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The zeros of complex differential-difference polynomials

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Abstract
This paper is devoted to considering the zeros of complex differential-difference polynomials of different types. Our results can be seen as the differential-difference analogues of Hayman conjecture (Ann. Math. 70:9-42, 1959).
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1 Introduction and main results

Let $f(z)$ be a meromorphic function in the complex domain. Assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [1, 2]. Recall that $a(z)$ is a small function with respect to $f(z)$, if $T(r,a) = S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Denote by $\rho(f)$ and $\rho_2(f)$ the order and the hyper-order of f. In this paper, c is a non-zero complex constant, n, k are positive integers, unless otherwise specified.

Hayman [3] conjectured that if f is a transcendental meromorphic function, then $f^n f'$ takes every finite non-zero value infinitely often. In fact, Hayman [3] proved that if f is a transcendental meromorphic function and $n \geq 3$, then $f^n f'$ takes every finite non-zero value infinitely often. Later, the case $n = 2$ was settled by Mues [4]. Bergweiler and Eremenko [5], Chen and Fang [6, Theorem 1] proved the case of $n = 1$, respectively. In the past years, the topic on the zeros of differential polynomials has always been an important research problem in value distribution of meromorphic functions. With the development of the difference analogues of Nevanlinna theory, some authors paid their attention to the zeros of difference polynomials. Laine and Yang [7, Theorem 2] firstly considered the zeros distribution of $f(z)^n f(z+c) - a$, where a is a non-zero constant, which can be seen as a difference analogue of Hayman conjecture. Recently, many authors were interested in the zeros distribution of difference polynomials of different types, such as [8–13].

A polynomial $Q(z,f)$ can be called a differential-difference polynomial in f whenever $Q(z,f)$ is a polynomial in $f(z)$, its shifts $f(z+c)$ and their derivatives, with small functions of $f(z)$ as the coefficients. It is interesting to consider the zeros of differential-difference polynomials. The aim of the paper is to explore the differences or analogues among the zeros of differential polynomials, difference polynomials, differential-difference polynomials. Liu *et al.* [14, Theorems 1.1 and 1.3] considered this problem and obtained the following result, where $\Delta_c f = f(z+c) - f(z)$.

Theorem A *Let f be a transcendental entire function of finite order and $a(z)$ be a non-zero small function with respect to $f(z)$. If $n \geq k+2$, then $[f(z)^n f(z+c)]^{(k)} - a(z)$ has infinitely many zeros. If f is not a periodic function with period c and $n \geq k+3$, then $[f(z)^n \Delta_c f]^{(k)} - a(z)$ has infinitely many zeros.*

If $a(z) \equiv 0$ in Theorem A, some results can be found in [15]. In this paper, we will consider the zeros of differential-difference polynomials of $f(z)^n f^{(k)}(z+c) - a(z)$ and $f(z)^n (\Delta_c f)^{(k)} - a(z)$.

Theorem 1.1 *Let f be a transcendental entire function of hyper-order $\rho_2(f) < 1$. If $n \geq 3$, then $f(z)^n f^{(k)}(z+c) - a(z)$ has infinitely many zeros, where $a(z)$ is a non-zero small function with respect to $f(z)$.*

Remark 1 (1) The condition that $a(z)$ is a non-zero small function cannot be removed, which can be seen by $f(z) = e^z$ and $e^c = 2$. Thus we get $f(z)^n f^{(k)}(z+c) = 2e^{(n+1)z}$ has no zeros.

(2) The condition $\rho_2(f) < 1$ cannot be deleted, which can be seen by $f(z) = e^{e^z}$ of $\rho_2(f) = 1$, thus $f(z)^n f'(z+c) + ne^z + P(z) = P(z)$ has finitely many zeros, where $e^c = -n$ and $P(z)$ is a non-zero polynomial. In fact, for any integer k, we can choose appropriate $\alpha(z)$ to make $f(z)^n f^{(k)}(z+c) + \alpha(z) + P(z) = P(z)$, $\alpha(z)$ is a polynomial in e^z.

If f is a finite order transcendental entire function, we prove the following result.

