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Barnes-type Daehee polynomials

Dae San Kim[1], Taekyun Kim[2*], Takao Komatsu[3] and Jong-Jin Seo[4]

*Correspondence: tkkim@kw.ac.kr
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract
In this paper, we consider Barnes-type Daehee polynomials of the first kind and of the second kind. From the properties of the Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.
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1 Introduction

In this paper, we consider the polynomials $D_n(x|a_1,\dots,a_r)$ and $\widehat{D}_n(x|a_1,\dots,a_r)$ called the Barnes-type Daehee polynomials of the first kind and of the second kind, whose generating functions are given by

$$\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^x=\sum_{n=0}^{\infty}D_n(x|a_1,\dots,a_r)\frac{t^n}{n!}, \tag{1}$$

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^x=\sum_{n=0}^{\infty}\widehat{D}_n(x|a_1,\dots,a_r)\frac{t^n}{n!}, \tag{2}$$

respectively, where $a_1,\dots,a_r\neq 0$. When $x=0$, $D_n(a_1,\dots,a_r)=D_n(0|a_1,\dots,a_r)$ and $\widehat{D}_n(a_1,\dots,a_r)=\widehat{D}_n(0|a_1,\dots,a_r)$ are called the Barnes-type Daehee numbers of the first kind and of the second kind, respectively.

Recall that the Daehee polynomials of the first kind and of the second kind of order r, denoted by $D_n^{(r)}(x)$ and $\widehat{D}_n^{(r)}(x)$, respectively, are given by the generating functions to be

$$\left(\frac{\ln(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!},$$

$$\left(\frac{(1+t)\ln(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}\widehat{D}_n^{(r)}(x)\frac{t^n}{n!},$$

respectively. If $a_1=\cdots=a_r=1$, then $D_n^{(r)}(x)=D_n(x|\underbrace{1,\dots,1}_{r})$ and $\widehat{D}_n^{(r)}(x)=\widehat{D}_n(x|\underbrace{1,\dots,1}_{r})$. Daehee polynomials were defined by the second author [1] and have been investigated in [2–4].

In this paper, we consider Barnes-type Daehee polynomials of the first kind and of the second kind. From the properties of the Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{3}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad (n \geq 0). \tag{4}$$

In particular,

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n, k \geq 0), \tag{5}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and therefore an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a *delta series*; if $O(f(t)) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{6}$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \qquad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!} \tag{7}$$

[5, Theorem 2.2.5]. Thus, by (7), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x + y). \tag{8}$$

Sheffer sequences are characterized by the generating function [5, Theorem 2.3.4].

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t),f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)} = \sum_{k=0}^{\infty}\frac{s_k(y)}{k!}t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim (g(t),f(t))$, we have the following equations [5, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{9}$$

$$s_n(x) = \sum_{j=0}^{n}\frac{1}{j!}\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j | x^n\rangle x^j, \tag{10}$$

$$s_n(x+y) = \sum_{j=0}^{n}\binom{n}{j}s_j(x)p_{n-j}(y), \tag{11}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim (1,f(t))$ and $q_n(x) \sim (1,g(t))$. Then the transfer formula [5, Corollary 3.8.2] is given by

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim (g(t),f(t))$ and $r_n(x) \sim (h(t),l(t))$, assume that

$$s_n(x) = \sum_{m=0}^{n}C_{n,m}r_m(x) \quad (n \geq 0).$$

Then we have [5, p.132]

$$C_{n,m} = \frac{1}{m!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}l(\bar{f}(t))^m \middle| x^n\right\rangle. \tag{12}$$

3 Main results

We now note that $D_n(x|a_1,\ldots,a_r)$ is the Sheffer sequence for

$$g(t) = \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{t}\right) \quad \text{and} \quad f(t) = e^t - 1.$$

Therefore,

$$D_n(x|a_1,\ldots,a_r) \sim \left(\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{t}\right), e^t-1\right). \tag{13}$$

$\widehat{D}_n(x|a_1,\ldots,a_r)$ is the Sheffer sequence for

$$g(t) = \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right) \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$\widehat{D}_n(x|a_1,\ldots,a_r) \sim \left(\prod_{j=1}^{r}\left(\frac{e^{a_j t}-1}{te^{a_j t}}\right), e^t-1\right). \tag{14}$$

