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Unbounded solution for a fractional boundary value problem

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Abstract
This paper concerns the existence of unbounded positive solutions of a fractional boundary value problem on the half line. By means of the properties of the Green function and the compression and expansion fixed point theorem (Kwong in Fixed Point Theory Appl. 2008:164537, 2008), sufficient conditions are obtained to guarantee the existence of a solution to the posed problem.
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1 Introduction

In this paper, we investigate the existence of solutions for a fractional boundary value problem (P) on the half line:

$$\begin{cases} {}^cD_{0^+}^q u(t) = f(t, u(t), u'(t)), \quad t > 0, \\ u(0) = u''(0) = 0, \qquad \lim_{t\to\infty} {}^cD_{0^+}^{q-1} u(t) = \alpha u(1), \end{cases} \tag{P}$$

where $f : [0, \infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $2 < q < 3$, $\alpha > 0$, ${}^cD_{0^+}^q$ denotes the Caputo fractional derivative. Note that few papers in the literature dealing with fractional differential equations considered the nonlinearity f in (P) depending on the derivative of u.

Since many problems in the natural sciences require a notion of positivity (only nonnegative densities, population sizes or probabilities make sense in real life), in the present study we discuss the existence of positive solutions for the problem (P). The proofs of the main results are based on the properties of the associated Green function, Leray-Schauder nonlinear alternative and Guo-Krasnosel'skii fixed point theorem on cone. Different methods are applied to investigate such boundary value problems, we can cite fixed point theory, topological degree methods, Mawhin theory, upper and lower solutions…; see [1–13].

Fractional boundary value problems on infinite intervals often appear in applied mathematics and physics. They can model some physical phenomena, such as the models of gas pressure in a semi-infinite porous medium; see [13]. The population growth model can also be characterized by a nonlinear fractional Volterra integrodifferential equation on the half line [14]. For more results on fractional differential equations in science and engineering and their applications we refer to [15–17].

Fractional boundary value problems on infinite intervals have been investigated by many authors; see [1, 2, 4, 5, 8, 11–13]. In [12], the authors proved the existence of unbounded solutions for the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t > 0, \\ u(0) = 0, \qquad \lim_{t\to\infty} D_{0^+}^{\alpha-1} u(t) = \alpha u(\eta), \end{cases}$$

by using Leray-Schauder nonlinear alternative. Here $1 < \alpha \leq 2$, and $D_{0^+}^{q}$ denotes the Riemann-Liouville fractional derivative.

In [5] by means of fixed point theorem on cone, the authors discussed the existence of multiple positive solutions for m-point fractional boundary value problems with p-Laplacian operator on infinite interval.

Further, in [13], the authors studied the following second order nonlinear differential equation on the half line:

$$\begin{cases} u''(t) + q(t) f(t, u(t), u'(t)) = 0, \quad t > 0, \\ \alpha u(0) - \beta u'(0) = 0, \qquad u'(\infty) = u_\infty \geq 0. \end{cases}$$

Applying a fixed point theorem and the monotone iterative technique, they proved the existence of positive solution.

The organization of this paper is as follows. In Section 2, we provide necessary background and properties of the Green function. The existence result is established under some sufficient conditions on the nonlinear term f. Section 3 is devoted to the existence of positive solutions on a cone. We conclude the paper with some examples.

2 Existence results

For the convenience of the readers, we first present some useful definitions and fundamental facts of fractional calculus theory, which can be found in [15, 18].

Definition 1 The Riemann-Liouville fractional integral of order α of a function g is defined by

$$I_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}}\, ds,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\, dt$ is the Gamma function, $\alpha > 0$.

Definition 2 The Caputo fractional derivative of order q of a function g is defined by

$$^{c}D_{a^+}^{q} g(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{q-n+1}}\, ds,$$

where $n = [q] + 1$ ($[q]$ is the entire part of q).

Lemma 3 *For $q > 0$, $g \in C([0,1])$, the homogeneous fractional differential equation $^{c}D_{a^+}^{q} g(t) = 0$ has a solution*

$$g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, and $n = [q] + 1$.

Lemma 4 *Let* $p, q \geq 0, f \in L_1[a,b]$. *Then* $I_{0^+}^p I_{0^+}^q f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^q I_{0^+}^p f(t)$ *and* ${}^cD_{a^+}^q I_{0^+}^q f(t) = f(t)$, *for all* $t \in [a,b]$.

Lemma 5 *Let* $\alpha, \beta > 0$ *and* $n = [\alpha] + 1$, *then the following relations hold*:

$$\begin{aligned} &{}^cD_{a^+}^\alpha t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \quad \beta > n, \\ &{}^cD_{a^+}^\alpha t^k = 0, \quad k = 0, 1, 2, \ldots, n-1. \end{aligned}$$

To prove the main results of this paper we need the following lemma.

Lemma 6 *Let* $y \in C(\mathbb{R}_+, \mathbb{R})$ *with* $\int_0^\infty y(s)\,ds < \infty$, *the linear nonhomogeneous boundary value problem*

$$\begin{aligned} &{}^cD_{a^+}^q u(t) = y(t), \quad t > 0, \\ &u(0) = u''(0) = 0, \qquad \lim_{t\to\infty} {}^cD_{0^+}^{q-1} u(t) = \alpha u(1), \end{aligned}$$

has a unique solution

$$u(t) = \int_0^\infty G(t,s) y(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\alpha} - \frac{t}{\Gamma(q)}(1-s)^{q-1}, & s \leq \min(t,1), \\ \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\alpha}, & 1 \leq s \leq t, \\ \frac{t}{\alpha} - \frac{t}{\Gamma(q)}(1-s)^{q-1}, & t \leq s \leq 1, \\ \frac{t}{\alpha}, & s \geq \max(t,1). \end{cases}$$