Theorem 1.2 *Let f be a finite order transcendental entire function. If $n \geq 2$, then $f(z)^n f^{(k)}(z+c) - a(z)$ has infinitely many zeros, where $a(z)$ is an entire function with $\rho(a) < \rho(f)$.*

Definition 1 Define that a polynomial $p(z)$ is a Borel exceptional polynomial of $f(z)$ when

$$\lambda\big(f(z) - p(z)\big) = \limsup_{r \to \infty} \frac{\log^+ N(r, \frac{1}{f(z)-p(z)})}{\log r} < \rho(f),$$

where $\lambda(f(z) - p(z))$ is the exponent of convergence of zeros of $f(z) - p(z)$.

Theorem 1.3 *Let f be a finite order transcendental entire function with a Borel exceptional polynomial $d(z)$. If $n \geq 1$, then $f(z)^n f^{(k)}(z+c) - b$ has infinitely many zeros, where b is a non-zero constant.*

Remark 2 (1) The condition that b is a non-zero constant cannot be removed, which can be seen by $f(z) = e^z$ which has a Borel exceptional value 0. Thus, we get $f(z)^n f^{(k)}(z+c) = e^c e^{(n+1)z}$ has no zeros.

(2) From the above three theorems, we can reduce the value of n with additional conditions. However, we hope that the condition $n \geq 3$ can be reduced to $n \geq 1$ in Theorem 1.1. Unfortunately, we have not succeeded in doing that.

If $f(z)$ is a transcendental meromorphic function, we obtain the next result.

Theorem 1.4 *Let f be a transcendental meromorphic function of hyper-order $\rho_2(f) < 1$. If $n \geq 2k + 6$, then $f(z)^n f^{(k)}(z + c) - a(z)$ has infinitely many zeros, where $a(z)$ is a non-zero small function with respect to $f(z)$.*

Using the similar method of proofs of Theorems 1.1 and 1.4 below, we can get the following result.

Theorem 1.5 *Let f be a transcendental meromorphic (entire) function of hyper-order $\rho_2(f) < 1$. If $n \geq 4k + 9$ ($n \geq 4$), then $f(z)^n(\Delta_c f)^{(k)} - a(z)$ has infinitely many zeros, where $a(z)$ is a non-zero small function with respect to $f(z)$.*

Finally, we recall the classical results due to Hayman [3, Theorems 8 and 9], which can be combined as follows.

Theorem B *Let f be a transcendental meromorphic function and $a \neq 0$, b be a finite complex constant. Then $f^n + af' - b$ has infinitely many zeros for $n \geq 5$. If f is transcendental entire, this holds for $n \geq 3$, resp. $n \geq 2$, if $b = 0$.*

We then proceed to consider the zeros of $f(z)^n + a(z)f^{(k)}(z + c) - b(z)$, which can be seen as the differential-difference analogues of Theorem B.

Theorem 1.6 *Let f be a transcendental entire function with finite order, let $a(z)$, $b(z)$ be small functions with respect to f. Then $f(z)^n + a(z)f^{(k)}(z + c) + b(z)$ has infinitely many zeros for $n \geq 3$, resp. $n \geq 2$, if $b(z) \equiv 0$.*

Remark 3 The condition $n \geq 3$ cannot be improved if $b(z) \not\equiv 0$, which can be seen by the function $f(z) = e^z + 1$ and $e^c = 2$, thus $f(z)^2 - f'(z + c) - 1 = e^{2z}$ has no zeros. The condition $n \geq 2$ cannot be improved if $b(z) \equiv 0$, which can be seen by the function $f(z) = e^z$ and $e^c = 2$, thus $f(z) - f'(z + c) = -e^z$ has no zeros.

2 Some lemmas

The difference analogue of logarithmic derivative lemma, given by Chiang and Feng [16, Corollary 2.5], Halburd and Korhonen [17, Theorem 2.1], plays an important part in considering the difference analogues of Nevanlinna theory. Afterwards, Halburd, Korhonen and Tohge improved the condition of growth from $\rho < \infty$ to $\rho_2(f) < 1$ as follows.

Lemma 2.1 [18, Theorem 5.1] *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$, $\varsigma < 1$, ε is a number small enough. Then*

$$m\left(r, \frac{f(z + c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}}\right) = S(r,f), \tag{2.1}$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.2 [18, Lemma 8.3] *Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function and let $s \in (0, \infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \varsigma < 1, \tag{2.2}$$

and $\delta \in (0, 1-\varsigma)$, *then*

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right) \tag{2.3}$$

for all r runs to infinity outside of a set of finite logarithmic measure.

From Lemma 2.2, then we get the following lemma.

