϕ is dense in X, then X is called an AD-space, thus AK implies AD. We know that the spaces c_0, cs and ℓ_p are AK-spaces, where $1 \leq p < \infty$. In addition to this, by $\mathcal{F}$ we denote the collection consisting of all non-empty and finite subsets of $\mathbb{N}$.

Another notion we need is that of matrix transformation. For this reason, in this paragraph, we shall be concerned with matrix transformation from a sequence space X to a sequence space Y. Given any infinite matrix $A = (a_{nk})$ of real numbers a_{nk}, where $n, k \in \mathbb{N}$, any sequence x, we write $Ax = ((Ax)_n)$, the A-transform of x, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. If $x \in X$ implies that $Ax \in Y$, then we say that A defines a matrix mapping from X into Y and denote it by $A : X \to Y$. By $(X : Y)$ we mean the class of all infinite matrices such that $A : X \to Y$.

The matrix domain has fundamental importance for this article. Therefore, the concept is presented in this paragraph. The λ_A is said to be matrix domain of an infinite matrix A for any subspace λ of the all real-valued sequence space $w(\mathbb{R})$ and is described as

$$\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}. \tag{1.1}$$

The new sequence space λ_A generated by the limitation matrix A from the space λ either includes the space λ or is included by the space λ, in general, *i.e.*, the space λ_A is the expansion or the contraction of the original space λ.

In order to establish a new brand sequence space, a triangle matrix was previously used. To obtain detailed information, one must search the articles [3–20]. These references will reflect the fact.

The layout of the rest of the present paper is as follows. At the beginning of Section 2, essential fundamental concepts and some historical materials are given; also the sequence spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ are introduced and they are proved to be linearly isomorphic to the sequence spaces c_0 and c, respectively. The goal of Section 3 is to derive some inclusion relations between them (the new defined spaces above). In Sections 4 and 5, the Schauder bases of the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ are obtained and the α-, β- and γ-duals of their generalizations (the generalized difference sequence spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ of non-absolute type) are determined, respectively. In Section 6, we characterize the matrix classes $(c^\lambda(\widetilde{B}) : \ell_p)$,

$(c_0^\lambda(\widetilde{B}) : \ell_p)$, $(c^\lambda(\widetilde{B}) : c)$, $(c^\lambda(\widetilde{B}) : c_0)$, $(c_0^\lambda(\widetilde{B}) : c)$ and $(c_0^\lambda(\widetilde{B}) : c_0)$, where $1 \leq p \leq \infty$. We also derive the properties of some other classes including Euler, difference, Riesz and Cesàro sequence spaces, using some old results. In the last section of the text, we note the significance of the present results in the literature related with difference sequence spaces and record some further suggestions.

2 The difference sequence spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ of non-absolute type

The difference sequence spaces are shortly analyzed here and we introduce sequence spaces both $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$, and show that these spaces are BK-spaces of non-absolute type $c^\lambda(\widetilde{B})$ and they are proved to be norm isomorphic to the well-known sequence spaces c_0 and c, respectively. For historical developments related to this approach, we must refer the reader to the articles [5, 10, 14, 16, 20] studied by many authors. We note here that research into this field is continuing.

From now on, let us assume that $\lambda = (\lambda_k)_{k=0}^\infty$ is a strictly increasing sequence of positive reals tending to infinity; in other words,

$$0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Here and after, we use the convention that any term with a negative subscript is equal to zero, *e.g.*, $\lambda_{-1} = 0$ and $x_{-1} = 0$.

Recently, Mursaleen and Noman [15] studied the sequence spaces c_0^λ and c^λ of non-absolute type, and later they introduced the difference sequence spaces $c_0^\lambda(\Delta)$ and $c^\lambda(\Delta)$ in [16] of non-absolute type. With the help of (1.1) the spaces $c_0^\lambda(\Delta)$ and $c^\lambda(\Delta)$ can be rewritten as follows: $c_0^\lambda(\Delta) = (c_0^\lambda)_\Delta$ and $c^\lambda(\Delta) = (c^\lambda)_\Delta$; respectively, where Δ denotes the band matrix representing the difference operator, *i.e.*, $\Delta x = (x_k - x_{k-1}) \in \omega$ for $x = (x_k) \in \omega$.

Let r and s be non-zero real numbers, and define the generalized difference matrix $B(r,s) = \{b_{nk}(r,s)\}$ by

$$b_{nk}(r,s) := \begin{cases} r, & k = n, \\ s, & k = n-1, \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

for all $k, n \in \mathbb{N}$. The $B(r,s)$-transform of a sequence $x = (x_k)$ is $B(r,s)(x) = rx_k + sx_{k-1}$ for all $k \in \mathbb{N}$. Now, we proceed slightly differently to Kızmaz [10] and the other authors following him, and employ a technique of obtaining a new sequence space by means of the matrix domain of a triangle limitation method.

Recently, Sönmez and Başar [17] have introduced the difference sequence spaces $c_0^\lambda(B)$ and $c^\lambda(B)$, which are the generalizations of the spaces $c_0^\lambda(\Delta)$ and $c^\lambda(\Delta)$ introduced by Mursaleen and Noman [16]. Again, the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ can be written as $c_0^\lambda(B) = (c_0^\lambda)_B$ and $c^\lambda(B) = (c^\lambda)_B$ using (1.1), where B denotes the generalized difference matrix $B(r,s) = \{b_{nk}(r,s)\}$ defined by (2.1).

Let $\tilde{r} = (r_n)_{n=0}^\infty$ and $\tilde{s} = (s_n)_{n=0}^\infty$ be given convergent sequences of positive real numbers. Define the sequential generalized difference matrix $B(\tilde{r},\tilde{s}) = \{b_{nk}(\tilde{r},\tilde{s})\}$ by

$$b_{nk}(\tilde{r},\tilde{s}) := \begin{cases} r_n, & k = n, \\ s_n, & k = n-1, \\ 0, & 0 \leq k < n-1 \text{ or } k > n \end{cases} \tag{2.2}$$

for all $k, n \in \mathbb{N}$, the set of natural numbers. We should record here that the matrix $B(\tilde{r}, \tilde{s})$ can be reduced to the generalized difference matrix $B(r, s)$ in the case $r_n = r$ and $s_n = s$ for all $n \in \mathbb{N}$. These choices are possible by the definition of sequential band matrix $B(\tilde{r}, \tilde{s})$. So, the results related to the matrix domain of the matrix $B(\tilde{r}, \tilde{s})$ are more general and more comprehensive than the corresponding consequences of the matrix domain of $B(r, s)$, and we include them.

We thus introduce the difference sequence spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$, which are the generalizations of the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ introduced by Sönmez and Başar [17], as follows:

$$c_0^\lambda(\widetilde{B}) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r_k x_k + s_{k-1} x_{k-1}) = 0 \right\},$$

$$c^\lambda(\widetilde{B}) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r_k x_k + s_{k-1} x_{k-1}) \text{ exists} \right\}.$$

With the notation of (1.1), we can redefine the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ as $c_0^\lambda(\widetilde{B}) = (c_0^\lambda)_{\widetilde{B}}$ and $c^\lambda(\widetilde{B}) = (c^\lambda)_{\widetilde{B}}$, where $\widetilde{B}$ denotes the sequential band matrix $B(\tilde{r}, \tilde{s}) = \{b_{nk}(\tilde{r}, \tilde{s})\}$ defined by (2.2).

Let us begin with the theorem which is one of our principal objects of study.