Proof By Lemmas 3 and 4, we obtain

$$u(t) = I_{0^+}^q y(t) + a + bt + ct^2.$$

The boundary conditions $u(0) = u''(0) = 0$, imply that

$$u(t) = I_{0^+}^q y(t) + bt;$$

applying Lemma 5 and the condition $\lim_{t\to\infty} {}^cD_{0^+}^{q-1} u(t) = \alpha u(1)$, we obtain

$$\begin{aligned} \lim_{t\to\infty} {}^cD_{0^+}^{q-1} u(t) &= \lim_{t\to\infty} \left(I_{0^+} y(t) + {}^cD_{0^+}^{q-1}(bt) \right) \\ &= \lim_{t\to\infty} I_{0^+} y(t) = \int_0^\infty y(s)\,ds, \\ u(1) = I_{0^+}^q y(1) + b, \end{aligned}$$

consequently

$$b = \frac{1}{\alpha} \int_0^\infty y(s)\,ds - I_{0^+}^q y(1) = \frac{1}{\alpha} \int_0^\infty y(s)\,ds - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} y(s)\,ds,$$

substituting b by its value, it yields

$$u(t) = \frac{1}{\Gamma(q)}\left[\int_0^t (t-s)^{q-1}y(s)\,ds - t\int_0^1 (1-s)^{q-1}y(s)\,ds\right] + \frac{t}{\alpha}\int_0^\infty y(s)\,ds.$$

The proof is complete. □

Lemma 7 *Assume that* $0 < \alpha \leq \Gamma(q)$, *then for all* $s, t \geq 0$ *we have*

$$0 \leq \frac{G(t,s)}{1+t^{q-1}} \leq \frac{2}{\alpha}, \qquad 0 \leq \frac{G_t(t,s)}{1+t^{q-1}} \leq \frac{4}{\alpha\Gamma(q-1)}.$$

Proof Simple computations give

$$G_t(t,s) = \begin{cases} \frac{(t-s)^{q-2}}{\Gamma(q-1)} + \frac{1}{\alpha} - \frac{1}{\Gamma(q)}(1-s)^{q-1}, & s \leq \min(t,1), \\ \frac{(t-s)^{q-2}}{\Gamma(q-1)} + \frac{1}{\alpha}, & 1 \leq s \leq t, \\ \frac{1}{\alpha} - \frac{1}{\Gamma(q)}(1-s)^{q-1}, & t \leq s \leq 1, \\ \frac{1}{\alpha}, & s \geq \max(t,1). \end{cases}$$

Let us consider the case $s \leq \min(t,1)$, then we get

$$\begin{aligned} G(t,s) &\geq \frac{t}{\alpha} - \frac{t}{\Gamma(q)}(1-s)^{q-1} \geq t\left(\frac{\Gamma(q)-\alpha}{\alpha\Gamma(q)}\right) \geq 0, \\ G_t(t,s) &\geq \frac{1}{\alpha} - \frac{1}{\Gamma(q)}(1-s)^{q-1} \geq 0. \end{aligned}$$

Firstly if $s \leq t \leq 1$, then

$$\begin{aligned} \frac{G(t,s)}{1+t^{q-1}} &\leq \frac{(t-s)^{q-1}+t}{\alpha(1+t^{q-1})} \leq \frac{2}{\alpha}, \\ \frac{G_t(t,s)}{1+t^{q-1}} &\leq \frac{\alpha(t-s)^{q-2}+\Gamma(q-1)}{\alpha(1+t^{q-1})\Gamma(q-1)} \leq \frac{\alpha+\Gamma(q-1)}{\alpha\Gamma(q-1)(1+t^{q-1})} \leq \frac{4}{\alpha\Gamma(q-1)}. \end{aligned}$$

Secondly if $s \leq 1 \leq t$, then

$$\begin{aligned} \frac{G(t,s)}{1+t^{q-1}} &\leq \frac{(t-s)^{q-1}+t}{\alpha(1+t^{q-1})} \leq \frac{t^{q-1}+t}{\alpha(1+t^{q-1})} \leq \frac{2t^{q-1}}{\alpha(1+t^{q-1})} \leq \frac{2}{\alpha}, \\ \frac{G_t(t,s)}{1+t^{q-1}} &\leq \frac{\alpha(t-s)^{q-2}+\Gamma(q-1)}{\alpha(1+t^{q-1})\Gamma(q-1)} \leq \frac{4}{\alpha\Gamma(q-1)}. \end{aligned}$$

Applying the same techniques to the other cases, the conclusion follows. □

In this paper, we will use the Banach space E defined by

$$E = \left\{u \in C^1(\mathbb{R}_+,\mathbb{R}), \lim_{t\to\infty}\frac{u(t)}{1+t^{q-1}} < \infty, \lim_{t\to\infty}\frac{u'(t)}{1+t^{q-1}} < \infty\right\}$$

and equipped with the norm $\|u\| = \max(\|u\|_\infty, \|u'\|_\infty)$, where $\|u\|_\infty = \sup_{t\geq 0}\frac{|u(t)|}{1+t^{q-1}}$ and $\mathbb{R}_+ = [0,\infty[$.

Define the integral operator $T:E\to E$ by

$$Tu(t)=\int_0^{\infty} G(t,s)f\bigl(s,u(s),u'(s)\bigr)\,ds,$$

so we have transformed the problem (P) to a Hammerstein integral equation by using the Green function.

Lemma 8 *The function $u\in E$ is solution of the boundary value problem* (P) *if and only if $Tu(t)=u(t)$, for all $t\in\mathbb{R}_+$.*

From this we see that to solve the problem (P) it remains to prove that the map T has a fixed point in E. Since the Arzela-Ascoli theorem cannot be applied in this situation, then, to prove that T is completely continuous, we need the following compactness criterion:

Lemma 9 [19] *Let $V=\{u\in C_\infty, \|u\|<l$, where $l>0\}$, $V(t)=\{\frac{u(t)}{1+t^{q-1}}, u\in V\}$, $V'(t)=\{\frac{u'(t)}{1+t^{q-1}}, u\in V\}$. V is relatively compact in E, if $V(t)$ and $V'(t)$ are both equicontinuous on any finite subinterval of $\mathbb{R}_+$ and equiconvergent at ∞, that is for any $\varepsilon>0$, there exists $\eta=\eta(\varepsilon)>0$ such that*

$$\left|\frac{u(t_1)}{1+t_1^{q-1}}-\frac{u(t_2)}{1+t_2^{q-1}}\right|<\varepsilon,\qquad \left|\frac{u'(t_1)}{1+t_1^{q-1}}-\frac{u'(t_2)}{1+t_2^{q-1}}\right|<\varepsilon,$$

$\forall u\in V$, $t_1,t_2\geq\eta$ (uniformly according to u).