Lemma 2.3 *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$. Then*

$$T\bigl(r, f(z+c)\bigr) = T(r,f) + S(r,f) \tag{2.4}$$

and

$$N\bigl(r, f(z+c)\bigr) = N(r,f) + S(r,f), N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r,f). \tag{2.5}$$

Lemma 2.4 *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$. Then*

$$m\left(r, \frac{f^{(k)}(z+c)}{f(z)}\right) = S(r,f) \tag{2.6}$$

for all r outside of a set of finite logarithmic measure.

Proof Combining the lemma of logarithmic derivative with Lemma 2.1, Lemma 2.3, we get

$$m\left(r, \frac{f^{(k)}(z+c)}{f(z)}\right) \le m\left(r, \frac{f^{(k)}(z+c)}{f(z+c)}\right) + m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f). \qquad \square$$

Lemma 2.5 [2, Theorems 1.22 and 1.24] *Let $f(z)$ be a transcendental meromorphic function. Then*

$$T\bigl(r, f^{(n)}\bigr) \le T(r,f) + n\overline{N}(r,f) + S(r,f), \tag{2.7}$$

$$N\left(r, \frac{1}{f^{(n)}}\right) \le N\left(r, \frac{1}{f}\right) + n\overline{N}(r,f) + S(r,f). \tag{2.8}$$

Lemma 2.6 *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$, and let $F(z) = f(z)^n f^{(k)}(z+c)$. Then*

$$(n-k-1)T(r,f) + S(r,f) \le T(r,F) \le (n+k+1)T(r,f) + S(r,f). \tag{2.9}$$

If $f(z)$ is a transcendental entire function of $\rho_2(f) < 1$, then

$$nT(r,f) + S(r,f) \le T(r,F) \le (n+1)T(r,f) + S(r,f). \tag{2.10}$$

Proof Remark that

$$\frac{1}{f(z)^{n+1}} = \frac{1}{F}\frac{f^{(k)}(z+c)}{f(z)}.$$

From Lemma 2.1, Lemma 2.3 and the standard Valiron-Mohon'ko theorem [2, 19] and f is a transcendental meromorphic function, then we obtain

$$\begin{aligned}(n+1)T(r,f) &= T\left(r,\frac{1}{f(z)^{n+1}}\right)+S(r,f)\\ &\le T\left(r,\frac{1}{F}\right)+T\left(r,\frac{f^{(k)}(z+c)}{f(z)}\right)+S(r,f)\\ &\le T(r,F)+N\left(r,\frac{f^{(k)}(z+c)}{f(z)}\right)+S(r,f)\\ &\le T(r,F)+N(r,f)+k\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\le T(r,F)+(k+2)T(r,f)+S(r,f).\end{aligned}\tag{2.11}$$

On the other hand, using Lemma 2.3, we have

$$\begin{aligned}T(r,F) &\le nT(r,f)+T\big(r,f^{(k)}(z+c)\big)+S(r,f)\\ &\le nT(r,f)+T(r,f)+k\overline{N}(r,f)+S(r,f)\\ &\le (n+k+1)T(r,f)+S(r,f).\end{aligned}\tag{2.12}$$

Thus, (2.9) follows from (2.11) and (2.12). If f is a transcendental entire function with $\rho_2(f)<1$, then

$$\begin{aligned}(n+1)T(r,f)+S(r,f) &= T\big(r,f(z)^{n+1}\big)=m\big(r,f(z)^{n+1}\big)\\ &\le m\left(r,\frac{Ff(z)}{f^{(k)}(z+c)}\right)\\ &\le T(r,F)+T\left(r,\frac{f^{(k)}(z+c)}{f(z)}\right)+S(r,f)\\ &\le T(r,F)+N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\le T(r,F)+T(r,f)+S(r,f).\end{aligned}\tag{2.13}$$

Thus, (2.10) follows from (2.12) and (2.13). □

Using the similar method as the proof of Lemma 2.6, we get the following result, which is important in the proof of Theorem 1.5.

Lemma 2.7 *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f)<1$, and let $G(z)=f(z)^n(\Delta_c f)^{(k)}$. Then*

$$(n-2k-2)T(r,f)+S(r,f)\le T(r,G)\le (n+2k+2)T(r,f)+S(r,f).\tag{2.14}$$

If $f(z)$ is a transcendental entire function of $\rho_2(f)<1$, then

$$nT(r,f)+S(r,f)\le T(r,G)\le (n+1)T(r,f)+S(r,f).\tag{2.15}$$

Remark (1) The right inequality of (2.14) cannot be improved, which can be seen by $f(z) = \frac{1}{1+e^z}$, $e^c = -1$, thus $f(z)^2(\Delta_c f)' = \frac{2e^{3z}+2e^z}{(1+e^z)^4(1-e^z)^2}$, which implies that $T(r, f(z)^2(\Delta_c f)') = 6T(r,f) + S(r,f)$.