Theorem 2.1 *The sets $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ are linear spaces together with coordinatewise addition and scalar multiplication; in other words, $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ represent the sequence spaces of generalized differences.*

Proof This result should also be fairly apparent. □

Let us return to explaining our main subject. In the other way around, the triangle matrix $\widetilde{\Lambda} = (\tilde{\lambda}_{nk})$ is defined by

$$\tilde{\lambda}_{nk} := \begin{cases} \frac{r_k(\lambda_k - \lambda_{k-1}) + s_k(\lambda_{k+1} - \lambda_k)}{\lambda_n}, & k < n, \\ r_n \frac{(\lambda_n - \lambda_{n-1})}{\lambda_n}, & k = n, \\ 0, & k > n \end{cases} \tag{2.3}$$

for all $n, k \in \mathbb{N}$. Using a simple calculus, we can derive the following equality:

$$(\widetilde{\Lambda} x)_n = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r_k x_k + s_{k-1} x_{k-1}) \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

For every $x = (x_k) \in \omega$ and with (1.1) we can conclude that $c_0^\lambda(\widetilde{B}) = (c_0)_{\widetilde{\Lambda}}$ and $c^\lambda(\widetilde{B}) = c_{\widetilde{\Lambda}}$ hold.

Moreover, we describe the sequence $y(\lambda) = \{y_k(\lambda)\}$ for each sequence $x = (x_k)$ and use it frequently in the future, as the $\widetilde{\Lambda}$-transform of x, that is, $y(\lambda) = \widetilde{\Lambda} x$ and also we get

$$y_k(\lambda) = \sum_{j=0}^{k-1} \frac{r_j(\lambda_j - \lambda_{j-1}) + s_j(\lambda_{j+1} - \lambda j)}{\lambda_k} x_j + r_k \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} x_k \quad \text{for all } k \in \mathbb{N}. \tag{2.5}$$

From now on, the summation in the range of 0 to $k-1$ will be equal to zero when $k = 0$.

Besides, equations of (2.4) and (2.5) give us a clue as to how to rewrite the following:

$$y_k(\lambda) = \frac{1}{\lambda_k}\sum_{j=0}^{k}(\lambda_j - \lambda_{j-1})(r_j x_j + s_{j-1}x_{j-1}) \quad \text{for all } k \in \mathbb{N}.$$

It is remarkable that the sequences $x = (x_k)$ and $y = (y_k)$ are connected by relation (2.5) everywhere in the paper.

Now, we will provide a complete proof for some of results obtained in this and the following sections so that the reader may be familiar with the ways the proofs are constructed and written. There are two fundamental theorems which help us. Let us now state the first one.

Theorem 2.2 *The difference sequence spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ are BK-spaces having the norm $\|x\|_{c_0^\lambda(\widetilde{B})} = \|x\|_{c^\lambda(\widetilde{B})} = \|\widetilde{\Lambda}x\|_\infty$; in other words, $\|x\|_{c_0^\lambda(\widetilde{B})} = \|x\|_{c^\lambda(\widetilde{B})} = \sup_{n\in\mathbb{N}} |(\widetilde{\Lambda}x)_n|$.*

Proof It is well known from previous arguments that c_0 and c are BK-spaces with respect to their natural norms and the matrix $\widetilde{\Lambda}$ is a triangle. For this reason, the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ are BK-spaces with the given norms. This, in fact, concludes the proof. □

Remark 2.3 It can easily be controlled that the absolute property is invalid on the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$; in other words, $\|x\|_{c_0^\lambda(\widetilde{B})} \neq \||x|\|_{c_0^\lambda(\widetilde{B})}$ and $\|x\|_{c^\lambda(\widetilde{B})} \neq \||x|\|_{c^\lambda(\widetilde{B})}$ for at least one sequence found in the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$. Thus, we can say that $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ are the sequence spaces of non-absolute type, in which $|x| = (|x_k|)$.

Here, let us give the definition of isomorphism. A bijective linear transformation $\tau : X \to Y$ is called an isomorphism from X to Y. When an isomorphism from Y to X exists, we say that X to Y are isomorphic and write $X \approx Y$.

It is time to give another very useful result for new difference sequence spaces defined above.

Theorem 2.4 *The newly defined non-absolute type sequence spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ are norm isomorphic to the well-known spaces c_0 and c, respectively; in other words, $c_0^\lambda(\widetilde{B}) \cong c_0$ and $c^\lambda(\widetilde{B}) \cong c$.*

Proof To start with this proof, a certain amount of linear algebra will be used. Showing the existence of a linear bijection between the spaces $c_0^\lambda(\widetilde{B})$ and c_0 proves the theorem. The transformation τ from $c_0^\lambda(\widetilde{B})$ to c_0 is defined by $x \mapsto y(\lambda)$, using the notation of (2.5). Then, $\tau x = y(\lambda) = \widetilde{\Lambda}x \in c_0$ for every $x \in c_0^\lambda(\widetilde{B})$ and it is routine to show that τ is linear. Further, it is obvious that $x = \theta$ whenever $\tau x = \theta$, which shows that τ is injective.

Let $y = (y_k) \in c_0$ and define the sequence $x = \{x_k(\lambda)\}$ by

$$x_k(\lambda) := \frac{1}{r_k}\sum_{j=0}^{k}\prod_{i=j}^{k-1}\left(-\frac{s_i}{r_i}\right)\sum_{i=j-1}^{j}(-1)^{j-i}\frac{\lambda_i}{\lambda_j - \lambda_{j-1}}y_i \quad \text{for all } k \in \mathbb{N}. \tag{2.6}$$

Clearly,

$$r_k x_k(\lambda) + s_{k-1}x_{k-1}(\lambda) = \sum_{i=k-1}^{k}(-1)^{k-i}\frac{\lambda_i}{\lambda_k - \lambda_{k-1}}y_i \quad \text{for all } k \in \mathbb{N}.$$

Let us make the following computation. We have by (2.4)

$$\begin{aligned}(\widetilde{\Lambda}x)_n &= \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(r_k x_k + s_{k-1}x_{k-1}) \\ &= \frac{1}{\lambda_n}\sum_{k=0}^{n}\sum_{i=k-1}^{k}(-1)^{k-i}\lambda_i y_i \\ &= y_n\end{aligned}$$

for every $n \in \mathbb{N}$. Thus, we have that $\widetilde{\Lambda}x = y$ and since $y \in c_0$, we conclude that $\widetilde{\Lambda}x \in c_0$. Thus, we deduce that $x \in c_0^\lambda(\widetilde{B})$ and $Tx = y$. Hence, T is surjective.

Moreover, one can easily see for every $x \in c_0^\lambda(\widetilde{B})$ that $\|Tx\|_\infty = \|y(\lambda)\|_\infty = \|\widetilde{\Lambda}x\|_\infty = \|x\|_{c_0^\lambda(\widetilde{B})}$, which means that T is norm preserving. Consequently, T is a linear bijection which shows that the spaces $c_0^\lambda(\widetilde{B})$ and c_0 are linearly isomorphic.

It is clear that if the spaces $c_0^\lambda(\widetilde{B})$ and c_0 are replaced by the spaces $c^\lambda(\widetilde{B})$ and c, respectively, then we obtain the fact $c^\lambda(\widetilde{B}) \cong c$, which proves our assertion. □

3 Some inclusion relations

In this section, we shall talk about several inclusion relations concerning the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$. The following theorems give some basic algebraic properties of the difference sequence spaces mentioned above.