We recall that a continuous mapping F from a subset M of a normed space X into another normed space Y is called completely continuous iff F maps bounded subset of M into relatively compact subset of Y.

Lemma 10 *Assume that $f\in C(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $f(t,0,0)\neq 0$ on any subinterval of $\mathbb{R}_+$ and there exist non-negative functions $h,k\in L^1(\mathbb{R}_+,\mathbb{R}_+)$ and $\psi_1,\psi_2\in C(\mathbb{R},\mathbb{R}_+^*)$ nondecreasing on $\mathbb{R}_+$, such that*

$$\bigl|f\bigl(t,(1+t^{q-1})x,(1+t^{q-1})\overline{x}\bigr)\bigr|\leq k(t)\psi_1\bigl(|x|\bigr)+h(t)\psi_2\bigl(|\overline{x}|\bigr),\quad \forall(t,x,\overline{x})\in\mathbb{R}_+\times\mathbb{R}^2, \tag{2.1}$$

then T is completely continuous. (*Here* $\mathbb{R}_+^*=]0,\infty[$.)

Proof The proof will be done in some steps.

Step 1: T is continuous. Let $(u_n)_{n\in N}\in E$ be a convergent sequence to u in E. Let $r_1>\max(\|u\|_\infty,\sup\|u_n\|_\infty)$ and $r_2>\max(\|u'\|_\infty,\sup\|u_n'\|_\infty)$, then we obtain with the help of Lemma 7, hypothesis (2.1) and some elementary inequalities

$$\begin{aligned}
&\int_0^{\infty}\frac{G(t,s)}{1+t^{q-1}}\bigl|f\bigl(s,u(s),u'(s)\bigr)-f\bigl(s,u_n(s),u_n'(s)\bigr)\bigr|\,ds\\
&\quad\leq\frac{2}{\alpha}\int_0^{\infty}\left|f\left(s,\frac{(1+t^{q-1})u(s)}{1+t^{q-1}},\frac{(1+t^{q-1})u'(s)}{1+t^{q-1}}\right)\right|\\
&\qquad-\left|f\left(s,\frac{(1+t^{q-1})u_n(s)}{1+t^{q-1}},\frac{(1+t^{q-1})u_n'(s)}{1+t^{q-1}}\right)\right|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2}{\alpha}\int_0^\infty \left(\left|f\left(s, \frac{(1+t^{q-1})u(s)}{1+t^{q-1}}, \frac{(1+t^{q-1})u'(s)}{1+t^{q-1}}\right)\right|\right. \\
&\quad \left.+\left|f\left(s, \frac{(1+t^{q-1})u_n(s)}{1+t^{q-1}}, \frac{(1+t^{q-1})u_n'(s)}{1+t^{q-1}}\right)\right|\right) ds \\
&\leq \frac{2}{\alpha}\int_0^\infty k(s)\psi_1\left(\frac{u(s)}{1+t^{q-1}}\right) + h(s)\psi_2\left(\frac{u'(s)}{1+t^{q-1}}\right) ds \\
&\quad + \frac{2}{\alpha}\int_0^\infty k(s)\psi_1\left(\frac{u_n(s)}{1+t^{q-1}}\right) + h(s)\psi_2\left(\frac{u_n'(s)}{1+t^{q-1}}\right) ds \\
&\leq \frac{4}{\alpha}\left(\psi_1(r_1)\int_0^\infty k(s)\,ds + \psi_2(r_2)\int_0^\infty h(s)\,ds\right) < \infty.
\end{aligned}
$$

Using similar techniques we prove that

$$
\int_0^\infty \frac{G_t(t,s)}{1+t^{q-1}}\left|f\left(s,u(s),u'(s)\right) - f\left(s,u_n(s),u_n'(s)\right)\right| ds < \infty,
$$

hence the integrals are convergent. With the help of Lebesgue dominated convergence theorem and the fact that f is continuous we get

$$
\begin{aligned}
\|Tu_n - Tu\|_\infty &= \sup_{t\geq 0}\int_0^\infty \frac{G(t,s)}{1+t^{q-1}}\left|f\left(s,u(s),u'(s)\right) - f\left(s,u_n(s),u_n'(s)\right)\right| ds \\
&\to 0 \quad \text{as } n\to\infty, \\
\left\|T'u_n - T'u\right\|_\infty &= \sup_{t\geq 0}\int_0^\infty \frac{G_t(t,s)}{1+t^{q-1}}\left|f\left(s,u(s),u'(s)\right) - f\left(s,u_n(s),u_n'(s)\right)\right| ds \\
&\to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

therefore

$$
\|Tu_n - Tu\| \to 0, \quad \text{as } n\to\infty.
$$

Step 2: T is relatively compact. Let $B_r = \{u\in E, \|u\| < r\}$, first let us show that TB_r is uniformly bounded. Let $u\in B_r$, taking (2.1) into account and the fact that ψ_1 and ψ_2 are nondecreasing on $\mathbb{R}_+$, it yields

$$
\begin{aligned}
\frac{|Tu(t)|}{1+t^{q-1}} &\leq \frac{2}{\alpha}\int_0^\infty \left|f\left(s,u(s),u'(s)\right)\right| ds \\
&= \frac{2}{\alpha}\int_0^\infty \left|f\left(s, \left(1+s^{q-1}\right)\frac{u(s)}{1+s^{q-1}}, \left(1+s^{q-1}\right)\frac{u'(s)}{1+s^{q-1}}\right)\right| ds \\
&\leq \frac{2}{\alpha}\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right).
\end{aligned}
$$

Consequently

$$
\|Tu\|_\infty \leq \frac{2}{\alpha}\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right). \tag{2.2}
$$

Similarly, we prove that

$$\|T'u\|_\infty \leq \frac{4}{\alpha\Gamma(q-1)}\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right); \tag{2.3}$$

this along with (2.3) yields

$$\|Tu\| \leq \frac{4}{\alpha\Gamma(q-1)}\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right) \tag{2.4}$$

thus TB_r is uniformly bounded.