(2) Inequality (2.15) cannot be improved. If $f(z) = e^z + z^2$, $e^c = 1$, thus $f(z)^n(\Delta_c f)' = 2c(e^z + z^2)^n$, which implies that $T(r, f(z)^n(\Delta_c f)') = nT(r,f) + S(r,f)$. If $f(z) = e^z$, $e^c = 2$, thus $f(z)^n(\Delta_c f)' = e^{(n+1)z}$, which implies that $T(r, f(z)^n(\Delta_c f)') = (n+1)T(r,f) + S(r,f)$.

The following two results are due to Yang and Yi, see [2].

Lemma 2.8 [2, Theorem 1.56] *Let f_1, f_2, f_3 be meromorphic functions such that f_1 is not a constant. If $f_1 + f_2 + f_3 = 1$ and if*

$$\sum_{j=1}^{3} N(r, 1/f_j) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < \big(\lambda + o(1)\big)T(r),$$

where $\lambda < 1$ and $T(r) := \max_{1\le j\le 3} T(r, f_j)$, then either $f_2 = 1$ or $f_3 = 1$.

Lemma 2.9 [2, Theorem 1.52] *If $f_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \ge 2)$, $g_j(z)$ $(j = 1, 2, \ldots, n)$ are entire functions satisfying*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$,

(ii) *the order of f_j is less than that of $e^{g_h(z)-g_k(z)}$ for $1 \le j \le n$, $1 \le h < k \le n$, then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.*

3 Proofs of Theorem 1.1 and Theorem 1.4

Denote $F(z) = f(z)^n f^{(k)}(z + c)$. From Lemma 2.6, then $F(z)$ is not a constant. Assume that $F(z) - a(z)$ has only finitely many zeros, from the second main theorem for three small functions [1, Theorem 2.5] and Lemma 2.5, then we get

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - a(z)}\right) + S(r,F) \\ &\le \overline{N}(r,f) + \overline{N}\big(r, f^{(k)}(z+c)\big) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}(z+c)}\right) + S(r,f) \\ &\le (k+4)T(r,f) + S(r,f). \end{aligned} \tag{3.1}$$

From Lemma 2.6, if $f(z)$ is a transcendental meromorphic function, we get

$$(n - k - 1)T(r,f) \le (k+4)T(r,f) + S(r,f),$$

which is a contradiction with $n \ge 2k + 6$. If $f(z)$ is a transcendental entire function, we get

$$nT(r,f) \le T(r,F) \le 2T(r,f) + S(r,f),$$

which is a contradiction with $n \ge 3$.

4 Proof of Theorem 1.2

From Theorem 1.1, we just need to prove the case that $n = 2$. Suppose contrary to the assertion that $f(z)^2 f^{(k)}(z + c) - a(z)$ has finitely many zeros, where $a(z)$ is an entire function with $\rho(a) < \rho(f)$. Then from the Hadamard factorization theorem, we have

$$f(z)^2 f^{(k)}(z+c) - a(z) = H(z)e^{Q(z)}, \tag{4.1}$$

where $H(z)$, $Q(z)$ are non-zero polynomials and $\deg Q \leq \rho(f)$. Differentiating (4.1) and eliminating $e^{Q(z)}$, we obtain

$$f(z)F(z,f) = q^*(z), \tag{4.2}$$

where

$$F(z,f) = 2f'(z)f^{(k)}(z+c)H(z) + H(z)f(z)f^{(k+1)}(z+c) - p^*(z)f(z)f^{(k)}(z+c),$$

and

$$p^*(z) = H'(z) + H(z)Q'(z), \qquad q^*(z) = a'(z)H(z) - a(z)H'(z) - a(z)H(z)Q'(z).$$