Theorem 3.1 *The inclusion $c_0^\lambda(\widetilde{B}) \subset c^\lambda(\widetilde{B})$ is strictly valid.*

Proof This proof of the theorem is fairly standard, so we must find an element which belongs to $c^\lambda(\widetilde{B})$ but which does not belong to $c_0^\lambda(\widetilde{B})$. Clearly, the inclusion $c_0^\lambda(\widetilde{B}) \subset c^\lambda(\widetilde{B})$ is valid. Let us illustrate that this inclusion is strict. To do this, consider the sequence $x = (x_k)$ given by $x_k = \frac{1}{r_k}[1 + \sum_{j=0}^{k}\prod_{i=j}^{k-1}(\frac{-s_i}{r_i})]$ for all $k \in \mathbb{N}$. Together with (2.4), we now have $(\widetilde{\Lambda}x)_n = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})$ for all $n \in \mathbb{N}$. Briefly, this tells us that $\widetilde{\Lambda}x = e$, and therefore $\widetilde{\Lambda}x \in c \setminus c_0$, where $e = (1,1,1,\ldots)$. In other words, the sequence x lies in $c^\lambda(\widetilde{B})$; however, it does not lie in $c_0^\lambda(\widetilde{B})$. That is, the inclusion $c_0^\lambda(\widetilde{B}) \subset c^\lambda(\widetilde{B})$ is strictly valid, so the claim is proved. □

Theorem 3.2 *The inclusion $c \subset c_0^\lambda(\widetilde{B})$ is strict when $s_{k-1} + r_k = 0$.*

Proof Let us assume that $s_{k-1} + r_k = 0$ and $x \in c$. In that case $B(\tilde{r},\tilde{s})x = (r_k x_k + s_{k-1}x_{k-1}) \in c_0$ and so $B(\tilde{r},\tilde{s})x \in c_0^\lambda$ due to the inclusion $c_0 \subset c_0^\lambda$. It is clear that $x \in c_0^\lambda(\widetilde{B})$. Because of this, the inclusion $c \subset c_0^\lambda(\widetilde{B})$ is valid. In addition to this, let us take the sequence $y = (y_k)$ given by $y_k = \sqrt{k+1}$ for all $k \in \mathbb{N}$. So, it is not hard to see that $y \notin c$. Then it can be obtained that $B(\tilde{r},\tilde{s})y \in c_0$. That is, $B(\tilde{r},\tilde{s})y \in c_0^\lambda$, which means that $y \in c_0^\lambda(\widetilde{B})$. Therefore, the sequence y is in $c_0^\lambda(\widetilde{B})$ but not in c. Consequently, the inclusion $c \subset c_0^\lambda(\widetilde{B})$ is strict. This marks the end of the proof. □

We should state here that it can be remembered that if $A \in (c:c)$ and $B \in (c:c)$, then $AB \in (c:c)$; in other words, $\widetilde{\Lambda} = (\tilde{\lambda}_{nk})$ is stronger than the ordinary convergence. Therefore we get the following.

Corollary 3.3 *The inclusions $c_0 \subset c_0^\lambda(\widetilde{B})$ and $c \subset c^\lambda(\widetilde{B})$ are strictly valid.*

It can easily be seen that the sequence y defined in the proof of Theorem 3.2 lies in $c_0^\lambda(\widetilde{B})$ but not in ℓ_∞. This motivates the following result.

Corollary 3.4 *The space ℓ_∞ does not include the space $c_0^\lambda(\widetilde{B})$ even though the spaces ℓ_∞ and $c_0^\lambda(\widetilde{B})$ overlap.*

In order to prove the theorem below, the following lemma [21, p.4] will be used.

Lemma 3.5 *$A \in (\ell_\infty : c_0)$ if and only if $\lim_{n\to\infty} \sum_k |a_{nk}| = 0$.*

Theorem 3.6 *The inclusion $\ell_\infty \subset c_0^\lambda(\widetilde{B})$ is strictly valid iff $z \in c_0^\lambda$, where the sequence $z = (z_k)$ is described by*

$$z_k = \left| \frac{r_k(\lambda_k - \lambda_{k-1}) + s_k(\lambda_{k+1} - \lambda_k)}{\lambda_k - \lambda_{k-1}} \right| \quad \textit{for all } k \in \mathbb{N}.$$

Proof Let the inclusion $\ell_\infty \subset c_0^\lambda(\widetilde{B})$ strictly hold. In this case $\widetilde{\Lambda}x \in c_0$ for every $x \in \ell_\infty$, and it follows that the matrix $\widetilde{\Lambda} = (\tilde{\lambda}_{nk})$ is in the class $(\ell_\infty : c_0)$. Therefore, by using Lemma 3.5 we have the following limit:

$$\lim_{n\to\infty} \sum_k |\tilde{\lambda}_{nk}| = 0. \tag{3.1}$$

Now, by joining the matrix $\widetilde{\Lambda} = (\tilde{\lambda}_{nk})$ given by (2.3), we can easily obtain that

$$\sum_k |\tilde{\lambda}_{nk}| = \frac{1}{\lambda_n} \sum_{k=0}^{n-1} \bigl| r_k(\lambda_k - \lambda_{k-1}) + s_k(\lambda_{k+1} - \lambda_k) \bigr| + |r_n| \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}, \tag{3.2}$$

$n \in \mathbb{N}$. In addition, if we consider equality (3.1), it gives us the following limits:

$$\lim_{n\to\infty} |r_n| \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} = 0 \tag{3.3}$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n-1} \bigl| r_k(\lambda_k - \lambda_{k-1}) + s_k(\lambda_{k+1} - \lambda_k) \bigr| = 0. \tag{3.4}$$

Now, we can write the following formula with a simple calculation:

$$\frac{1}{\lambda_n} \sum_{k=0}^{n-1} \bigl| r_k(\lambda_k - \lambda_{k-1}) + s_k(\lambda_{k+1} - \lambda_k) \bigr| = \frac{\lambda_{n-1}}{\lambda_n} \left[\frac{1}{\lambda_{n-1}} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) z_k \right] \tag{3.5}$$

for every $n \geq 1$ due to the fact that $\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} = 1$ by (3.3). By passing to limit as $n \to \infty$ in (3.5), it is easy to see that together with (3.4)

$$\lim_{n\to\infty} \frac{1}{\lambda_{n-1}} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) z_k = 0. \tag{3.6}$$

This clearly indicates that $z = (z_k) \in c_0^\lambda$.

To prove the converse, let us assume that $z = (z_k) \in c_0^\lambda$. In this condition, it is obvious that (3.6) holds. Moreover, we can easily obtain

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k=0}^{n-1}\bigl|r_k(\lambda_k-\lambda_{k-1})+s_k(\lambda_{k+1}-\lambda_k)\bigr| &= \frac{1}{\lambda_n}\sum_{k=0}^{n-1}(\lambda_k-\lambda_{k-1})z_k \\ &\leq \frac{1}{\lambda_{n-1}}\sum_{k=0}^{n-1}(\lambda_k-\lambda_{k-1})z_k \qquad (3.7)\end{aligned}$$

for every $n \geq 1$. If we consider equality (3.7) with inequality (3.6), then the results in condition (3.4) hold. In the other way around, if we apply the triangle inequality, then we obtain the following revised form:

$$\left|\frac{1}{\lambda_n}\sum_{k=0}^{n-1}r_k(\lambda_k-\lambda_{k-1})-s_k(\lambda_k-\lambda_{k+1})\right| \leq \frac{1}{\lambda_n}\sum_{k=0}^{n-1}\bigl|r_k(\lambda_k-\lambda_{k-1})-s_k(\lambda_k-\lambda_{k+1})\bigr|$$

for every $n \geq 1$. Then this inequality gives $\lim_{n\to\infty}[\sum_{k=0}^{n-1} r_k(\lambda_k-\lambda_{k-1})+s_k(\lambda_{k+1}-\lambda_k)]/\lambda_n = 0$ with the aid of (3.4). Especially, if we take $r_k = 1$ and $s_k = 1$ for all $k \in \mathbb{N}$, then it is obviously seen that $\lim_{n\to\infty}[\lambda_n - \lambda_{n-1} - \lambda_0]/\lambda_n = 0$, which means that (3.3) holds. Thus, one can easily deduce by equality (3.2) that (3.1) holds. From Lemma 3.5 it can be obtained that $\widetilde{\Lambda} \in (\ell_\infty : c_0)$. Therefore, it is not hard to see that the inclusion $\ell_\infty \subset c_0^\lambda(\widetilde{B})$ strictly holds by using Corollary 3.4. In fact, this is exactly what we want to prove. □

4 The bases for the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$

In this section, we give two sequences of the points of the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ forming the bases for those spaces.