3.1 Explicit expressions

Recall that Barnes' multiple Bernoulli polynomials $B_n(x|a_1,\ldots,a_r)$ are defined by the generating function as

$$\frac{t^r}{\prod_{j=1}^{r}(e^{a_j t}-1)}e^{xt} = \sum_{n=0}^{\infty} B_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}, \tag{15}$$

where $a_1,\ldots,a_r \neq 0$ [6–8]. Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \geq 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m.$$

Theorem 1

$$D_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} S_1(n,m)B_m(x|a_1,\ldots,a_r) \tag{16}$$

$$= \sum_{j=0}^{n}\left(\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)D_{n-l}(a_1,\ldots,a_r)\right)x^j \tag{17}$$

$$= \sum_{m=0}^{n}\binom{n}{m}D_{n-m}(a_1,\ldots,a_r)(x)_m, \tag{18}$$

$$\widehat{D}_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} S_1(n,m)B_m(x+a_1+\cdots+a_r|a_1,\ldots,a_r) \tag{19}$$

$$= \sum_{j=0}^{n}\left(\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{D}_{n-l}(a_1,\ldots,a_r)\right)x^j \tag{20}$$

$$= \sum_{m=0}^{n}\binom{n}{m}\widehat{D}_{n-m}(a_1,\ldots,a_r)(x)_m. \tag{21}$$

Proof Since

$$\prod_{j=1}^{r}\left(\frac{e^{a_j t}-1}{t}\right)D_n(x|a_1,\ldots,a_r) \sim \left(1, e^t-1\right) \tag{22}$$

and

$$(x)_n \sim \left(1, e^t-1\right), \tag{23}$$

we have

$$\begin{aligned} D_n(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r}\left(\frac{t}{e^{a_jt}-1}\right)(x)_n \\ &= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{t}{e^{a_jt}-1}\right)x^m \\ &= \sum_{m=0}^{n} S_1(n,m)B_m(x|a_1,\ldots,a_r). \end{aligned}$$

So, we get (16).

Similarly, by

$$\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\widehat{D}_n(x|a_1,\ldots,a_r) \sim \left(1, e^t-1\right) \tag{24}$$

and (23), we have

$$\begin{aligned} \widehat{D}_n(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)(x)_n \\ &= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)x^m \\ &= \sum_{m=0}^{n} S_1(n,m)e^{(a_1+\cdots+a_r)t}\prod_{j=1}^{r}\left(\frac{t}{e^{a_jt}-1}\right)x^m \\ &= \sum_{m=0}^{n} S_1(n,m)B_m(x+a_1+\cdots+a_r|a_1,\ldots,a_r). \end{aligned}$$

Therefore, we get (19).

By (10) with (13), we get

$$D_n(x|a_1,\ldots,a_r) = \sum_{j=0}^{n}\frac{1}{j!}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^j \middle| x^n\right\rangle x^j.$$

Since

$$\begin{aligned} &\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^j \middle| x^n\right\rangle \\ &\quad= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| \left(\ln(1+t)\right)^j x^n\right\rangle \\ &\quad= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| j!\sum_{l=j}^{\infty} S_1(l,j)\frac{t^l}{l!}x^n\right\rangle \\ &\quad= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| x^{n-l}\right\rangle \end{aligned}$$

$$
\begin{aligned}
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \sum_{i=0}^{\infty}D_i(a_1,\ldots,a_r)\frac{t^i}{i!}\middle| x^{n-l}\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)D_{n-l}(a_1,\ldots,a_r),
\end{aligned}
$$

we obtain (17).

Similarly, by (10) with (14), we get

$$
\widehat{D}_n(x|a_1,\ldots,a_r) = \sum_{j=0}^{n}\frac{1}{j!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^j\middle| x^n\right\rangle x^j.
$$

Since

$$
\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^j\middle| x^n\right\rangle \\
&\quad= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| \bigl(\ln(1+t)\bigr)^j x^n\right\rangle \\
&\quad= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| x^{n-l}\right\rangle \\
&\quad= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \sum_{i=0}^{\infty}\widehat{D}_i(a_1,\ldots,a_r)\frac{t^i}{i!}\middle| x^{n-l}\right\rangle \\
&\quad= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{D}_{n-l}(a_1,\ldots,a_r),
\end{aligned}
$$

we obtain (20).