We affirm that $F(z,f)$ cannot vanish identically. Indeed, if $F(z,f) \equiv 0$, then

$$a'(z)H(z) - a(z)H'(z) - a(z)H(z)Q'(z) \equiv 0,$$

which implies that

$$\frac{a'(z)}{a(z)} - \frac{H'(z)}{H(z)} = Q'(z).$$

By integrating the above equation, we have

$$\frac{a(z)}{H(z)} = Ae^{Q(z)},$$

where A is a non-zero constant. Since $a(z)$ is an entire function with $\rho(a) < \rho(f)$ and $H(z)$ is a non-zero polynomial. Thus, we get $\deg Q < \rho(f)$. Therefore $f(z)^2 f^{(k)}(z+c) = (A+1)H(z)e^{Q(z)}$, from Lemma 2.1 and Lemma 2.5 we get

$$\begin{aligned} 2T(r,f) &= T\left(r, \frac{H(z)e^{Q(z)}}{f^{(k)}(z+c)}\right) + O(1) \\ &= O\left(r^{\rho(f)-1+\varepsilon}\right) + T(r,f) + S(r,f), \end{aligned} \tag{4.3}$$

which contradicts the assumption that $f(z)$ is transcendental of finite order $\rho(f)$. From (4.2), we get

$$T\big(r, f(z)F(z,f)\big) = S(r,f). \tag{4.4}$$

From the Clunie lemma [19, Theorem 2.4.2], we get

$$m\big(r, F(z,f)\big) = S(r,f), \tag{4.5}$$

obviously, $F(z,f)$ is an entire function. Thus, from (4.4) and (4.5), we get $T(r,f) = S(r,f)$, which is a contradiction.

5 Proof of Theorem 1.3

If $d(z) \not\equiv 0$ is a Borel exceptional polynomial of $f(z)$, thus the value 1 is a Borel exceptional value of $\frac{f(z)}{d(z)}$, if $d(z) \equiv 0$, then the value 0 is a Borel exceptional value of $f(z)$, then $f(z)$ must have positive integer order [2, p.106, Corollary]. Without loss of generality, assume that $\rho(f) = s$, s is a positive integer, then the transcendental entire function $f(z)$ can be written as $f(z) = d(z) + h(z)e^{\alpha z^s}$, where α is a nonzero constant and $h(z)$ is a nonzero entire function with $\lambda(h) \le \rho(h) < \rho(f) = s$. Hence,

$$f(z+c) = d(z+c) + h(z+c)e^{\alpha(z+c)^s} = d(z+c) + h_1(z)e^{\alpha z^s}, \tag{5.1}$$

where

$$h_1(z) = h(z+c)e^{\alpha(C_s^1 z^{s-1}c + C_s^2 z^{s-2}c^2 + \cdots + C_s^{s-1}zc^{s-1} + c^s)}. \tag{5.2}$$

From Theorem 1.2, we need to prove the case of $n = 1$ only. Suppose that $f(z)f^{(k)}(z+c) - b$ has finitely many zeros, from the Hadamard factorization theorem, then we assume that

$$f(z)f^{(k)}(z+c) - b = A(z)e^{\beta z^s}, \tag{5.3}$$

where $A(z)$ is an entire function with order $\rho(A) < s$ and has finitely many zeros, β is a nonzero constant. Thus, we get

$$\bigl(d(z) + h(z)e^{\alpha z^s}\bigr)\bigl[d(z+c) + h_1(z)e^{\alpha z^s}\bigr]^{(k)} - b = A(z)e^{\beta z^s}. \tag{5.4}$$

Let $B_1(z) = s\alpha h_1(z)z^{s-1} + h_1'(z)$, $B_2(z) = s\alpha B_1(z)z^{s-1} + B_1'(z), \ldots, B_k(z) = s\alpha B_{k-1}(z)z^{s-1} + B_{k-1}'(z)$. Thus, we have

$$\bigl[h(z)d^{(k)}(z+c) + d(z)B_k(z)\bigr]e^{\alpha z^s} + h(z)B_k(z)e^{2\alpha z^s} - A(z)e^{\beta z^s} = b - d(z)d^{(k)}(z+c). \tag{5.5}$$

Since $d(z)$ is a nonzero polynomial, then we get $d(z)d^{(k)}(z+c) - b \not\equiv 0$. Let $f_1 = \frac{[h(z)d^{(k)}(z+c)+d(z)B_k(z)]e^{\alpha z^s}}{b-d(z)d^{(k)}(z+c)}$, $f_2 = \frac{h(z)B_k(z)e^{2\alpha z^s}}{b-d(z)d^{(k)}(z+c)}$, $f_3 = \frac{-A(z)e^{\beta z^s}}{b-d(z)d^{(k)}(z+c)}$. Thus, we get $f_1 + f_2 + f_3 = 1$. Since $\rho(h(z)) < s$, $\rho(A(z)) < s$, which implies that $\rho(B_k(z)) < s$, we get f_1, f_2, f_3 are not constants, which is a contradiction with Lemma 2.8. The proof of Theorem 1.3 is completed.

