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Existence and uniqueness of mild solutions for a class of nonlinear fractional evolution equation

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Abstract

In this paper, we discuss a class of fractional evolution equations with the Riemann-Liouville fractional derivative and obtain the existence and uniqueness of mild solutions by using some classical fixed point theorem. Then we give some examples to demonstrate the main results.

MSC: 30C45; 30C80

Keywords: fractional derivative; evolution equation; Riemann-Liouville integral; existence; uniqueness

1 Introduction

The nonlinear fractional evolution equation is a general form for fractional ordinary differential equations, fractional partial differential equations, and fractional functional differential equations related to the time variable. We can widely find the applications in several fields of sciences and technology. Many real phenomena in those fields can be described very successfully by models using mathematical tools of fractional calculus, such as dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, modeling of earthquakes, fluid dynamics, traffic models with fractional derivative, measurements of viscoelastic material properties, modeling of viscoplasticity, control theory, and economy (see [1–4]). There has been a great deal of interest in the solutions of fractional evolution equations in infinite dimensional space. One is referred to the monographs of N'Guérékata *et al.* [5], Mophou *et al.* [6, 7], Liu *et al.* [8, 9], EI-Borai [10], Zhou *et al.* [11, 12], and the references therein. But to the best of the author's knowledge, most of them have researched the fractional evolution equation with Caputo derivative and the initial conditions such as $x(0) = x_0$ and so on; there are few papers on the Riemann-Liouville fractional derivative. We note that on a series of examples from the field of viscoelasticity, Heymans and Podlubny [13] have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives, and that it is possible to obtain initial values for such initial conditions by appropriate measurements or observations. In this paper, we discuss the existence and uniqueness of the following fractional evolution equation with Riemann-Liouville fractional derivative:

$$\begin{cases} D_0^q x(t) = Ax(t) + f(t, x(t)), & t \in (0, T], \\ I_0^{1-q} x(0) = x_0, \end{cases} \tag{1.1}$$

where D^q is the Riemann-Liouville fractional derivative of order $0 < q < 1$, A is the infinitesimal generator of a q-resolvent family $S_q(t)$ defined on a Banach space X and $x(t) \in C_{1-q}([0, T], X)$. The function $f : [0, T] \times C_{1-q}([0, T], X) \to X$ is given and satisfies some conditions which are weak compared to the existing results and the conclusion is generalized.

2 Definitions and preliminary results

In this section, we introduce preliminary facts which are used throughout this paper. Let us denote by $C([0, T], X)$ the space of all X-valued continuous functions defined on $[0, T]$, which turns out to be a Banach space with the norm

$$\|x\| = \sup_{t \in [0,T]} \|x(t)\|.$$

We define similarly another Banach space $C_{1-q}([0, T], X)$, which X-valued function $x(t)$ is continuous on $(0, T]$ and $t^{1-q}x(t)$ is continuous on $[0, T]$ with the norm

$$\|x\|_{1-q} = \sup_{t \in [0,T]} t^{1-q} \|x(t)\|.$$

$\mathcal{L}(X)$ is the space of all linear and bounded operators on X. The definitions and results of the fractional calculus reported below are not exhaustive but rather oriented to the subject of this paper. For the proofs, which are omitted, we refer the reader to [14, 15] or other texts on basic fractional calculus. As x is an abstract function with values in X, the integrals which appear in Definition 2.1 and Definition 2.2 are taken in Bochner's sense.

Definition 2.1 (see [15]) The fractional primitive of order $q > 0$ of function $x(t) \in C_{1-q}([0, T], X)$ is given by

$$I_0^q x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s)\, ds.$$

From [16] we know $I_0^q x(t)$ exists for all $q > 0$, when $x \in C_{1-q}([0, T], X)$ and $I_0^{1-q} x(0)$ is bounded; notice also that when $x \in C([0, T], X)$ then $I_0^q x(t) \in C[0, T]$, and, moreover, $I_0^{1-q} x(0) = 0$.