Next, we show that TB_r is equicontinuous on any compact interval of $\mathbb{R}_+$. Let $u \in B_r$, $t_1, t_2 \in [a,b]$, $0 \leq a < b < \infty$, $t_1 \leq t_2$ we have

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{q-1}} - \frac{Tu(t_1)}{1+t_1^{q-1}}\right| \\
&\quad \leq \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{q-1}} - \frac{G(t_1,s)}{1+t_1^{q-1}}\right| \left|f\left(s,u(s),u'(s)\right)\right| ds \\
&\quad \leq \int_0^\infty \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{q-1}} \left|f\left(s,u(s),u'(s)\right)\right| ds \\
&\qquad + \int_0^\infty \frac{G(t_1,s)(t_2^{q-1}-t_1^{q-1})}{(1+t_1^{q-1})(1+t_2^{q-1})} \left|f\left(s,u(s),u'(s)\right)\right| ds \\
&\quad \leq \frac{2}{\alpha}\left(\frac{(t_2-t_1)}{1+t_2^{q-1}} + \frac{(t_2^{q-1}-t_1^{q-1})}{(1+t_1^{q-1})(1+t_2^{q-1})}\right)\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right),
\end{aligned}$$

which approaches zero uniformly when $t_1 \to t_2$. On the other hand we have

$$\begin{aligned}
&\left|\frac{T'u(t_2)}{1+t_2^{q-1}} - \frac{T'u(t_1)}{1+t_1^{q-1}}\right| \\
&\quad \leq \int_0^\infty \left|\frac{G_t(t_2,s)}{1+t_2^{q-1}} - \frac{G_t(t_1,s)}{1+t_1^{q-1}}\right| \left|f\left(s,u(s),u'(s)\right)\right| ds \\
&\quad \leq \int_0^\infty \frac{|G_t(t_2,s)-G_t(t_1,s)|}{1+t_2^{q-1}} \left|f\left(s,u(s),u'(s)\right)\right| ds \\
&\qquad + \int_0^\infty \frac{G_t(t_1,s)(t_2^{q-1}-t_1^{q-1})}{(1+t_1^{q-1})(1+t_2^{q-1})} \left|f\left(s,u(s),u'(s)\right)\right| ds.
\end{aligned} \tag{2.5}$$

Let us estimate the second integral on the right hand side of the inequality (2.5):

$$\begin{aligned}
&\int_0^\infty \frac{G_t(t_1,s)(t_2^{q-1}-t_1^{q-1})}{(1+t_1^{q-1})(1+t_2^{q-1})} \left|f\left(s,u(s),u'(s)\right)\right| ds \\
&\quad \leq \frac{4}{\alpha\Gamma(q-1)} \frac{(t_2^{q-1}-t_1^{q-1})}{(1+t_1^{q-1})(1+t_2^{q-1})}\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right),
\end{aligned}$$

which approaches zero uniformly when $t_1 \to t_2$. Now we analyze the first integral on the right hand side of inequality (2.5) in different cases when the compact $[a,b]$ contains 1 or not.

If $t_1 < t_2 \leq 1$, then

$$\begin{aligned}
&\int_0^\infty \frac{|G_t(t_2,s) - G_t(t_1,s)|}{1 + t_2^{q-1}} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\quad = \int_0^{t_1} \frac{|(t_2 - s)^{q-2} - (t_1 - s)^{q-2}|}{\Gamma(q-1)(1 + t_2^{q-1})} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\qquad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\quad \leq \frac{(q-2)(t_2 - t_1)}{\Gamma(q-1)(1 + t_2^{q-1})} \left(\psi_1(r) \int_0^{t_1} k(s)\, ds + \psi_2(r) \int_0^{t_1} h(s)\, ds\right) \\
&\qquad + \frac{1}{\Gamma(q-1)(1 + t_2^{q-1})} \left(\psi_1(r) \int_{t_1}^{t_2} k(s)\, ds + \psi_2(r) \int_{t_1}^{t_2} h(s)\, ds\right),
\end{aligned}$$

which approaches zero uniformly when $t_1 \to t_2$.

If $t_1 < 1 \leq t_2$, then

$$\begin{aligned}
&\int_0^\infty \frac{|G_t(t_2,s) - G_t(t_1,s)|}{1 + t_2^{q-1}} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\quad = \int_0^{t_1} \frac{(t_2 - s)^{q-2} - (t_1 - s)^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\qquad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\quad \leq \frac{(t_2 - t_1)^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left(\psi_1(r) \int_0^{t_1} k(s)\, ds + \psi_2(r) \int_0^{t_1} h(s)\, ds\right) \\
&\qquad + \frac{(t_2)^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left(\psi_1(r) \int_{t_1}^{t_2} k(s)\, ds + \psi_2(r) \int_{t_1}^{t_2} h(s)\, ds\right) \\
&\quad \to 0, \quad \text{uniformly as } t_1 \to t_2.
\end{aligned}$$

If $1 \leq t_1 < t_2$, then

$$\begin{aligned}
&\int_0^\infty \frac{|G_t(t_2,s) - G_t(t_1,s)|}{1 + t_2^{q-1}} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\quad = \int_0^{t_1} \frac{(t_2 - s)^{q-2} - (t_1 - s)^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\qquad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left|f\left(s, u(s), u'(s)\right)\right| ds \\
&\quad \leq \frac{(t_2 - t_1)^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left(\psi_1(r) \int_0^{t_1} k(s)\, ds + \psi_2(r) \int_0^{t_1} h(s)\, ds\right) \\
&\qquad + \frac{{t_2}^{q-2}}{\Gamma(q-1)(1 + t_2^{q-1})} \left(\psi_1(r) \int_{t_1}^{t_2} k(s)\, ds + \psi_2(r) \int_{t_1}^{t_2} h(s)\, ds\right) \\
&\quad \to 0, \quad \text{uniformly as } t_1 \to t_2.
\end{aligned}$$

Thus T is equicontinuous on the compact $[a,b]$.