The concept of convergence of a series can be used to define a basis as follows. Let $(X, \|\cdot\|_X)$ be a normed space. Then the sequence (b_n) in X is called a Schauder basis for X if for every $x \in X$ there exists a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty}\bigl\|x-(\alpha_0 b_0+\alpha_1 b_1+\cdots+\alpha_n b_n)\bigr\|_X = 0.$$

In this case, the series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and is written as $x = \sum_k \alpha_k b_k$.

Because of the fact that the transformation T defined from $c_0^\lambda(\widetilde{B})$ to c_0 in the proof of Theorem 2.4 is an isomorphism, the inverse image of the basis $\{e^{(k)}\}_{k=0}^\infty$ of the space c_0 is the basis for the newly defined space $c_0^\lambda(\widetilde{B})$. Thus, the subsequent theorem can be easily stated.

Theorem 4.1 *Let $\alpha_k(\lambda) = \widetilde{\Lambda}_k(x)$ for all $k \in \mathbb{N}$ and $l = \lim_{k\to\infty}\widetilde{\Lambda}_k(x)$. Define the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{k=0}^\infty$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(\lambda) := \begin{cases} \prod_{i=k}^{n-1}(\frac{-s_i}{r_{i+1}})[\frac{\lambda_k}{r_k(\lambda_k-\lambda_{k-1})}+\frac{\lambda_k}{s_k(\lambda_{k+1}-\lambda_k)}], & k < n, \\ \frac{1}{r_k}\frac{\lambda_k}{(\lambda_k-\lambda_{k-1})}, & k = n, \\ 0, & k > n. \end{cases} \qquad (4.1)$$

Then the following statements hold:

(a) *The sequence* $\{b^{(k)}(\lambda)\}_{k=0}^{\infty}$ *is a basis for the space* $c_0^{\lambda}(\widetilde{B})$ *and any* $x \in c_0^{\lambda}(\widetilde{B})$ *has a unique representation of the form* $x = \sum_k \alpha_k(\lambda) b^{(k)}(\lambda)$.
(b) *The sequence* $\{b, b^{(0)}(\lambda), b^{(1)}(\lambda), \ldots\}$ *is a basis for the space* $c^{\lambda}(\widetilde{B})$ *and any* $x \in c^{\lambda}(\widetilde{B})$ *has a unique representation of the form* $x = lb + \sum_k [\alpha_k(\lambda) - l] b^{(k)}(\lambda)$, *where* $b = (b_k) = \{\sum_{j=0}^{k}(1/r_j)\prod_{i=0}^{j-1}(-s_i/r_i)\}_{k=0}^{\infty}$.

Finally, it easily follows from Theorem 2.2 that $c_0^{\lambda}(\widetilde{B})$ and $c^{\lambda}(\widetilde{B})$ are the Banach spaces with their natural norms. Thus, by Theorem 4.1 we can obtain the following.

Corollary 4.2 *The difference sequence spaces* $c_0^{\lambda}(\widetilde{B})$ *and* $c^{\lambda}(\widetilde{B})$ *are separable.*

5 The α-, β- and γ-duals of the spaces $c_0^{\lambda}(\widetilde{B})$ and $c^{\lambda}(\widetilde{B})$

The concept of multiplier space plays an important role in the present section. To state the α-, β- and γ-duals of the generalized difference sequence spaces $c_0^{\lambda}(\widetilde{B})$ and $c^{\lambda}(\widetilde{B})$ of non-absolute type, we give the terminology of a multiplier space.

The set $S(\lambda, \mu)$ described as follows is known as the *multiplier space* of any sequence spaces λ and μ,

$$S(\lambda, \mu) = \{a = (a_k) \in \omega : ax = (a_k x_k) \in \mu \text{ for all } x = (x_k) \in \lambda\}.$$

It can be observed for a sequence space φ with $\mu \subset \varphi$ and $\varphi \subset \lambda$ that the inclusions $S(\lambda, \mu) \subset S(\lambda, \varphi)$ and $S(\lambda, \mu) \subset S(\varphi, \mu)$ hold, respectively.

When evaluating the multiplier space $S(\lambda, \mu)$, the α-, β- and γ-duals of a sequence space λ, which are respectively denoted by λ^{α}, λ^{β} and λ^{γ}, are defined by

$$S^{\alpha} = S(\lambda, \ell_1), \qquad \lambda^{\beta} = S(\lambda, cs) \quad \text{and} \quad \lambda^{\gamma} = S(\lambda, bs).$$

It is obvious that $\lambda^{\alpha} \subset \lambda^{\beta} \subset \lambda^{\gamma}$. Also it can be seen that the inclusions $\lambda^{\alpha} \subset \mu^{\alpha}$, $\lambda^{\beta} \subset \mu^{\beta}$ and $\lambda^{\gamma} \subset \mu^{\gamma}$ hold whenever $\mu \subset \lambda$.

The α-dual, β-dual and γ-dual are also referred to as *Köthe-Toeplitz dual*, *generalized Köthe-Toeplitz dual* and *Garling dual*, respectively [1].

Let us now state the following lemmas (see [22]). In this way, the results will be used in the proofs of our Theorems 5.5 to 5.8.

Lemma 5.1 $A = (a_{nk}) \in (c_0 : \ell_1) = (c : \ell_1)$ *iff* $\sup_{K \in \mathcal{F}} \sum_n |\sum_{k \in K} a_{nk}| < \infty$.

Lemma 5.2 $A = (a_{nk}) \in (c_0 : c)$ *iff*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad \textit{for each fixed } k \in \mathbb{N}, \tag{5.1}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{5.2}$$

Lemma 5.3 $A = (a_{nk}) \in (c : c)$ *iff* (5.1) *and* (5.2) *hold, and*

$$\lim_{n\to\infty} \sum_k a_{nk} \quad \textit{exists}. \tag{5.3}$$

Lemma 5.4 $A = (a_{nk}) \in (c : \ell_{\infty}) = (c_0 : \ell_{\infty})$ *iff* (5.2) *holds.*

Now, it is time to give the following theorem.

Theorem 5.5 *The α-dual of the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$ is given by the following set*:

$$h_1^\lambda = \left\{ a = (a_n) \in \omega : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} h_{nk}^\lambda \right| < \infty \right\},$$

here the matrix $H^\lambda = (h_{nk}^\lambda)$ is described with the help of the sequence $a = (a_n) \in \omega$ given by

$$h_{nk}^{(\lambda)} = \begin{cases} \prod_{i=k}^{n-1}(\frac{-s_i}{r_{i+1}})[\frac{\lambda_k}{r_k(\lambda_k-\lambda_{k-1})} + \frac{\lambda_k}{s_k(\lambda_{k+1}-\lambda_k)}]a_n, & k < n, \\ \frac{\lambda_n}{r_n(\lambda_n-\lambda_{n-1})}a_n, & k = n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

Proof The essential idea in this proof is the usage of the definition of the γ-dual. Let us assume that $a = (a_n) \in \omega$. In this condition, we can easily obtain the following equality:

$$a_n x_n = \frac{1}{r_n} \sum_{k=0}^{n} \prod_{i=k}^{n-1} \left(\frac{-s_i}{r_i} \right) \sum_{i=k-1}^{k} (-1)^{k-i} \frac{\lambda_i}{\lambda_k - \lambda_{k-1}} a_n y_i = \left(H^\lambda y \right)_n \quad \text{for all } n \in \mathbb{N} \tag{5.4}$$

from relations (2.5) and (2.6). We use the newly obtained notation result in $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in c_0^\lambda(\widetilde{B})$ or $c^\lambda(\widetilde{B})$ iff $H^\lambda y \in \ell_1$ whenever $y = (y_k) \in c_0$ or c with the help of (5.4). This indicates that the sequence $a = (a_n) \in \{c_0^\lambda(\widetilde{B})\}^\alpha$ or $a = (a_n) \in \{c^\lambda(\widetilde{B})\}^\alpha$ iff $H^\lambda \in (c_0 : \ell_1) = (c : \ell_1)$. Thus, we derive with the aid of Lemma 5.4 by writing H^λ in place of A that $a = (a_n) \in \{c_0^\lambda(\widetilde{B})\}^\alpha = \{c^\lambda(\widetilde{B})\}^\alpha$ iff $\sup_{K\in\mathcal{F}} \sum_n |\sum_{k\in K} h_{nk}^\lambda| < \infty$. Briefly, this tells us the consequence that $\{c_0^\lambda(\widetilde{B})\}^\alpha = \{c^\lambda(\widetilde{B})\}^\alpha = h_1^\lambda$. This conclusion is what was sought for. □