Definition 2.2 (see [15]) The fractional derivative of order $0 < q < 1$ of a function $x(t) \in C_{1-q}([0, T], X)$ is given by

$$D_0^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^t (t-s)^{-q} x(s)\, ds.$$

We have $D_0^q I_0^q x(t) = x(t)$ for all $x(t) \in C_{1-q}([0, T], X)$.

Lemma 2.3 (see [15]) *Let $0 < q < 1$. If we assume $x(t) \in C_{1-q}([0,T],X)$, then the fractional differential equation*

$$D_0^q x(t) = 0$$

has $x(t) = ct^{q-1}$, $c \in R$, as solutions.

From this lemma we can obtain the following law of composition.

Lemma 2.4 (see [15]) *Assume that $x(t) \in C_{1-q}([0,T],X)$ with a fractional derivative of order $0 < q < 1$ that belongs to $C_{1-q}([0,T],X)$. Then*

$$I_0^q D_0^q x(t) = x(t) + ct^{q-1}$$

for some $c \in R$. When the function x is in $C([0,T],X)$, then $c = 0$.

Recall that the Laplace transform of a function $f \in L^1(R_+,X)$ is defined by $\mathfrak{L}(f(t)) = \int_0^\infty e^{-\lambda t} f(t)\,dt$, $\operatorname{Re}(\lambda) > \omega$, if the integral is absolutely convergent for $\operatorname{Re}(\lambda) > \omega$.

Theorem 2.5 (see [17]) *Let E be a closed, convex and bounded and nonempty subset of a Banach space X and $N : E \to E$ be a completely continuous operator. Then N has at least one fixed point in E.*

Definition 2.6 (see [7]) Let A be a closed and linear operator with domain $D(A)$ defined on Banach space X and $q > 0$. Let $\rho(A)$ be the resolvent set of A. We call A the generator of a q-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S_q : R_+ \to \mathcal{L}(X)$ satisfying $S_q(0) = I$ such that $\{\lambda^q : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$\left(\lambda^q I - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S_q(t) x\,dt, \quad \operatorname{Re}(\lambda) > \omega, x \in X.$$

In this case, $S_q(t)$ is called the q-resolvent family generated by A.

Remark 2.7 Note that if A is the generator of an q-resolvent family $S_q(t)$ then the Laplace transform of $S_q(t)$ is $\mathfrak{L}(S_q(t)) = (\lambda^q I - A)^{-1}$.

Then using the fact that

$$\mathfrak{L}\left(D_0^q x(t)\right) = \lambda^q \mathfrak{L}\left(x(t)\right) - I_0^{1-q} x(0),$$

we deduce for the Laplace transform of (1.1)

$$\lambda^q \mathfrak{L}\left(x(t)\right) - I_0^{1-q} x(0) = A \mathfrak{L}\left(x(t)\right) + \mathfrak{L}\left(f\left(t, x(t)\right)\right),$$

and then

$$\mathfrak{L}\left(x(t)\right) = \left(\lambda^q I - A\right)^{-1} x_0 + \left(\lambda^q I - A\right)^{-1} \mathfrak{L}\left(f\left(t, x(t)\right)\right).$$

Consequently,

$$x(t) = S_q(t)x_0 + \int_0^t S_q(t-s)f\bigl(s,x(s)\bigr)\,ds,$$

if A generates the q-resolvent family $S_q(t)$.

Throughout this work f will be a continuous function $[0,T] \times C_{1-q}([0,T],X) \to X$.