Next, we obtain

$$
\begin{aligned}
D_n(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty}D_i(y|a_1,\ldots,a_r)\frac{t^i}{i!}\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| \sum_{m=0}^{\infty}(y)_m\frac{t^m}{m!}x^n\right\rangle \\
&= \sum_{m=0}^{n}(y)_m\binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| x^{n-m}\right\rangle \\
&= \sum_{m=0}^{n}\binom{n}{m}D_{n-m}(a_1,\ldots,a_r)(y)_m.
\end{aligned}
$$

Thus, we get the identity (18).

Similarly,

$$\begin{aligned}\widehat{D}_n(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{D}_i(y|a_1,\ldots,a_r)\frac{t^i}{i!} \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \sum_{m=0}^{\infty}(y)_m\frac{t^m}{m!}x^n \right\rangle \\ &= \sum_{m=0}^{n}(y)_m\binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| x^{n-m} \right\rangle \\ &= \sum_{m=0}^{n}\binom{n}{m}\widehat{D}_{n-m}(a_1,\ldots,a_r)(y)_m.\end{aligned}$$

Thus, we get the identity (21). □

3.2 Sheffer identity

Theorem 2

$$D_n(x+y|a_1,\ldots,a_r) = \sum_{j=0}^{n}\binom{n}{j}D_j(x|a_1,\ldots,a_r)(y)_{n-j}, \tag{25}$$

$$\widehat{D}_n(x+y|a_1,\ldots,a_r) = \sum_{j=0}^{n}\binom{n}{j}\widehat{D}_j(x|a_1,\ldots,a_r)(y)_{n-j}. \tag{26}$$

Proof By (13) with

$$\begin{aligned}p_n(x) &= \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{t}\right)D_n(x|a_1,\ldots,a_r) \\ &= (x)_n \sim \left(1, e^t-1\right),\end{aligned}$$

using (11), we have (25).

By (14) with

$$\begin{aligned}p_n(x) &= \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\widehat{D}_n(x|a_1,\ldots,a_r) \\ &= (x)_n \sim \left(1, e^t-1\right),\end{aligned}$$

using (11), we have (26). □

3.3 Difference relations

Theorem 3

$$D_n(x+1|a_1,\ldots,a_r) - D_n(x|a_1,\ldots,a_r) = nD_{n-1}(x|a_1,\ldots,a_r), \tag{27}$$

$$\widehat{D}_n(x+1|a_1,\ldots,a_r) - \widehat{D}_n(x|a_1,\ldots,a_r) = n\widehat{D}_{n-1}(x|a_1,\ldots,a_r). \tag{28}$$

Proof By (9) with (13), we get

$$\left(e^t-1\right)D_n(x|a_1,\ldots,a_r)=nD_{n-1}(x|a_1,\ldots,a_r).$$

By (8), we have (27).

Similarly, by (9) with (14), we get

$$\left(e^t-1\right)\widehat{D}_n(x|a_1,\ldots,a_r)=n\widehat{D}_{n-1}(x|a_1,\ldots,a_r).$$

By (8), we have (28). □

3.4 Recurrence

Theorem 4

$$\begin{aligned}D_{n+1}(x|a_1,\ldots,a_r)&=xD_n(x-1|a_1,\ldots,a_r)\\&\quad-\sum_{m=0}^{n}\left(\sum_{i=m}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\&\quad\left.\times B_{i+1-m}(-a_j)^{i+1-m}D_{n-l}(a_1,\ldots,a_r)\right)(x-1)^m,\end{aligned}\tag{29}$$

$$\begin{aligned}\widehat{D}_{n+1}(x|a_1,\ldots,a_r)&=\left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_n(x-1|a_1,\ldots,a_r)\\&\quad-\sum_{m=0}^{n}\left(\sum_{i=m}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\&\quad\left.\times B_{i+1-m}(-a_j)^{i+1-m}\widehat{D}_{n-l}(a_1,\ldots,a_r)\right)(x-1)^m,\end{aligned}\tag{30}$$

where B_n is the nth ordinary Bernoulli number.