Theorem 5.6 *Define the sets h_2^λ, h_3^λ, h_4^λ and h_5^λ as follows*:

$$h_2^\lambda = \left\{ a = (a_k) \in \omega : \sum_{j=k+1}^{\infty} \prod_{i=0}^{n-j} \left(\frac{-s_i}{r_{i+1}} \right) a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$h_3^\lambda = \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n-1} \left| \tilde{a}_k(n) \right| < \infty \right\},$$

$$h_4^\lambda = \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \left| \frac{1}{r_n} \frac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n \right| < \infty \right\},$$

$$h_5^\lambda = \left\{ a = (a_k) \in \omega : \sum_k \frac{1}{r_k} \sum_{j=0}^{k} \prod_{i=j}^{k-1} \left(\frac{-s_i}{r_i} \right) a_k \text{ converges} \right\},$$

where

$$\tilde{a}_k(n) = \lambda_k \left[\frac{a_k}{r_k(\lambda_k - \lambda_{k-1})} + \left(\frac{1}{r_k(\lambda_k - \lambda_{k-1})} + \frac{1}{s_k(\lambda_{k+1} - \lambda_k)} \right) \sum_{j=k+1}^{n} \prod_{j=k+1}^{n-j} \left(\frac{-s_i}{r_{i+1}} \right) a_j \right]$$

for $k < n$.

Then

(i) $\{c_0^\lambda(\widetilde{B})\}^\beta = h_2^\lambda \cap h_3^\lambda \cap h_4^\lambda$,
(ii) $\{c^\lambda(\widetilde{B})\}^\beta = h_2^\lambda \cap h_3^\lambda \cap h_4^\lambda \cap h_5^\lambda$.

Proof According to the definition of β-dual, it is not too difficult to show that condition (i) holds. For this, we deal with the following equality:

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left\{\frac{1}{r_k}\sum_{j=0}^{k}\prod_{i=j}^{k-1}\left(\frac{-s_i}{r_i}\right)\left[\sum_{i=j-1}^{j}(-1)^{j-i}\frac{\lambda_i}{\lambda_j-\lambda_{j-1}}y_i\right]\right\}a_k \\ &= \sum_{k=0}^{n-1}\lambda_k\left[\frac{a_k}{r_k(\lambda_k-\lambda_{k-1})}+\left(\frac{1}{r_k(\lambda_k-\lambda_{k-1})}+\frac{1}{s_k(\lambda_{k+1}-\lambda_k)}\right)\right. \\ &\quad\times\left.\sum_{j=k+1}^{n}\prod_{j=k+1}^{n}\left(\frac{-s_i}{r_{i+1}}\right)a_j\right]y_k+\frac{1}{r_n}\frac{\lambda_n}{(\lambda_n-\lambda_{n-1})}a_n y_n \\ &= \sum_{k=0}^{n-1}\tilde{a}_k(n)y_k+\frac{1}{r_n}\frac{\lambda_n}{(\lambda_n-\lambda_{n-1})}a_n y_n \\ &= T_n^\lambda(y)\quad\text{for all } n\in\mathbb{N}\end{aligned}\tag{5.5}$$

from elementary calculus where the matrix $T^\lambda = (t_{nk}^\lambda)$ is defined by

$$t_{nk}^\lambda := \begin{cases}\tilde{a}_k(n), & k<n,\\ \frac{1}{r_n}\frac{\lambda_n}{(\lambda_n-\lambda_{n-1})}a_n, & k=n,\\ 0, & k>n\end{cases}$$

for all $k,n\in\mathbb{N}$. We are now ready to start the proof with the help of (5.5). One can easily deduce $ax=(a_k x_k)\in cs$ whenever $x=(x_k)\in c_0^\lambda(\widetilde{B})$ iff $T^\lambda y\in c$ whenever $y=(y_k)\in c_0$. This means that $a=(a_k)\in\{c_0^\lambda(\widetilde{B})\}^\beta$ iff $T^\lambda\in(c_0:c)$. Therefore, by using Lemma 5.2, we derive from (5.1) and (5.2) that

$$\sum_{j=k+1}^{\infty}\prod_{i=0}^{n-j}\left(\frac{-s_i}{r_{i+1}}\right)a_j\quad\text{exists for each } k\in\mathbb{N},\tag{5.6}$$

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n-1}\left|\tilde{a}_k(n)\right|<\infty,\tag{5.7}$$

$$\sup_{n\in\mathbb{N}}\left|\frac{1}{r_n}\frac{\lambda_n}{(\lambda_n-\lambda_{n-1})}a_n\right|<\infty.\tag{5.8}$$

Therefore, we conclude that $\{c_0^\lambda(\widetilde{B})\}^\beta = h_2^\lambda\cap h_3^\lambda\cap h_4^\lambda$.

First of all, the assertion (ii) of the theorem has exactly the same idea as in the first part of it, the proof of the second part can be obtained similarly. It comes fairly easily from Lemma 5.3 with the aid of (5.5) that $a=(a_k)\in\{c^\lambda(\widetilde{B})\}^\beta$ iff $T^\lambda\in(c:c)$. Thus, conditions (5.6), (5.7) and (5.8) are valid from (5.1) and (5.2).

Moreover, the following equality can be directly written:

$$\sum_{k=0}^{n}\frac{1}{r_k}\sum_{j=0}^{k}\prod_{i=j}^{k-1}\left(\frac{-s_i}{r_i}\right)a_k=\sum_{k=0}^{n-1}\tilde{a}_k(n)+\frac{1}{r_n}\frac{\lambda_n}{(\lambda_n-\lambda_{n-1})}a_n=\sum_k t_{nk}^\lambda$$

holds for all $n \in \mathbb{N}$. Therefore, by using (5.3), we have that

$$\left\{ \frac{1}{r_k} \sum_{j=0}^{k} \prod_{i=j}^{k-1} \left(\frac{-s_i}{r_i} \right) a_k \right\} \in cs. \tag{5.9}$$

Consequently, it is clear that $\{c^\lambda(\widetilde{B})\}^\beta = h_2^\lambda \cap h_3^\lambda \cap h_4^\lambda \cap h_5^\lambda$, which gives the desired result. □

Remark 5.7 We may note by combining (5.9) with conditions (5.7) and (5.8) that $\{\sum_{j=0}^{k} \prod_{i=j}^{k-1} (\frac{-s_i}{r_i}) a_k / r_k\} \in bs$ for every sequence $a = (a_k) \in \{c_0^\lambda(\widetilde{B})\}^\beta$.

Finally, we conclude this section with the following theorem which determines the γ-dual of the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$.

Theorem 5.8 *The set $h_3^\lambda \cap h_4^\lambda$ is the γ-dual of the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$.*

Proof The proof of this theorem can also be proved in a much similar way to the proof of Theorem 5.6 using Lemma 5.4 instead of Lemma 5.2, thus it is left to the reader. □

6 Certain matrix mappings related to the spaces $c_0^\lambda(\widetilde{B})$ and $c^\lambda(\widetilde{B})$

One of the most important ideas is matrix transformation in this work. Therefore, we focus on this concept in the present section.