Definition 2.8 A function $x \in C_{1-q}([0,T],X)$ is said to be a mild solution of (1.1) if x satisfies

$$x(t) = S_q(t)x_0 + \int_0^t S_q(t-s)f\bigl(s,x(s)\bigr)\,ds. \tag{2.1}$$

Since we have the uniqueness of the Laplace transform, a 1-resolvent family is the same as a C_0-semigroup whereas a 2-resolvent family corresponds to the concept of sine family; see [18]. We note that q-resolvent families are a particular case of (q,k)-regularized families introduced in [19]. These have been studied in a series of several papers in recent years (see [20, 21] and so on). According to [19] a q-resolvent family $S_q(t)$ corresponds to a regularized family. For more details on the q-resolvent family, we refer to [22] and the references therein. We also refer to [6] for more information as regards the resolvent or solution operator. As in the situation of C_0-semigroups we have diverse relations of a q-resolvent family and its generator. So we can assume the following condition to present the first result in this paper.

3 Results

We present now our first result.

Theorem 3.1 *Assume that*

(H_0) *There exist* $M > 0$ *and* $\delta > 0$ *such that* $\|S_q(t)\| \le Me^{\delta t}$;

(H_1) *There exist a constant* $l \in (0,q)$ *and* $u(t) \in L^{\frac{1}{l}}([0,T],R^+)$ *such that*

$$\bigl\|f(t,x) - f(t,y)\bigr\| \le u(t)\|x-y\|, \quad t \in [0,T], (x,y) \in X,$$

where $T^{1-q}e^{\delta T} < (\frac{q-l}{1-l})^{1-l}\frac{1}{Mu^*}$ *and* $u^* = (\int_0^T (u(s))^{\frac{1}{l}}\,ds)^l$.

Then (1.1) has a unique mild solution on $[0,T]$.

Proof Let

$$l_f = \max_{0\le t\le T}\bigl\|f(t,0)\bigr\|.$$

Consider the operator $N : C_{1-q}([0,T],X) \to C_{1-q}([0,T],X)$ defined by

$$Nx(t) = S_q(t)x_0 + \int_0^t S_q(t-s)f\bigl(s,x(s)\bigr)\,ds.$$

Let $B_R = \{x(\cdot) \in C_{1-q}([0,T],X) : \|x\|_{1-q} \leq R\}$, which is a bounded and closed subset of $C_{1-q}([0,T],X)$. For any $x(\cdot) \in B_R$, we have

$$\begin{aligned}
&\|Nx(t)\|_{1-q} \\
&\quad = \sup_{0\leq t\leq T} t^{1-q}\|Nx(t)\| \\
&\quad = \sup_{0\leq t\leq T} t^{1-q}\left\|S_q(t)x_0 + \int_0^t S_q(t-s)f(s,x(s))\,ds\right\| \\
&\quad \leq \sup_{0\leq t\leq T} t^{1-q}\left(Me^{\delta t}\|x_0\| + \int_0^t \|S_q(t-s)\|\|f(s,x(s)) - f(s,0) + f(s,0)\|\,ds\right) \\
&\quad \leq T^{1-q}\left(Me^{\delta T}\|x_0\| + Me^{\delta T}\int_0^t u(s)s^{q-1}s^{1-q}\|x\|\,ds + Me^{\delta T}l_f T\right) \\
&\quad \leq T^{1-q}\left(Me^{\delta T}\|x_0\| + Me^{\delta T}\|x\|_{1-q}u^*\left(\frac{1-l}{q-l}\right)^{1-l} + Me^{\delta T}l_f T\right) \\
&\quad \leq T^{1-q}\left(Me^{\delta T}\|x_0\| + Me^{\delta T}Ru^*\left(\frac{1-l}{q-l}\right)^{1-l} + Me^{\delta T}l_f T\right).
\end{aligned}$$

Now let

$$\begin{aligned}
&T^{1-q}\left(Me^{\delta T}\|x_0\| + Me^{\delta T}Ru^*\left(\frac{1-l}{q-l}\right)^{1-l} + Me^{\delta T}l_f T\right) < R, \\
&T^{1-q}Me^{\delta T}\|x_0\| + T^{2-q}Me^{\delta T}l_f < \left(1 - T^{1-q}Me^{\delta T}u^*\left(\frac{1-l}{q-l}\right)^{1-l}\right)R.
\end{aligned}$$

The right-hand side will be positive if

$$T^{1-q}e^{\delta T} < \left(\frac{q-l}{1-l}\right)^{1-l}\frac{1}{Mu^*}. \tag{3.1}$$

Therefore, N maps the ball of B_R of radius R into itself, when T satisfies (3.1).