6 Proof of Theorem 1.6

Let $\psi := \frac{a(z)f^{(k)}(z+c)+b(z)}{f(z)^n}$. We proceed to proving that $\psi + 1$ has infinitely many zeros, thus $f(z)^n + a(z)f^{(k)}(z+c) + b(z)$ has infinitely many zeros. If f is a transcendental entire function with finite order, we will prove

$$T(r,\psi) \ge (n-1)T(r,f) + S(r,f). \tag{6.1}$$

Applying the first main theorem and Lemma 2.3, Lemma 2.5, we observe that

$$\begin{aligned} T\bigl(r,f(z)^n\bigr) &= T\left(r,\psi\cdot\frac{1}{a(z)f^{(k)}(z+c)+b(z)}\right) + O(1) \\ &\le T(r,\psi) + T\bigl(r,a(z)f^{(k)}(z+c)+b(z)\bigr) + O(1) \\ &\le T(r,\psi) + T(r,f) + S(r,f). \end{aligned} \tag{6.2}$$

From (6.2), we easily obtain (6.1). We will estimate the zeros and poles of ψ,

$$\overline{N}(r,\psi) \le \overline{N}\left(r,\frac{1}{f}\right) + S(r,f), \tag{6.3}$$

and

$$\overline{N}\left(r,\frac{1}{\psi}\right) \le \overline{N}\left(r,\frac{1}{a(z)f^{(k)}(z+c)+b(z)}\right) + S(r,f). \tag{6.4}$$

Using the second main theorem, Lemma 2.3, Lemma 2.5, we get

$$\begin{aligned}(n-1)T(r,f) &\le T(r,\psi) + S(r,f)\\ &\le \overline{N}(r,\psi) + \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{\psi+1}\right) + S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\psi+1}\right)\\ &\quad + \overline{N}\left(r,\frac{1}{a(z)f^{(k)}(z+c)+b(z)}\right) + S(r,f)\\ &\le 2T(r,f) + \overline{N}\left(r,\frac{1}{\psi+1}\right) + S(r,f).\end{aligned} \tag{6.5}$$

Since $n \ge 4$, then (6.5) implies that $\psi + 1$ has infinitely many zeros. In what follows, we will prove that if f is a transcendental entire function with finite order and $b(z) \not\equiv 0$, then n can be reduced to $n \ge 3$.

We suppose that $f(z)^n + a(z)f^{(k)}(z+c) + b(z)$ has finitely many zeros, from the Hadamard factorization theorem, then there exist two polynomials $r(z)$ and $p(z)$ such that

$$f(z)^n + a(z)f^{(k)}(z+c) + b(z) = r(z)e^{p(z)}. \tag{6.6}$$

Differentiating (6.6) and eliminating $e^{p(z)}$, we obtain

$$\begin{aligned}&f(z)^{n-1}\left(nf'(z) - \left(p'(z) + \frac{r'(z)}{r(z)}\right)f(z)\right) + a'(z)f^{(k)}(z+c) + a(z)f^{(k+1)}(z+c) + b'(z)\\ &\quad = \left(p'(z) + \frac{r'(z)}{r(z)}\right)\left(a(z)f^{(k)}(z+c) + b(z)\right).\end{aligned} \tag{6.7}$$

If $nf' - (p' + r'/r)f \equiv 0$, then $f(z)^n = Cr(z)e^{p(z)}$. Thus, from (6.6), we get

$$\frac{C-1}{C}f(z)^n + a(z)f^{(k)}(z+c) + b(z) = 0. \tag{6.8}$$

Thus $C = 1$, otherwise, $nT(r,f) = T(r,a(z)f^{(k)}(z+c)+b(z)) \le T(r,f) + S(r,f)$, which is a contradiction with $n \ge 3$. Hence $a(z)f^{(k)}(z+c) + b(z) = 0$, which is also a contradiction. Thus $nf' - (p' + r'/r)f \not\equiv 0$. Since $n \ge 3$, we may apply the Clunie lemma [19, Lemma 2.4.2], Lemma 2.4 and (6.7) to conclude that

$$T\left(r, nf' - \left(p' + \frac{r'}{r}\right)f\right) = S(r,f)$$

and

$$T\left(r,f\left(nf' - \left(p' + \frac{r'}{r}\right)f\right)\right) = S(r,f).$$

Combining the above two estimates, we obtain $T(r,f) = S(r,f)$, a contradiction.