Proof By applying

$$s_{n+1}(x)=\left(x-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x)\tag{31}$$

[5, Corollary 3.7.2] with (13), we get

$$D_{n+1}(x|a_1,\ldots,a_r)=xD_n(x-1|a_1,\ldots,a_r)-e^{-t}\frac{g'(t)}{g(t)}D_n(x|a_1,\ldots,a_r).$$

Now,

$$\begin{aligned}\frac{g'(t)}{g(t)}&=\left(\ln g(t)\right)'\\&=\left(\sum_{j=1}^{r}\ln\left(e^{a_jt}-1\right)-r\ln t\right)'\end{aligned}$$

$$= \sum_{j=1}^{r} \frac{a_j e^{a_j t}}{e^{a_j t} - 1} - \frac{r}{t}$$
$$= \frac{\sum_{j=1}^{r} \prod_{i \neq j} (e^{a_i t} - 1)(a_j t e^{a_j t} - e^{a_j t} + 1)}{t \prod_{j=1}^{r} (e^{a_j t} - 1)}.$$

Since

$$\sum_{j=1}^{r} \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r = \frac{\sum_{j=1}^{r} \prod_{i \neq j} (e^{a_i t} - 1)(a_j t e^{a_j t} - e^{a_j t} + 1)}{\prod_{j=1}^{r} (e^{a_j t} - 1)}$$
$$= \frac{\frac{1}{2} (\sum_{j=1}^{r} a_1 \cdots a_{j-1} a_j^2 a_{j+1} \cdots a_r) t^{r+1} + \cdots}{(a_1 \cdots a_r) t^r + \cdots}$$
$$= \frac{1}{2} \left(\sum_{j=1}^{r} a_j \right) t + \cdots$$

is a series with order ≥ 1, by (17) we have

$$D_{n+1}(x|a_1, \ldots, a_r)$$
$$= x D_n(x-1|a_1, \ldots, a_r) - e^{-t} \frac{\sum_{j=1}^{r} \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r}{t} D_n(x|a_1, \ldots, a_r)$$
$$= x D_n(x-1|a_1, \ldots, a_r) - e^{-t} \frac{\sum_{j=1}^{r} \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r}{t} \left(\sum_{i=0}^{n} \sum_{l=i}^{n} \binom{n}{l} S_1(l,i) D_{n-l}(a_1, \ldots, a_r) x^i \right)$$
$$= x D_n(x-1|a_1, \ldots, a_r) - \sum_{i=0}^{n} \sum_{l=i}^{n} \binom{n}{l} S_1(l,i) D_{n-l}(a_1, \ldots, a_r) e^{-t} \left(\sum_{j=1}^{r} \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r \right) \frac{x^{i+1}}{i+1}.$$

Since

$$e^{-t} \left(\sum_{j=1}^{r} \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r \right) x^{i+1} = e^{-t} \left(\sum_{j=1}^{r} \sum_{m=0}^{\infty} \frac{(-1)^m B_m a_j^m}{m!} t^m - r \right) x^{i+1}$$
$$= e^{-t} \left(\sum_{j=1}^{r} \sum_{m=0}^{i+1} \binom{i+1}{m} B_m (-a_j)^m x^{i+1-m} - r x^{i+1} \right)$$
$$= \sum_{j=1}^{r} \sum_{m=1}^{i+1} \binom{i+1}{m} B_m (-a_j)^m (x-1)^{i+1-m}$$
$$= \sum_{j=1}^{r} \sum_{m=0}^{i} \binom{i+1}{m} B_{i+1-m} (-a_j)^{i+1-m} (x-1)^m, \tag{32}$$

we have

$$D_{n+1}(x|a_1, \ldots, a_r) = x D_n(x-1|a_1, \ldots, a_r)$$
$$- \sum_{i=0}^{n} \sum_{l=i}^{n} \sum_{j=1}^{r} \sum_{m=0}^{i} \frac{1}{i+1} \binom{n}{l} \binom{i+1}{m} S_1(l,i)$$

$$
\begin{aligned}
&\times B_{i+1-m}(-a_j)^{i+1-m}D_{n-l}(a_1,\ldots,a_r)(x-1)^m \\
&= xD_n(x-1|a_1,\ldots,a_r) \\
&\quad -\sum_{m=0}^{n}\left(\sum_{i=m}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right. \\
&\quad \left.\times B_{i+1-m}(-a_j)^{i+1-m}D_{n-l}(a_1,\ldots,a_r)\right)(x-1)^m,
\end{aligned}
$$

which is the identity (29).