Next we show that N is a contraction on B_R. For this, let us take $x(\cdot), y(\cdot) \in B_R$, then we get

$$\begin{aligned}
&\|Nx(t) - Ny(t)\|_{1-q} \\
&\quad = \sup_{0\leq t\leq T} t^{1-q}\|Nx(t) - Ny(t)\| \\
&\quad = \sup_{0\leq t\leq T} t^{1-q}\left\|\int_0^t S_q(t-s)\big(f(s,x(s)) - f(s,y(s))\big)\,ds\right\| \\
&\quad \leq \sup_{0\leq t\leq T} t^{1-q}\int_0^t \|S_q(t-s)\|\|f(s,x(s)) - f(s,y(s))\|\,ds \\
&\quad \leq T^{1-q}Me^{\delta T}\int_0^t u(s)\|x(s) - y(s)\|\,ds \\
&\quad \leq T^{1-q}Me^{\delta T}\int_0^t u(s)s^{q-1}s^{1-q}\|x(s) - y(s)\|\,ds \\
&\quad \leq T^{1-q}Me^{\delta T}\int_0^t u(s)s^{q-1}\,ds\|x - y\|_{1-q}
\end{aligned}$$

$$
\begin{aligned}
&\le T^{1-q} M e^{\delta T} \left(\int_0^t \left(u(s)\right)^{\frac{1}{l}}\, ds \right)^l \left(\int_0^t \left(s^{q-1}\right)^{\frac{1}{1-l}}\, ds \right)^{1-l} \|x-y\|_{1-q} \\
&\le T^{1-q} M e^{\delta T} u^* \left(\frac{1-l}{q-l} \right)^{1-l} \|x-y\|_{1-q}.
\end{aligned}
$$

From the condition (3.1), we conclude that N is a contraction. Therefore, N has a unique fixed point in B_R. So (2.1) is the unique mild solution of (1.1) on $[0,T]$. □

Now we assume that

(H_2) There exist a constant $l \in (0,q)$ and a function $u(t) \in L^{\frac{1}{l}}([0,T],(0,\infty))$ such that

$$\left\| f\left(t, x(t)\right) \right\| \le u(t)\left(1 + \|x\|^{\rho}\right),$$

where $0 \le \rho < 1$;

(H_3) The q-resolvent family $S_q(t)x$ is equicontinuous.

Theorem 3.2 *Assume that* (H_0), (H_2), *and* (H_3) *hold. Then* (1.1) *has at least one mild solution on* $[0,T]$.

Proof Define

$$
\begin{aligned}
&Nx(t) = S_q(t)x_0 + \int_0^t S_q(t-s) f\left(s, x(s)\right) ds, \\
&B_r = \left\{ x(t) \in C_{1-q}\left([0,T], X\right), \|x\|_{1-q} \le r \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
&r \ge \max\left\{ 2(Me^{\delta T} T^{1-q} \|x_0\| + T^{2-q-l} M e^{\delta T} u^*, k^{\frac{1}{1-\rho}} \right\}, \\
&k = 2Me^{\delta T} u^* \left(\frac{1-l}{\rho q - \rho + 1 - l} \right)^{1-l} T^{\rho q - \rho + 2 - l - q}.
\end{aligned}
$$

Observe that B_r is a closed, bounded, and convex subset of Banach space X.