It remains to prove the case $n = 2$ and $b(z) = 0$. Thus (6.7) now takes the form

$$\begin{aligned} &f(z)\left(2f'(z) - \left(p'(z) + \frac{r'(z)}{r(z)}\right)f(z)\right) \\ &\quad = -a'(z)f^{(k)}(z+c) - a(z)f^{(k+1)}(z+c) + \left(p'(z) + \frac{r'(z)}{r(z)}\right)\left(a(z)f^{(k)}(z+c)\right). \end{aligned} \tag{6.9}$$

Similarly as the case $n \geq 3$, we also conclude that $\phi := 2f' - (p' + r'/r)f \not\equiv 0$. We have

$$T\left(r,\phi(z)\right) = S(r,f).$$

Differentiating $\phi(z)$, we obtain

$$2f'' - \left(p' + \frac{r'}{r}\right)f' - \left(p' + \frac{r'}{r}\right)'f = \phi' = \frac{\phi'}{\phi}\phi = \frac{\phi'}{\phi}\left(2f' - \left(p' + \frac{r'}{r}\right)f\right),$$

and so

$$2f'' - \left(p' + \frac{r'}{r} + 2\frac{\phi'}{\phi}\right)f' - \left(p'' - p'\frac{\phi'}{\phi} + \left(\frac{r'}{r}\right)' - \frac{r'}{r}\frac{\phi'}{\phi}\right)f = 0.$$

This can be written as

$$2\left(\frac{f'}{f}\right)' + 2\left(\frac{f'}{f}\right)^2 - \left(p' + \frac{r'}{r} + 2\frac{\phi'}{\phi}\right)\frac{f'}{f} - \left(p'' + \left(\frac{r'}{r}\right)' - \left(p + \frac{r'}{r}\right)\frac{\phi'}{\phi}\right) = 0. \tag{6.10}$$

We proceed to show that $N(r,1/f) = S(r,f)$. Suppose that z_0 is a zero of f with multiplicity k. If $k \geq 2$, then z_0 is a zero of ϕ, the contribution to $N(r,1/f)$ is $S(r,f)$. If the zero of f is simple and we must have that $p' + \frac{r'}{r} + 2\frac{\phi'}{\phi}$ vanishes at z_0, which implies $N(r,\frac{1}{f}) = S(r,f)$. Therefore, we can assume that $f(z)$ takes the form $f(z) = \varphi(z)e^{\alpha z^s}$, where $q(z)$ is a polynomial and $N(r,\frac{1}{\varphi}) = S(r,f)$ and $\rho(\varphi) < s$. Substituting this expression into (6.6), we obtain

$$\varphi(z)^2e^{2\alpha z^s} + a(z)\left[\varphi(z+c)e^{\alpha(z+c)^s}\right]^{(k)} = r(z)e^{p(z)} := g(z)e^{\beta z^s},$$

where $g(z)$ is an entire function with $\rho(g) < s$, β is a constant. If $\beta = 2\alpha$, which implies that $[\varphi(z+c)e^{\alpha(z+c)^s}]^{(k)} \equiv 0$, which is impossible. If $\beta = \alpha$, then $\varphi(z) \equiv 0$, which also is impossible. Thus, $\beta \neq 2\alpha$ and $\beta \neq \alpha$, from Lemma 2.9, we get $\varphi(z)^2e^{2\alpha z^s} \equiv 0$, $a(z)[\varphi(z+c)e^{\alpha(z+c)^s}]^{(k)} \equiv 0$ and $g(z)e^{\beta z^s} \equiv 0$, which are impossible. Thus, we have completed the proof of Theorem 1.6.

Remark 4 Inequality (6.1) is not valid for $f(z)$ is a transcendental meromorphic function, which can be seen by $f(z) = \tan z$, thus $\psi := \frac{\tan'(z+\pi)-1}{\tan^n z} = \frac{1}{\tan^{n-2} z}$. Thus, $T(r,\psi) = (n-2)T(r,f) + S(r,f)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have achieved equal contributions to this paper. All authors read and approved the final version of the manuscript.

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