Step 3: T is equiconvergent at ∞. Since

$$\int_0^\infty \left|f\left(s,u(s),u'(s)\right)\right| ds \leq \left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right) < \infty$$

we have

$$\lim_{t\to+\infty}\left|\frac{Tu(t)}{1+t^{q-1}}\right| \leq \frac{1}{\Gamma(q)}\int_0^\infty \left|f\left(s,u(s),u'(s)\right)\right| ds < \infty, \qquad \lim_{t\to+\infty}\left|\frac{T'u(t)}{1+t^{q-1}}\right| = 0,$$

consequently T is equiconvergent at ∞. The proof is complete. □

Now, we can give an existence result.

Theorem 11 *Assume that the hypotheses of Lemma* 10 *hold and that there exists* $r > 0$*, such that*

$$\frac{4}{\alpha\Gamma(q-1)}\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right) < r. \tag{2.6}$$

Then the fractional boundary value problem (P) *has at least one nontrivial solution* $u^* \in E$.

To prove this theorem, we apply the Leray-Schauder nonlinear alternative.

Lemma 12 [20] *Let* F *be a Banach space and* Ω *a bounded open subset of* F, $0 \in \Omega$. *Let* $T : \overline{\Omega} \to F$ *be a completely continuous operator. Then either there exist* $x \in \partial\Omega$, $\lambda > 1$, *such that* $T(x) = \lambda x$, *or there exists a fixed point* $x^* \in \overline{\Omega}$ *of* T.

Proof of Theorem 11 From the proof of Lemma 10, we know that T is a completely continuous operator. Now we apply the nonlinear alternative of Leray-Schauder to prove that T has at least one nontrivial solution in E. Let $u \in \partial B_r$, such that $u = \lambda Tu$, $0 < \lambda < 1$; we get with the help of (2.4):

$$\|u\| = \lambda\|Tu\| \leq \|Tu\| \leq \frac{4}{\alpha\Gamma(q-1)}\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right). \tag{2.7}$$

This together with (2.6) implies

$$r = \|u\| \leq \frac{4}{\alpha\Gamma(q-1)}\left(\psi_1(r)\int_0^\infty k(s)\,ds + \psi_2(r)\int_0^\infty h(s)\,ds\right) < r,$$

which contradicts the fact that $u \in \partial B_r$. Lemma 12 allows one to conclude that the operator T has a fixed point $u^* \in \overline{B}_r$ and then the fractional boundary value problem (P) has a nontrivial solution $u^* \in E$. The proof is complete. □

3 Positive solutions

To study the existence of positive solution of the problem (P), first, we will introduce a positive cone constituted of continuous positive functions or some suitable subset of it. Second, we will impose suitable assumptions on the nonlinear terms such that the hypotheses of the cone theorem are satisfied. Third, we will apply a fixed point theorem to conclude the existence of a positive solution in the annular region.

Definition 13 A function u is called positive solution of the problem (P) if $u(t) \geq 0$, $\forall t \in \mathbb{R}_+$, and it satisfies the boundary conditions in (P).

Definition 14 A nonempty subset P of a Banach space E is called a cone if P is convex, closed, and satisfies the conditions

(i) $\alpha x \in P$ for all $x \in P$ and $\alpha \in R_+$;

(ii) $x; -x \in P$ imply $x = 0$.

Lemma 15 *Assume that* $0 < \alpha < \Gamma(q)$, *then for* $0 < \tau_1 \leq t \leq \tau_2$ *and* $s > 0$ *we have*

$$\frac{G(t,s)}{1+t^{q-1}} \geq \frac{\gamma\tau_1}{1+\tau_2^{q-1}} \quad \text{and} \quad \frac{G_t(t,s)}{1+t^{q-1}} \geq \frac{\gamma}{1+\tau_2^{q-1}},$$

where $\gamma = \frac{1}{\alpha} - \frac{1}{\Gamma(q)} > 0$.

Proof The proof is easy, we omit it. □

Let us make the following hypotheses on the nonlinear term f:

(H) $f \in C(\mathbb{R}_+^3, \mathbb{R}_+)$, $\int_0^\infty f(t,u,v)\,dt < \infty$, $f(t,u,v) = a(t)g(t,u,v)$, where $a \in L_1(\mathbb{R}_+, \mathbb{R}_+)$, $g \in C(\mathbb{R}_+^3, \mathbb{R}_+)$ and $0 < \int_{\tau_1}^{\tau_2} a(t)\,dt < \infty$.

Define the cone K by

$$K = \left\{ u \in E, u(t) \geq 0, u'(t) \geq 0, \forall t \geq 0, \min_{t\in[\tau_1,\tau_2]} \frac{u(t)+u'(t)}{1+t^{q-1}} \geq \gamma_1 \|u\| \right\},$$

where $\gamma_1 = \frac{\gamma\alpha(1+\tau_1)\Gamma(q-1)}{4(1+\tau_2^2)}$.

Lemma 16 *We have* $TK \subset K$.