Now we prove that $N : B_r \to B_r$. For any $x \in B_r$, we have

$$
\begin{aligned}
&\left\| Nx(t) \right\|_{1-q} \\
&\quad = \sup_{0 \le t \le T} \left\| t^{1-q} Nx(t) \right\| \\
&\quad = \sup_{0 \le t \le T} \left\| t^{1-q} S_q(t) x_0 + t^{1-q} \int_0^t S_q(t-s) f\left(s, x(s)\right) ds \right\| \\
&\quad \le \sup_{0 \le t \le T} \left(t^{1-q} M e^{\delta T} \|x_0\| + t^{1-q} M e^{\delta T} \int_0^t u(s)\left(1 + \|x\|^{\rho}\right) ds \right) \\
&\quad \le T^{1-q} M e^{\delta T} \|x_0\| + T^{1-q} M e^{\delta T} \int_0^t u(s)\, ds + t^{1-q} M e^{\delta T} \int_0^t u(s) s^{q\rho - \rho} \left\| s^{1-q} x \right\|^{\rho} ds \\
&\quad \le T^{1-q} M e^{\delta T} \|x_0\| + T^{2-q-l} M e^{\delta T} u^* + M e^{\delta T} u^* \left(\frac{1-l}{\rho q - \rho + 1 - l} \right)^{1-l} T^{\rho q - \rho + 2 - l - q} r^{\rho}
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{r}{2} + \frac{r}{2} \\
&= r.
\end{aligned}
$$

Notice that $N(x(t))$ is continuous on $[0, T]$, therefore $N : B_r \to B_r$.

In view of the continuity of f, it is easy to show that the operator N is continuous. Now we show that N is a completely continuous operator. For each $x \in B_r$, let $t_1, t_2 \in [0, T]$ with $t_2 > t_1$. Then

$$
\begin{aligned}
&\|Nx(t_1) - Nx(t_2)\|_{1-q} \\
&\quad = \sup_{0 \le t \le T} \| t^{1-q} (Nx(t_1) - Nx(t_2)) \| \\
&\quad = \sup_{0 \le t \le T} t^{1-q} \bigg\| S_q(t_1)x_0 + \int_0^{t_1} S_q(t_1 - s) f(s, x(s))\, ds \\
&\qquad - S_q(t_2)x_0 - \int_0^{t_2} S_q(t_2 - s) f(s, x(s))\, ds \bigg\| \\
&\quad \le \sup_{0 \le t \le T} t^{1-q} \bigg(\| S_q(t_1)x_0 - S_q(t_2)x_0 \| \\
&\qquad + \bigg\| \int_0^{t_1} S_q(t_1 - s) f(s, x(s))\, ds - \int_0^{t_2} S_q(t_2 - s) f(s, x(s))\, ds \bigg\| \bigg) \\
&\quad \le I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \sup_{0 \le t \le T} t^{1-q} \| S_q(t_1)x_0 - S_q(t_2)x_0 \|, \\
I_2 &= \sup_{0 \le t \le T} t^{1-q} \bigg\| \int_0^{t_1} \big(S_q(t_1 - s) - S_q(t_2 - s) \big) f(s, x(s))\, ds \bigg\|, \\
I_3 &= \sup_{0 \le t \le T} t^{1-q} \bigg\| \int_{t_1}^{t_2} S_q(t_2 - s) f(s, x(s))\, ds \bigg\|.
\end{aligned}
$$

Actually, I_1 and I_2 tend to 0 as $t_1 \to t_2$ independently of $x \in B_r$. Indeed, since $S_q(t)$ is equicontinuous, we have $\|S_q(t_1)x_0 - S_q(t_2)x_0\| \to 0$, $\|S_q(t_1)f(s, x(s)) - S_q(t_2)f(s, x(s))\| \to 0$. Hence $I_1 \to 0$.