It is appropriate to characterize certain classes of the matrix mappings. Therefore, we emphasize the matrix classes such as $(c^\lambda(\widetilde{B}) : \ell_p)$, $(c_0^\lambda(\widetilde{B}) : \ell_p)$, $(c^\lambda(\widetilde{B}) : c)$, $(c^\lambda(\widetilde{B}) : c_0)$, $(c_0^\lambda(\widetilde{B}) : c)$ and $(c_0^\lambda(\widetilde{B}) : c_0)$, where $1 \le p \le \infty$. We also characterize some other classes including the Riesz, difference, Euler and Cesàro sequence spaces.

For the sake of simplicity, here and in what follows, we shall write

$$\tilde{a}_{nk}(m) = \lambda_k \left[\frac{a_{nk}}{r_k(\lambda_k - \lambda_{k-1})} + \left(\frac{1}{r_k(\lambda_k - \lambda_{k-1})} + \frac{1}{s_k(\lambda_{k+1} - \lambda_k)} \right) \sum_{j=k+1}^{m} \prod_{i=0}^{n-j} \left(\frac{-s_i}{r_{i+1}} \right) a_{nj} \right]$$

if $k < m$,

$$\tilde{a}_{nk} = \lambda_k \left[\frac{a_{nk}}{r_k(\lambda_k - \lambda_{k-1})} + \left(\frac{1}{r_k(\lambda_k - \lambda_{k-1})} + \frac{1}{s_k(\lambda_{k+1} - \lambda_k)} \right) \sum_{j=k+1}^{\infty} \prod_{i=0}^{n-j} \left(\frac{-s_i}{r_{i+1}} \right) a_{nj} \right].$$

For an infinite matrix $A = (a_{nk})$, and we state here that the series on the right-hand side in the above equality are convergent for all $k, m, n \in \mathbb{N}$.

The results of the following lemmas will be used in the proofs of our theorems.

Lemma 6.1 [21] *The matrix mappings between the BK-spaces are continuous.*

Lemma 6.2 [22] $A = (a_{nk}) \in (c : \ell_p)$ *iff* $\sup_{K \in \mathcal{F}} \sum_n |\sum_{k \in K} a_{nk}|^p < \infty$ $(1 \le p < \infty)$.

Lemma 6.3 [22] $A = (a_{nk}) \in (c : c_0)$ *iff* (5.2) *holds and*

$$\lim_{n \to \infty} a_{nk} = 0 \quad \textit{for all } k \in \mathbb{N}, \tag{6.1}$$

$$\lim_{n \to \infty} \sum_k a_{nk} = 0.$$

Lemma 6.4 [22] *$A = (a_{nk}) \in (c_0 : c_0)$ iff (5.2) and (6.1) hold.*

These motivate the following theorems related to the matrix transformations.

Theorem 6.5 *Let us assume that $A = (a_{nk})$ is an infinite matrix defined on the complex field. In that case, the following statements are valid.*

(i) *Let $1 \leq p < \infty$. Then $A \in (c^{\lambda}(\widetilde{B}) : \ell_p)$ if and only if*

$$\sum_{j=k+1}^{\infty} \prod_{i=0}^{n-j} \left(\frac{-s_i}{r_{i+1}}\right) a_{nj} \quad \text{exists for each fixed } k \in \mathbb{N}, \tag{6.2}$$

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \tilde{a}_{nk} \right|^p < \infty, \tag{6.3}$$

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^{m-1} \left|\tilde{a}_{nk}(m)\right| < \infty \quad \text{for all } n \in \mathbb{N}, \tag{6.4}$$

$$\left\{ \frac{1}{r_k} \sum_{j=0}^{k} \prod_{i=j}^{k-1} \left(\frac{-s_i}{r_i}\right) a_{nk} \right\}_{k=0}^{\infty} \in cs \quad \text{for each fixed } n \in \mathbb{N}, \tag{6.5}$$

$$\lim_{k \to \infty} \frac{\lambda_k}{r_k(\lambda_k - \lambda_{k-1})} a_{nk} = a_n \quad \text{for each fixed } n \in \mathbb{N}, \tag{6.6}$$

$$(a_n) \in \ell_p. \tag{6.7}$$

(ii) *$A \in (c^{\lambda}(B) : \ell_{\infty})$ if and only if (6.5) and (6.6) hold, and*

$$\sup_{n \in \mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty, \tag{6.8}$$

$$(a_n) \in \ell_{\infty}. \tag{6.9}$$

Proof For proving the sufficiency of the theorem, let us assume that conditions (6.2)-(6.7) hold and take any $x = (x_k) \in c^{\lambda}(\widetilde{B})$. In this condition, using Theorem 5.6 we obtain that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{c^{\lambda}(\widetilde{B})\}^{\beta}$ for all $n \in \mathbb{N}$. This requires the existence of the A-transform of x, that is, Ax exists. Moreover, it is obviously seen that the associated sequence $y = (y_k)$ lies in the space c. Furthermore, if we combine Lemma 6.2 together with condition (6.3) we see that the matrix $\widetilde{A} = (\tilde{a}_{nk})$ is in the class $(c : \ell_p)$, where $1 \leq p < \infty$.

Now, let us consider the following equality obtained from relation (2.5) from the mth partial sum of the series $\sum_k a_{nk} x_k$:

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} \tilde{a}_{nk}(m) y_k + \frac{\lambda_m}{r_m(\lambda_m - \lambda_{m-1})} a_{nm} y_m \quad \text{for all } n, m \in \mathbb{N}. \tag{6.10}$$

Thus, since $y \in c$ and $\widetilde{A} \in (c : \ell_p)$, this clearly indicates that the series $\sum_k \tilde{a}_{nk} y_k$ converges for every $n \in \mathbb{N}$. Moreover, it follows by (6.2) that the series $\sum_{j=k+1}^{\infty} \prod_{i=0}^{n-j} (\frac{-s_i}{r_{i+1}}) a_{nj}$ converges for all $n, k \in \mathbb{N}$ and hence $\tilde{a}_{nk}(m) \to \tilde{a}_{nk}$ as $m \to \infty$. Then we can derive from (6.10) as $m \to \infty$ with the aid of (6.6) that

$$\sum_k a_{nk} x_k = \sum_k \tilde{a}_{nk} y_k + l a_n \quad \text{for all } n \in \mathbb{N}, \tag{6.11}$$

which can be shortly written as follows:

$$(Ax)_n = (\widetilde{A}y)_n + la_n \quad \text{for all } n \in \mathbb{N}. \tag{6.12}$$

This newly obtained formula results in the fact that by taking p-norm,

$$\|Ax\|_p \leq \|\widetilde{A}y\|_p + |l| \left\| (a_n) \right\|_p < \infty.$$

This shows that $Ax \in \ell_p$. That is, $A \in (c^\lambda(\widetilde{B}) : \ell_p)$.

Now, in order to verify the converse claim, let us assume that $A \in (c^\lambda(\widetilde{B}) : \ell_p)$, where $1 \leq p < \infty$. In this condition, Ax exists for every $x \in c^\lambda(\widetilde{B})$ and it is not difficult to see that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c^\lambda(\widetilde{B})\}^\beta$ for all $n \in \mathbb{N}$. Using Theorem 5.6, one can immediately see the necessity of conditions (6.4) and (6.5).