Proof Taking Lemma 7 into account, we get

$$\frac{Tu(t)}{1+t^{q-1}} \leq \frac{2}{\alpha}\int_0^\infty f\bigl(s,u(s),u'(s)\bigr)\,ds, \qquad \frac{T'u(t)}{1+t^{q-1}} \leq \frac{4}{\alpha\Gamma(q-1)}\int_0^\infty f\bigl(s,u(s),u'(s)\bigr)\,ds,$$

thus

$$\|Tu\| \leq \frac{4}{\alpha\Gamma(q-1)}\int_0^\infty f\bigl(s,u(s),u'(s)\bigr)\,ds.$$

Lemma 15 implies for all $t \in [\tau_1, \tau_2]$

$$\frac{Tu(t)}{1+t^{q-1}} \geq \frac{\gamma\tau_1}{1+\tau_2^{q-1}}\int_0^\infty f\bigl(s,u(s),u'(s)\bigr)\,ds \geq \frac{\gamma\alpha\tau_1\Gamma(q-1)}{4(1+\tau_2^{q-1})}\|Tu\|,$$
$$\frac{T'u(t)}{1+t^{q-1}} \geq \frac{\gamma}{1+\tau_2^{q-1}}\int_0^\infty f\bigl(s,u(s),u'(s)\bigr)\,ds \geq \frac{\gamma\alpha\Gamma(q-1)}{4(1+\tau_2^{q-1})}\|Tu\|.$$

Therefore,

$$\min_{t\in[\tau_1,\tau_2]} \frac{Tu(t)+T'u(t)}{1+t^{q-1}} \geq \frac{\gamma\alpha(1+\tau_1)\Gamma(q-1)}{4(1+\tau_2^{q-1})}\|Tu\|.$$

□

Let us introduce the following notation:

$$A^{\delta} = \lim_{u+v\to\delta} \sup_{t\geq 0} \frac{g(t,(1+t^{q-1})u,(1+t^{q-1})v)}{u+v},$$
$$A_{\delta} = \lim_{u+v\to\delta} \inf_{t\geq 0} \frac{g(t,(1+t^{q-1})u,(1+t^{q-1})v)}{u+v} \quad \bigl(\delta = 0^{+} \text{ or } +\infty\bigr).$$

Theorem 17 *Under the hypothesis* (H) *and if* $0 < \alpha < \Gamma(q)$, *then the fractional boundary value problem* (P) *has at least one positive solution in the case* $A^0 = 0$ *and* $A_\infty = \infty$.

To prove Theorem 17, we apply the well-known Guo-Krasnosel'skii fixed point theorem on cone.

Theorem 18 [21, 22] *Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\mathcal{A}: K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K,$$

be a completely continuous operator such that

(i) (*Expansive form*) $\|\mathcal{A}u\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) (*Compressive form*) $\|\mathcal{A}u\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Proof of Theorem 17 From $A^0 = 0$, we deduce that for any $\varepsilon > 0$, there exists $R_1 > 0$, such that if $0 < u + v \leq R_1$, then $g(t,(1+t^{q-1})u,(1+t^{q-1})v) \leq \varepsilon(u+v)$, $\forall t \geq 0$. Let $\Omega_1 = \{u \in E, \|u\| < \frac{R_1}{2}\}$ and $u \in K \cap \partial\Omega_1$, by Lemma 7 we get

$$\begin{aligned}
\frac{Tu(t)}{1+t^{q-1}} &\leq \frac{2}{\alpha} \int_0^\infty a(s) g\bigl(s, u(s), u'(s)\bigr)\, ds \\
&= \frac{2}{\alpha} \int_0^\infty a(s) g\left(s, \bigl(1+s^{q-1}\bigr)\frac{u(s)}{1+s^{q-1}}, \bigl(1+s^{q-1}\bigr)\frac{u'(s)}{1+s^{q-1}}\right) ds \\
&\leq \frac{2\varepsilon}{\alpha} \int_0^\infty a(s) \left(\frac{u(s)}{1+s^{q-1}} + \frac{u'(s)}{1+s^{q-1}}\right) ds \\
&\leq \frac{4\varepsilon}{\alpha} \|u\| \int_0^\infty a(s)\, ds.
\end{aligned}$$

Similarly we obtain

$$\frac{T'u(t)}{1+t^{q-1}} \leq \frac{8\varepsilon}{\alpha\Gamma(q-1)} \|u\| \int_0^\infty a(s)\, ds,$$

therefore,

$$\|Tu\| \leq \frac{8\varepsilon}{\alpha\Gamma(q-1)} \|u\| \int_0^\infty a(s)\, ds.$$

Choosing $\varepsilon \leq \frac{\alpha\Gamma(q-1)}{8\int_0^\infty a(s)\, ds}$, it yields $\|Tu\| \leq \|u\|$, for any $u \in K \cap \partial\Omega_1$.

Now, since $A_\infty = \infty$, then for any $M > 0$, there exists $R > 0$, such that $\forall t \geq 0$

$$g\big(t, \big(1 + t^{q-1}\big)u, \big(1 + t^{q-1}\big)v\big) \geq M(u + v)$$

for $u + v \geq R$. Let $R_2 > \max\{\frac{R_1}{2}, \frac{R}{\gamma_1}\}$ and denote by $\Omega_2 = \{u \in E : \|u\| < R_2\}$. For $u \in K \cap \partial\Omega_2$ and $t \in [\tau_1, \tau_2]$, we obtain

$$\frac{u(t) + u'(t)}{1 + t^{q-1}} \geq \min_{[\tau_1,\tau_2]} \frac{u(t) + u'(t)}{1 + t^{q-1}} \geq \gamma_1 \|u\| = \gamma_1 R_2 > R.$$