In view of (H_0), (H_2), and $1 - l + \rho(1 - q) > 0$, we have

$$
\begin{aligned}
&\sup_{0 \le t \le T} t^{1-q} \| \big(S_q(t_1 - s) - S_q(t_2 - s) \big) f(s, x(s)) \| \\
&\quad \le 2T^{1-q} M e^{\delta T} \| f(s, x(s)) \| \\
&\quad \le 2T^{1-q} M e^{\delta T} u(s) \big(1 + \| x(s) \|^{\rho} \big) \\
&\quad = 2T^{1-q} M e^{\delta T} u(s) + 2T^{1-q} M e^{\delta T} u(s) s^{\rho(q-1)} \| s^{1-q} x(s) \|^{\rho} \\
&\quad = 2T^{1-q} M e^{\delta T} u(s) + 2T^{1-q} M e^{\delta T} u(s) s^{\rho(q-1)} \|x\|_{1-q}^{\rho} \\
&\quad \le 2T^{1-q} M e^{\delta T} u(s) + 2r^{\rho} T^{1-q} M e^{\delta T} u(s) s^{\rho(q-1)}
\end{aligned}
$$

and since

$$\left(\int_0^T \left(s^{\rho(q-1)}\right)^{\frac{1}{1-l}} ds\right)^{1-l} = \left(\int_0^T s^{\frac{\rho(q-1)}{1-l}} ds\right)^{1-l} = \left(\frac{1-l}{1-l+\rho(q-1)}\right)^{1-l} T^{1-l+\rho(q-1)},$$

so $s^{\rho(q-1)} \in L^{\frac{1}{1-l}}([0,T],\mathbb{R}^+)$. According to the Hölder inequality we can obtain $u(s)s^{\rho(q-1)} \in L^1([0,T],\mathbb{R}^+)$, then

$$2T^{1-q}Me^{\delta T}u(s) + 2r^\rho T^{1-q}Me^{\delta T}u(s)s^{\rho(q-1)} \in L^1\big([0,T],\mathbb{R}^+\big).$$

In view of the Lebesgue's dominated convergence theorem, we can deduce that $I_2 \to 0$ as $t_1 \to t_2$,

$$\begin{aligned} I_3 &= \sup_{0\le t\le T} t^{1-q}\left\|\int_{t_1}^{t_2} S_q(t_2-s)f\big(s,x(s)\big)\,ds\right\| \\ &\le T^{1-q}Me^{\delta T}\int_{t_1}^{t_2} u(s)\big(1+\|x(s)\|^\rho\big)\,ds \\ &= T^{1-q}Me^{\delta T}\int_{t_1}^{t_2} u(s)\,ds + T^{1-q}Me^{\delta T}\int_{t_1}^{t_2} u(s)s^{\rho(q-1)}\|s^{1-q}x(s)\|^\rho\,ds \\ &\le T^{1-q}Me^{\delta T}u^*\left((t_2-t_1)^{1-l} + \left(\frac{1-l}{1-l+\rho(q-1)}\right)^{1-l}\big(t_2^{1-l+\rho(q-1)} - t_1^{1-l+\rho(q-1)}\big)\right), \end{aligned}$$

$I_3 \to 0$ as $t_1 \to t_2$. Hence N maps bounded sets of X into equicontinuous sets of X.

Next we will prove the operator N maps B_r into a relatively compact set in X. Indeed from the equicontinuity of $S_q(t)$ and $N: B_r \to B_r$, according to the Arzela-Ascoli theorem the set $\{Nx(t): x \in B_r\}$ is relatively compact in X, for every $t \in [0,T]$. Therefore, we can obtain N is a completely continuous operator. Thus the conclusion of Theorem 2.5 implies that (1.1) has at least one mild solution on $[0,T]$. □

Now we assume that

(H_4) There exist a constant $l \in (0,q)$ and a function $u(t) \in L^{\frac{1}{l}}([0,T],(0,\infty))$ such that

$$\big\|f\big(t,x(t)\big)\big\| \le u(t)\big(1+\|x\|\big)$$

and $Me^{\delta T}u^*(\frac{1-l}{\rho q-\rho+1-l})^{1-l}T^{\rho q-\rho+2-l-q} < 1$.