Also, we get by using Lemma 6.1 that there is a constant $M > 0$ such that

$$\|Ax\|_p \leq M\|x\|_{c^\lambda(\widetilde{B})} \tag{6.13}$$

holds for all $x \in c^\lambda(\widetilde{B})$ because of the fact that $c^\lambda(\widetilde{B})$ and ℓ_p are BK-spaces. Now, $K \in \mathcal{F}$. In this case the sequence $z = \sum_{k\in K} b^{(k)}(\lambda)$ lies $c^\lambda(\widetilde{B})$, where the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n\in\mathbb{N}}$ is defined by (4.1) for every fixed $k \in \mathbb{N}$. We have

$$\|z\|_{c^\lambda(\widetilde{B})} = \left\| \widetilde{\Lambda}(z) \right\|_\infty = \left\| \sum_{k\in K} \widetilde{\Lambda}\left(b^{(k)}(\lambda)\right) \right\|_\infty = \left\| \sum_{k\in K} e^{(k)} \right\|_\infty = 1,$$

because for each fixed $k \in \mathbb{N}$ the equality $\widetilde{\Lambda}(b^{(k)}(\lambda)) = e^{(k)}$ holds. Moreover, we can easily derive the following equation using (4.1):

$$(Az)_n = \sum_{k\in K} A_n\left(b^{(k)}(\lambda)\right) = \sum_{k\in K} \sum_j a_{nj} b_j^{(k)}(\lambda) = \sum_{k\in K} \tilde{a}_{nk}$$

for every $n \in \mathbb{N}$. Therefore, we obtain the following inequality for any $K \in \mathcal{F}$:

$$\left(\sum_n \left| \sum_{k\in K} \tilde{a}_{nk} \right|^p \right)^{1/p} \leq M$$

due to the fact that inequality (6.13) is met for the sequence $z \in c^\lambda(\widetilde{B})$. This result requires that inequality (6.3) is necessary. In conclusion, the statement $\widetilde{A} = (\tilde{a}_{nk}) \in (c : \ell_p)$ is obtained following Lemma 6.2.

First, let us assume that $y = (y_k) \in c \setminus c_0$ and take into account the sequence $x = (x_k)$ given by (2.6) for each $k \in \mathbb{N}$. Next, the sequences x and y are joined with relation (2.5), that is, $x \in c^\lambda(\widetilde{B})$ such that $y = \widetilde{\Lambda}x$. Thus, there exist both Ax and $\widetilde{A}y$. The newly obtained results show the convergence both of the series $\sum_k a_{nk}x_k$ and $\sum_k \tilde{a}_{nk}y_k$ for each $n \in \mathbb{N}$. Then we can conclude that

$$\lim_{m\to\infty} \sum_{k=0}^{m-1} \tilde{a}_{nk}(m) y_k = \sum_k \tilde{a}_{nk} y_k \quad \text{for all } n \in \mathbb{N}.$$

In conclusion, making $m \to \infty$ in (6.10), we conclude that

$$\lim_{m\to\infty} \frac{\lambda_m}{r_m(\lambda_m - \lambda_{m-1})} a_{nm} y_m \quad \text{exists for each fixed } n \in \mathbb{N},$$

and, moreover, because of the fact that $y = (y_k) \in c \setminus c_0$, we also have

$$\lim_{m\to\infty} \frac{\lambda_m}{r_m(\lambda_m - \lambda_{m-1})} a_{nm} \quad \text{exists for each fixed } n \in \mathbb{N}.$$

This result requires that the limit given by (6.6) is necessary. Thus, relation (6.12) holds.

Finally, the necessity of (6.7) immediately follows from (6.12) owing to the fact that $Ax \in \ell_p$ and $\widetilde{A}y \in \ell_p$. This represents the desired proof of part (i) of the theorem.

One can prove part (ii) using a similar way as that in the proof of part (i) with Lemma 5.4 in place of Lemma 6.2; the details are left to the reader. □

Remark 6.6 The following limit exists:

$$\lim_{m\to\infty} \sum_{k=0}^{m-1} \left|\tilde{a}_{nk}(m)\right| = \sum_k |\tilde{a}_{nk}|$$

for each $n \in \mathbb{N}$, using (6.8). This newly obtained result informs us that condition (6.8) requires condition (6.4).

Here, it may be recalled that the claim $(c_0 : \ell_p) = (c : \ell_p)$ for $1 \le p \le \infty$ is valid (see [22, pp.7-8]). In conclusion, using Theorem 5.6 and Lemmas 6.2 and 5.4, we immediately have the following theorem.

Theorem 6.7 *Let us assume that $A = (a_{nk})$ is an infinite matrix defined on the complex field. In that case, the following statements are valid.*

(i) *Let us suppose that $1 \le p < \infty$. In that case, $A \in (c_0^\lambda(\widetilde{B}) : \ell_p)$ is valid iff* (6.3) *and* (6.4) *hold, and*

$$\sum_{j=k}^{\infty} \prod_{i=0}^{n-j} \left(\frac{-s_i}{r_{i+1}}\right) a_{nj} \quad \textit{exists for all } n, k \in \mathbb{N}, \tag{6.14}$$

$$\left\{ \frac{\lambda_k}{r_k(\lambda_k - \lambda_{k-1})} a_{nk} \right\}_{k=0}^{\infty} \in \ell_\infty \quad \textit{for all } n \in \mathbb{N}. \tag{6.15}$$

(ii) $A \in (c_0^\lambda(\widetilde{B}) : \ell_\infty)$ *is valid iff all of* (6.8) *and* (6.14) *and* (6.15) *hold.*

Proof Since the proof of this theorem can be obtained by using the same way as that used in the proof of Theorem 6.5, we leave it to the reader. □

Theorem 6.8 $A = (a_{nk}) \in (c^\lambda(\widetilde{B}) : c)$ *is valid iff* (6.5), (6.6) *and* (6.8) *hold, and*

$$\lim_{n\to\infty} a_n = a, \tag{6.16}$$

$$\lim_{n\to\infty} \tilde{a}_{nk} = \alpha_k \quad \textit{for each } k \in \mathbb{N}, \tag{6.17}$$

$$\lim_{n\to\infty} \sum_k \tilde{a}_{nk} = \alpha. \tag{6.18}$$

Proof First of all, let us prove the sufficiency of the conditions. For this, let us assume that A satisfies conditions (6.5), (6.6), (6.8), (6.16), (6.17) and (6.18), and take any $x \in c^\lambda(B)$. Condition (6.8) requires condition (6.4) for all $n \in \mathbb{N}$; we have $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c^\lambda(\widetilde{B})\}^\beta$ by Theorem 5.6, *i.e.*, Ax exists. It is also seen from (6.8) and (6.17) that $\sum_{j=0}^{k}|\alpha_j| \leq \sup_{m\in\mathbb{N}}\sum_j |\tilde{a}_{nk}| < \infty$ is valid for every $k \in \mathbb{N}$. This results in the fact that $(\alpha_k) \in \ell_1$ and the series $\sum_k \alpha_k(y_k - l)$ is convergent, where $y = (y_k) \in c$ is the sequence connected with $x = (x_k)$ via the relation given by (2.5) in such a way that $y_k \to l$ when $k \to \infty$. Furthermore, when Lemma 5.3 is combined with conditions (6.8), (6.17) and (6.18), it is clearly seen that the matrix $\widetilde{A} = (\tilde{a}_{nk})$ lies in the class $(c : c)$.

Now, if we think in a similar way as in the proof of Theorem 6.5, we easily have that relation (6.11) is valid and can be rewritten as follows:

$$\sum_k a_{nk}x_k = \sum_k \tilde{a}_{nk}(y_k - l) + l\sum_k \tilde{a}_{nk} + la_n \quad \text{for all } n \in \mathbb{N}. \tag{6.19}$$

In this case, by letting $n \to \infty$ in (6.19), we observe that the first term on the right-hand side tends to $\sum_k \alpha_k(y_k - l)$ with the help of (6.8) and (6.17). Similarly, the second and the last term tend to $l\alpha$ by (6.18) and la by (6.16), respectively. In conclusion, we get $\lim_{n\to\infty}(Ax)_n = \sum_k \alpha_k(y_k - l) + l(\alpha + a)$, and this shows that $Ax \in c$; in other words, $A \in (c^\lambda(\widetilde{B}) : c)$.