Using Lemma 15 and the fact that $u \in K$, we obtain for all $t \in [\tau_1, \tau_2]$

$$\begin{aligned}
\frac{Tu(t)}{1 + t^{q-1}} &\geq \frac{\gamma \tau_1}{1 + \tau_2^{q-1}} \int_0^\infty a(s) g\big(s, u(s), u'(s)\big)\, ds \\
&= \frac{\gamma \tau_1}{1 + \tau_2^{q-1}} \int_0^\infty a(s) g\left(s, \big(1 + s^{q-1}\big)\frac{u(s)}{1 + s^{q-1}}, \big(1 + s^{q-1}\big)\frac{u'(s)}{1 + s^{q-1}}\right) ds \\
&\geq \frac{\gamma \tau_1}{1 + \tau_2^{q-1}} M \int_0^\infty a(s) \left(\frac{u(s)}{1 + s^{q-1}} + \frac{u'(s)}{1 + s^{q-1}}\right) ds \\
&\geq \frac{\gamma \tau_1}{1 + \tau_2^{q-1}} M \int_{\tau_1}^{\tau_2} a(s) \frac{u(s) + u'(s)}{1 + s^{q-1}}\, ds \\
&\geq \frac{\gamma \tau_1}{1 + \tau_2^{q-1}} M \min_{t \in [\tau_1,\tau_2]} \frac{u(t) + u'(t)}{1 + t^{q-1}} \int_{\tau_1}^{\tau_2} a(s)\, ds \\
&\geq \frac{\gamma \gamma_1 \tau_1}{1 + \tau_2^{q-1}} M \|u\| \int_{\tau_1}^{\tau_2} a(s)\, ds,
\end{aligned}$$

similarly, we get

$$\frac{T'u(t)}{1 + t^{q-1}} \geq \frac{\gamma \gamma_1}{1 + \tau_2^{q-1}} M \|u\| \int_{\tau_1}^{\tau_2} a(s)\, ds.$$

Thus

$$\|Tu\| \geq \max(1, \tau_1) \frac{\gamma \gamma_1}{1 + \tau_2^{q-1}} M \|u\| \int_{\tau_1}^{\tau_2} a(s)\, ds.$$

Let us choose M such that

$$M \geq \frac{1 + \tau_2^{q-1}}{\gamma \gamma_1 \max(1, \tau_1) \int_{\tau_1}^{\tau_2} a(s)\, ds},$$

then we obtain

$$\|Tu\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_2.$$

The first statement of Theorem 18 implies that T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. The proof is complete. □

Now define the function

$$A^*:\mathbb{R}_+\to\mathbb{R}_+,$$
$$A^*(r)=\max\Big\{\sup_{t\geq 0} g\big(t,\big(1+t^{q-1}\big)u,\big(1+t^{q-1}\big)v\big),(u+v)\in[0,r]\Big\},$$
$$A_0^*=\lim_{r\to 0^+}\frac{A^*(r)}{r},\qquad A_\infty^*=\lim_{r\to+\infty}\frac{A^*(r)}{r}.$$

It is proved in [9] that:

Lemma 19 *If g is continuous then $A_0^*=A^0$ and $A_\infty^*=A^\infty$.*

Theorem 20 *Under the hypothesis* (H) *and if $0<\alpha<\Gamma(q)$ and g is decreasing according to the both variables, then the problem* (P) *has at least one nontrivial positive solution in the cone K, in the case $A_0=+\infty$ and $A^\infty=0$.*

Proof Since $A_0=+\infty$, then for $M\geq\frac{1+\tau_2^{q-1}}{4\tau_1\gamma\int_0^\infty a(s)\,ds}>0$, there exists $r_1>0$, such that if $0<u+v\leq r_1$, then for all $t\geq 0$, we have

$$g\big(t,\big(1+t^{q-1}\big)u,\big(1+t^{q-1}\big)v\big)\geq M(u+v).$$

Let $\Omega_1=\{u\in E,\|u\|<\frac{r_1}{2}\}$, we should prove the second statement of Theorem 18. Suppose $u_1\in P\cap\partial\Omega_1$, then

$$\begin{aligned}\|Tu_1\|&\geq\frac{Tu_1(t)}{(1+t^{q-1})}\\&\geq\frac{\gamma\tau_1}{1+\tau_2^{q-1}}\int_0^\infty a(s)g\left(s,\big(1+s^{q-1}\big)\frac{u_1(s)}{(1+s^{q-1})},\big(1+s^{q-1}\big)\frac{u_1'(s)}{(1+s^{q-1})}\right)ds.\end{aligned}$$

Now from the fact that g is decreasing, we get

$$\begin{aligned}\|Tu_1\|&\geq\frac{\gamma\tau_1}{1+\tau_2^{q-1}}\int_0^\infty a(s)g\big(s,\big(1+s^{q-1}\big)\|u_1\|,\big(1+s^{q-1}\big)\|u_1\|\big)\,ds\\&\geq\frac{2\gamma\tau_1\|u_1\|}{1+\tau_2^{q-1}}M\int_0^\infty a(s)\,ds.\end{aligned}$$

Thus

$$\|Tu_1\|\geq\|u_1\|\quad\text{on } P\cap\partial\Omega_1.$$

From $A^\infty=0$ and Lemma 19, we get $A_\infty^*=0$, so for $\varepsilon\leq\frac{\alpha\Gamma(q-1)}{8\int_0^\infty a(s)\,ds}$, there exists $R>0$, such that if $r\geq R$, then $A^*(r)\leq\varepsilon r$. Let $\Omega_2=\{u\in E,\|u\|<r_2\}$, where $r_2>\max(\frac{r_1}{2},\frac{R}{2})$, then $\overline{\Omega_1}\subset\Omega_2$. Suppose that $u_2\in P\cap\partial\Omega_2$, then it yields

$$\begin{aligned}\frac{Tu_2(t)}{1+t^{q-1}}&\leq\frac{2}{\alpha}\int_0^\infty a(s)g\left(s,\big(1+s^{q-1}\big)\frac{u_2(s)}{(1+s^{q-1})},\big(1+s^{q-1}\big)\frac{u_2'(s)}{(1+s^{q-1})}\right)ds\\&\leq\frac{2A^*(2r_2)}{\alpha}\int_0^\infty a(s)\,ds\leq\frac{4r_2\varepsilon}{\alpha}\int_0^\infty a(s)\,ds.\end{aligned}$$

On the other hand we get

$$\begin{aligned}\frac{T'u_2(t)}{1+t^{q-1}} &\leq \frac{4}{\alpha\Gamma(q-1)}\int_0^\infty a(s)g\left(s,\left(1+s^{q-1}\right)\frac{u_2(s)}{(1+s^{q-1})},\left(1+s^{q-1}\right)\frac{u_2'(s)}{(1+s^{q-1})}\right)ds \\ &\leq \frac{4A^*(2r_2)}{\alpha\Gamma(q-1)}\int_0^\infty a(s)\,ds \leq \frac{8r_2\varepsilon}{\alpha\Gamma(q-1)}\int_0^\infty a(s)\,ds.\end{aligned}$$