Corollary 3.3 *Assume that* (H_0), (H_3), *and* (H_4) *hold. Then* (1.1) *has at least one mild solution on* $[0,T]$.

The proof of the Corollary 3.3 is similar to Theorem 3.2.

4 Example

Example 4.1 Let $q = \frac{1}{2}$, $T = 1$, $x_0 = 1$. Consider the following fractional evolution equation:

$$\begin{cases} D_0^{\frac{1}{2}}x(t) = Ax(t) + f(t,x(t)), & t \in (0,1], \\ I_0^{\frac{1}{2}}x(0) = 1. \end{cases} \tag{4.1}$$

Assume that $x \in C_{\frac{1}{2}}[0,1]$, $A : D(A) \subset C_{\frac{1}{2}}[0,1] \to C_{\frac{1}{2}}[0,1]$, defined by $Ax = x'$, $x \in D(A)$, where $D(A) = \{x \in C_{\frac{1}{2}}[0,1] \mid x' \in C_{\frac{1}{2}}[0,1]\}$.

It is well know that A is an infinitesimal generator of a semigroup $\{S_{\frac{1}{2}}(t), t \geq 0\}$ in $C_{\frac{1}{2}}[0,1]$ and given by $S_{\frac{1}{2}}(t)x(s) = x(t+s)$, for $x \in C_{\frac{1}{2}}[0,1]$, $S_{\frac{1}{2}}(t)$ is a strongly continuous semigroup on $C_{\frac{1}{2}}[0,1]$ and $\|S_{\frac{1}{2}}(t)\| \leq 1$. We choose $f(t,x(t)) = \frac{1}{64}t^{\frac{2}{3}}x(t)$.

Obviously, $f(t,x(t))$ is continuous on $[0,1]$ since $x \in C_{\frac{1}{2}}[0,1]$,

$$\left\|f\bigl(t,x(t)\bigr) - f\bigl(t,y(t)\bigr)\right\| = \left\|\frac{1}{64}t^{\frac{2}{3}}\bigl(x(t)-y(t)\bigr)\right\| \leq \frac{1}{64}t^{\frac{2}{3}}\left\|x(t)-y(t)\right\| \leq \frac{1}{64}t^{\frac{2}{3}}\|x-y\|.$$

We know $\frac{1}{64}t^{\frac{2}{3}} \in L^4([0,1],R^+)$, $u^* = (\frac{3}{11})^{\frac{1}{4}}$,

$$\frac{(1-l)Me^{\delta T}u^*}{q-l}T^{\frac{1-2l+ql}{1-l}} = \frac{3}{64}\left(\frac{3}{11}\right)^{\frac{1}{4}} < 1.$$

So (4.1) has a unique mild solution on $[0,1]$ by Theorem 3.1.

Example 4.2 Let $q = \frac{1}{2}$, $T = 1$, $x_0 = 1$. Consider the following fractional evolution equation:

$$\begin{cases} D_0^{\frac{1}{2}}x(t) = Ax(t) + f(t,x(t)), \quad t \in (0,1], \\ I_0^{\frac{1}{2}}x(0) = 1. \end{cases} \tag{4.2}$$

Assume that $x \in C_{\frac{1}{2}}[0,1]$, $A : D(A) \subset C_{\frac{1}{2}}[0,1] \to C_{\frac{1}{2}}[0,1]$, defined by $Ax = x'$, $x \in D(A)$, where $D(A) = \{x \in C_{\frac{1}{2}}[0,1] \mid x' \in C_{\frac{1}{2}}[0,1]\}$.

It is well known that A is an infinitesimal generator of a semigroup $\{S_{\frac{1}{2}}(t), t \geq 0\}$ in $C_{\frac{1}{2}}[0,1]$ and given by $S_{\frac{1}{2}}(t)x(s) = x(t+s)$, for $x \in C_{\frac{1}{2}}[0,1]$, $S_{\frac{1}{2}}(t)$ is a strongly continuous semigroup on $C_{\frac{1}{2}}[0,1]$, $\|S_{\frac{1}{2}}(t)\| \leq 1$, and we assume that $S_{\frac{1}{2}}(t)x$ is equicontinuous.