Conversely, let us assume that $A \in (c^\lambda(\widetilde{B}) : c)$. In that case, since the inclusion $c \subset \ell_\infty$ is valid, it is obvious that $A \in (c^\lambda(\widetilde{B}) : \ell_\infty)$. Thus, the necessity of conditions (6.5), (6.6) and (6.8) is clear from Theorem 6.5. Moreover, let us consider the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n\in\mathbb{N}} \in c^\lambda(\widetilde{B})$ described by (4.1) for every fixed $k \in \mathbb{N}$. Thus, it is obvious that $Ab^{(k)}(\lambda) = \{\tilde{a}_{nk}\}_{n\in\mathbb{N}}$. Next it is seen that $\{\tilde{a}_{nk}\}_{n\in\mathbb{N}} \in c$ for each $k \in \mathbb{N}$, and this illustrates the necessity of (6.17). Now, let us assume that $z = \sum_k b^{(k)}(\lambda)$. In this case, the linear transformation $T : c^\lambda(\widetilde{B}) \to c$, described as in the proof of Theorem 2.4, is continuous by analogy; and, moreover, $\widetilde{\Lambda}(b^{(k)}(\lambda)) = e^{(k)}$ is valid for each fixed $k \in \mathbb{N}$. Thus, we obtain that $\widetilde{\Lambda}_n(z) = \sum_k \widetilde{\Lambda}_n(b^{(k)}(\lambda)) = \sum_k \delta_{nk} = 1$ for each $n \in \mathbb{N}$ and this result demonstrates that $\widetilde{\Lambda}(z) = e \in c$ and hence $z \in c^\lambda(\widetilde{B})$. In the other way around, since $c^\lambda(\widetilde{B})$ and c are the BK-spaces, Lemma 6.1 requires that the matrix mapping $A : c^\lambda(\widetilde{B}) \to c$ is continuous. Therefore, for every $n \in \mathbb{N}$, we have $(Az)_n = \sum_k A_n(b^{(k)}(\lambda)) = \sum_k \tilde{a}_{nk}$. This result represents the necessity of (6.18).

Next, it follows that $\widetilde{A} = (\tilde{a}_{nk}) \in (c : c)$ by (6.8), (6.17) and (6.18) together with Lemma 5.3. Thus, using (6.5) and (6.6), it is seen that relation (6.12) is valid for all $x \in c^\lambda(\widetilde{B})$ and $y \in c$. Moreover, x and y are connected with relation (2.5) where $y_k \to l$ when $k \to \infty$.

In the last step, the necessity of (6.16) is immediately seen (6.12) due to the fact that $Ax \in c$ and $\widetilde{A}x \in c$. This last step concludes the proof. □

Theorem 6.9 *The statement* $A = (a_{nk}) \in (c^\lambda(\widetilde{B}) : c_0)$ *is valid iff* (6.5), (6.6) *and* (6.8), *and the following conditions hold*:

$$\lim_{n\to\infty} a_n = 0,$$

$$\lim_{n\to\infty} \tilde{a}_{nk} = 0 \quad \textit{for each } k \in \mathbb{N},$$

$$\lim_{n\to\infty} \sum_k \tilde{a}_{nk} = 0.$$

Proof The present theorem can be easily proved in a similar way used in the proof of Theorem 6.8 with Lemma 6.3 in place of Lemma 5.3, we leave it to the reader. □

Theorem 6.10 *The statement* $A = (a_{nk}) \in (c_0^\lambda(\widetilde{B}) : c)$ *is valid iff conditions* (6.8), (6.14), (6.15) *and* (6.17) *hold*.

Proof The proof can be easily obtained with Lemma 5.2, Theorem 5.6 and part (ii) of Theorem 6.7. □

Theorem 6.11 *The statement* $A = (a_{nk}) \in (c_0^\lambda(\widetilde{B}) : c_0)$ *is valid iff conditions* (6.8), (6.14), (6.15) *and* (6.17) *hold with* $\alpha_k = 0$ *for each* $k \in \mathbb{N}$.

Proof The proof is obvious when Lemma 6.4, Theorems 5.6 and 6.10 are evaluated. □

Competing interests
The author declares that he has no competing interests.

Acknowledgements
This paper was solely written by the author without any support from any organization. Certain results of this study were partially presented at the *Algerian-Turkish International Days on Mathematics* (*ATIM* 2013) held on 12-14 September, 2013 in Istanbul, Turkey at Fatih University.

Received: 14 January 2014 Accepted: 26 May 2014 Published: 04 Jun 2014

References
1. Başar, F: Summability Theory and Its Applications. Bentham Science Publishers, İstanbul (2012). ISBN:978-1-60805-252-3
2. Wilansky, A: Functional Analysis. Blaisdell, Boston (1964)
3. Altay, B, Başar, F: The matrix domain and the fine spectrum of the difference operator Δ on the sequence space ℓ_p ($0 < p < 1$). Commun. Math. Anal. **2**(2), 1-11 (2007)
4. Altay, B, Başar, F, Mursaleen, M: On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I. Inf. Sci. **176**(10), 1450-1462 (2006)
5. Altay, B, Başar, F, Malkowsky, E: Matrix transformations on sequence spaces related to strong Cesàro summability and boundedness. Appl. Math. Comput. **211**(2), 255-264 (2009)
6. Aydın, C, Başar, F: Some new sequence spaces which include the spaces ℓ_p and ℓ_∞. Demonstr. Math. **38**(3), 641-656 (2005)
7. Candan, M: Domain of the double sequential band matrix in the classical sequence spaces. J. Inequal. Appl. **2012**, 281 (2012)
8. Candan, M: Almost convergence and double sequential band matrix. Acta Math. Sci. Ser. B **34**(2), 354-366 (2014)
9. Choudhary, B, Nanda, S: Functional Analysis with Applications. Wiley, New Delhi (1989)
10. Kızmaz, H: On certain sequence spaces. Can. Math. Bull. **24**(2), 169-176 (1981)
11. Malkowsky, E: Recent results in the theory of matrix transformations in sequence spaces. Mat. Vesn. **49**, 187-196 (1997)
12. Malkowsky, E, Rakočević, V, Živković-Zlatanović, S: Matrix transformations between the sequence spaces BP^p and certain BK spaces. Bull. - Acad. Serbe Sci. Arts, Cl. Sci. Math. Nat., Sci. Math. **27**, 33-46 (2002)
13. Malkowsky, E, Savaş, E: Matrix transformations between sequence spaces of generalized weighted means. Appl. Math. Comput. **147**, 333-345 (2004)
14. Malkowsky, E, Mursaleen, M, Suantai, S: The dual spaces of sets of difference sequences of order *m* and matrix transformations. Acta Math. Sin. Engl. Ser. **23**(3), 521-532 (2007)
15. Mursaleen, M, Noman, AK: On the spaces of λ-convergent and bounded sequences. Thai J. Math. **8**(2), 311-329 (2010)
16. Mursaleen, M, Noman, AK: On some new difference sequence spaces of non-absolute type. Math. Comput. Model. **52**, 603-617 (2010)
17. Sönmez, A, Başar, F: Generalized difference spaces of non-absolute type of convergent and null sequences. Abstr. Appl. Anal. **2012**, Article ID 435076 (2012)
18. Ng, P-N, Lee, P-Y: Cesàro sequence spaces of non-absolute type. Comment. Math. Prace Mat. **20**(2), 429-433 (1978)
19. Sarıgöl, MA: On difference sequence spaces. J. Karadeniz Tech. Univ., Fac. Arts Sci., Ser. Math.-Phys. **10**, 63-71 (1987)
20. Sönmez, A: Some new sequence spaces derived by the domain of the triple band matrix. Comput. Math. Appl. **62**, 641-650 (2011)
21. Wilansky, A: Summability Through Functional Analysis. North-Holland Mathematics Studies, vol. 85. North-Holland, Amsterdam (1984)
22. Stieglitz, M, Tietz, H: Matrix transformationen von folgenraumen eine ergebnisübersicht. Math. Z. **154**, 1-16 (1977)

10.1186/1687-1847-2014-163
Cite this article as: Candan: Domain of the double sequential band matrix in the spaces of convergent and null sequences. *Advances in Difference Equations* 2014, 2014:163