Therefore

$$\|Tu_2\| \leq \frac{8r_2\varepsilon}{\alpha\Gamma(q-1)}\int_0^\infty a(s)\,ds < r_2 = \|u_2\|$$

then from the second statement of Theorem 18, T has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$. The proof is complete. □

Remark If $\int_0^\infty f(t,u(t),v(t))\,dt<\infty$, then every positive solution of the problem (P) is unbounded. Indeed

$$\begin{aligned}u(t) &= \frac{1}{\Gamma(q)}\left[\int_0^t (t-s)^{q-1}f\left(s,u(s),u'(s)\right)ds - t\int_0^1 (1-s)^{q-1}f\left(s,u(s),u'(s)\right)ds\right] \\ &\quad + \frac{t}{\alpha}\int_0^\infty f\left(s,u(s),u'(s)\right)ds \\ &\geq t\left(\frac{1}{\alpha}\int_0^\infty f\left(s,u(s),u'(s)\right)ds - \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}f\left(s,u(s),u'(s)\right)ds\right) \\ &\geq t\left(\frac{1}{\alpha}-\frac{1}{\Gamma(q)}\right)\int_0^\infty f\left(s,u(s),u'(s)\right)ds,\end{aligned}$$

therefore our conclusion follows.

Example 21 Let us consider the problem (P) with

$$g(t,u,v) = \frac{(u+v)^2}{(1+t^{q-1})^2}, \qquad a(t) = \frac{1}{1+t^2}, \qquad q = \frac{5}{2}, \qquad \alpha = 1,$$

by direct calculation we obtain $\Gamma(\frac{5}{2}) = 1.3293 > \alpha$, $\int_{\tau_1}^{\tau_2}\frac{1}{1+s^2}\,ds = \arctan\tau_2 - \arctan\tau_1 > 0$, $A^0 = 0$ and $A_\infty = \infty$. Clearly hypothesis (H) is satisfied, so by Theorem 17 there exists at least one nontrivial positive solution in the cone K.

Example 22 Let us reconsider the above example with

$$g(t,u,v) = \frac{1+t^{q-1}}{1+u+v}.$$

Easily we check the hypothesis (H) and find that g is decreasing with respect to u and v. Furthermore we have the case $A_0 = +\infty$ and $A^\infty = 0$. Thus by Theorem 20 there exists at least one nontrivial positive solution in the cone K.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final draft.

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References

1. Agarwal, RP, O'Regan, D: Infinite Interval Problems for Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht (2001)
2. Agarwal, RP, O'Regan, D: Boundary value problems of nonsingular type on the semi-infinity interval. Tohoku Math. J. **51**, 391-397 (1999)
3. Guezane-Lakoud, A, Khaldi, R: Solvability of a three-point fractional nonlinear boundary value problem. Differ. Equ. Dyn. Syst. **20**(4), 395-403 (2012)
4. Liang, S, Zhang, J: Existence of three positive solutions of *m*-point boundary value problems for some nonlinear fractional differential equations on an infinite interval. Comput. Math. Appl. **61**, 3343-3354 (2011)
5. Liang, S, Shi, S: Existence of multiple positive solutions for *m*-point fractional boundary value problems with *p*-Laplacian operator on infinite interval. J. Appl. Math. Comput. **38**, 687-707 (2012)
6. Liu, Y, Ahmad, B, Agarwal, RP: Existence of solutions for a coupled system of nonlinear fractional differential equations with fractional boundary conditions on the half-line. Adv. Differ. Equ. **2013**, Article ID 46 (2013)
7. Nyamoradi, N, Baleanu, D, Agarwal, RP: On a multipoint boundary value problem for a fractional order differential inclusion on an infinite interval. Adv. Math. Phys. **2013**, Article ID 823961 (2013)
8. Su, X, Zhang, S: Unbounded solutions to a boundary value problem of fractional order on the half-line. Comput. Math. Appl. **61**, 1079-1087 (2011)
9. Wang, H: On the numbers of positive solutions of a nonlinear systems. J. Math. Anal. Appl. **281**, 287-306 (2003)
10. Webb, JRL, Infante, G: Positive solutions of nonlocal boundary value problems involving integral conditions. NoDEA Nonlinear Differ. Equ. Appl. **15**(1-2), 45-67 (2008)
11. Yan, BQ, O'Regan, D, Agarwal, RP: Unbounded solutions for singular boundary value problems on the semi-infinite interval: upper and lower solutions and multiplicity. J. Comput. Appl. Math. **197**, 365-386 (2006)
12. Zhao, X, Ge, W: Unbounded solutions for a fractional boundary value problems on the infinite interval. Acta Appl. Math. **109**, 495-505 (2010)
13. Zhang, X: Existence of positive solutions for boundary value problems for second order nonlinear differential equations on the half line. Electron. J. Differ. Equ. **141**, 1-10 (2009)
14. Xu, H: Analytical approximations for a population growth model with fractional order. Commun. Nonlinear Sci. Numer. Simul. **14**(5), 1978-1983 (2009)
15. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
16. Mainardi, F: Fractals and Fractional Calculus in Continuum Mechanics. Springer, New York (1997)
17. Podlubny, I: Fractional Differential Equations Mathematics in Sciences and Engineering. Academic Press, New York (1999)
18. Lakshmikantham, V, Vatsala, AS: Basic theory of fractional differential equations. Nonlinear Anal. **69**(8), 2677-2682 (2008)
19. Corduneanu, C: Integral Equations and Stability of Feedback Systems. Academic Press, New York (1973)
20. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)
21. Guo, DJ, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Notes and Reports in Mathematics in Science and Engineering, vol. 5. Academic Press, Boston (1988)
22. Kwong, MK: On Krasnoselskii's cone fixed point theorem. Fixed Point Theory Appl. **2008**, Article ID 164537 (2008)

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