We choose $f(t,x(t)) = \frac{1}{64}t^{\frac{2}{3}}x(t)^\rho$, $0 \leq \rho \leq 1$. Obviously, $f(t,x(t))$ is continuous on $[0,1]$ since $x \in C_{\frac{1}{2}}[0,1]$ and

$$\begin{aligned} \left\|f\bigl(t,x(t)\bigr)\right\| &= \left\|\frac{1}{64}t^{\frac{2}{3}}x(t)^\rho\right\| \\ &= \frac{1}{64}t^{\frac{2}{3}}\left\|t^{\frac{1}{2}}x(t)\right\|^\rho t^{-\frac{1}{2}\rho} \\ &\leq \frac{1}{64}t^{\frac{2}{3}-\frac{1}{2}\rho}\|x\|_{\frac{1}{2}}^\rho \leq \frac{1}{64}t^{\frac{2}{3}-\frac{1}{2}\rho}\bigl(1 + \|x\|_{\frac{1}{2}}^\rho\bigr). \end{aligned}$$

We know $\frac{1}{64}t^{\frac{2}{3}-\frac{1}{2}\rho} \in L^4([0,1],R^+)$, $u^* = (\frac{3}{5})^{\frac{1}{4}}$,

$$\frac{(1-l)Me^{\delta T}u^*}{q-l}T^{\frac{1-2l+ql}{1-l}} = \frac{3}{64}\left(\frac{3}{5}\right)^{\frac{1}{4}} < 1.$$

So (4.2) has at least one mild solution on $[0,1]$ by Theorem 3.2 and Corollary 3.3.

Example 4.3 To illustrate our results, we give another more concrete example of application. We consider the following fractional anomalous diffusion equation:

$$\begin{cases} \partial_t^q z(t,x) = \partial_x^2 z(t,x) + f(t,z(t,x)), & t \in (0,T], x \in [0,\pi], \\ I_0^{1-q} z(0,x) = x_0, \\ z(t,0) = z(t,\pi) = 0. \end{cases} \tag{4.3}$$

To study this system in the abstract form (1.1), we choose the space $X = L^2[0,\pi]$ and the operator A defined by $Az = z''$, with domain $D(A) = \{z \in L^2[0,\pi] : z, z'$ absolutely continuous, $z'' \in X, z(0) = z(\pi) = 0\}$.

Then A generates a uniformly bounded analytic semigroup which satisfies the condition (H_0), (H_3). Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with the corresponding normalized eigenvectors $\gamma_n(x) = (2/\pi)^{1/2}\sin(nx)$. Then the following properties hold.

(i) If $z \in D(A)$, then

$$Az = \sum_{n=1}^{\infty} n^2 \langle z, \gamma_n \rangle \gamma_n.$$

(ii) For each $z \in X$,

$$A^{-\frac{1}{2}} z = \sum_{n=1}^{\infty} \frac{1}{n} \langle z, \gamma_n \rangle \gamma_n.$$

In particular, $\|A^{-\frac{1}{2}}\| = 1$.

(iii) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} z = \sum_{n=1}^{\infty} n \langle z, \gamma_n \rangle \gamma_n$$

on the space $D(A^{\frac{1}{2}}) = \{z(\cdot) \in X, A^{\frac{1}{2}} z \in X\}$.

If the nonlinear term $f(t,z)$ satisfies the condition (H_1), then (4.3) has a unique mild solution on $[0,T]$ by Theorem 3.1. If the nonlinear term $f(t,z)$ satisfies the conditions (H_2) and (H_4), then (4.3) has at least one mild solution on $[0,T]$ by Theorem 3.2 and Corollary 3.3.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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