Next, by applying (31) with (14), we get

$$
\widehat{D}_{n+1}(x|a_1,\ldots,a_r) = x\widehat{D}_n(x-1|a_1,\ldots,a_r) - e^{-t}\frac{g'(t)}{g(t)}\widehat{D}_n(x|a_1,\ldots,a_r).
$$

Now,

$$
\begin{aligned}
\frac{g'(t)}{g(t)} &= \bigl(\ln g(t)\bigr)' \\
&= \left(\sum_{j=1}^{r}\ln\bigl(e^{a_jt}-1\bigr) - r\ln t - \left(\sum_{j=1}^{r}a_j\right)t\right)' \\
&= \sum_{j=1}^{r}\frac{a_je^{a_jt}}{e^{a_jt}-1} - \frac{r}{t} - \sum_{j=1}^{r}a_j.
\end{aligned}
$$

By (20) we have

$$
\begin{aligned}
&\widehat{D}_{n+1}(x|a_1,\ldots,a_r) \\
&\quad = \left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_n(x-1|a_1,\ldots,a_r) \\
&\qquad - e^{-t}\frac{\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r}{t}\widehat{D}_n(x|a_1,\ldots,a_r) \\
&\quad = \left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_n(x-1|a_1,\ldots,a_r) \\
&\qquad - e^{-t}\frac{\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r}{t}\left(\sum_{i=0}^{n}\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)\widehat{D}_{n-l}(a_1,\ldots,a_r)x^i\right) \\
&\quad = \left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_n(x-1|a_1,\ldots,a_r) \\
&\qquad - \sum_{i=0}^{n}\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)\widehat{D}_{n-l}(a_1,\ldots,a_r)e^{-t}\left(\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r\right)\frac{x^{i+1}}{i+1}.
\end{aligned}
$$

By (32), we have the identity (30). □

3.5 Differentiation

Theorem 5

$$\frac{d}{dx}D_n(x|a_1,\ldots,a_r) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}D_l(x|a_1,\ldots,a_r), \tag{33}$$

$$\frac{d}{dx}\widehat{D}_n(x|a_1,\ldots,a_r) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{D}_l(x|a_1,\ldots,a_r). \tag{34}$$

Proof We shall use

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\langle\bar{f}(t)|x^{n-l}\rangle s_l(x)$$

(*cf.* [5, Theorem 2.3.12]). Since

$$\begin{aligned}\langle\bar{f}(t)|x^{n-l}\rangle = \langle\ln(1+t)|x^{n-l}\rangle &= \left\langle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\middle|x^{n-l}\right\rangle\\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\langle t^m|x^{n-l}\rangle\\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l}\\ &= (-1)^{n-l-1}(n-l-1)!,\end{aligned}$$

with (13), we have

$$\begin{aligned}\frac{d}{dx}D_n(x|a_1,\ldots,a_r) &= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!D_l(x|a_1,\ldots,a_r)\\ &= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}D_l(x|a_1,\ldots,a_r),\end{aligned}$$

which is the identity (33). Similarly, with (14), we have the identity (34). □

3.6 More relations

The classical Cauchy numbers c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

(see *e.g.* [9, 10]).

Theorem 6

$$\begin{aligned}D_n(x|a_1,\ldots,a_r) &= xD_{n-1}(x-1|a_1,\ldots,a_r)\\ &\quad + \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_lD_{n-l}(x-1|a_1,\ldots,a_r)\end{aligned}$$

$$-\frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jc_lD_{n-l}(x+a_j-1|a_1,\ldots,a_r,a_j), \tag{35}$$

$$\begin{aligned}\widehat{D}_n(x|a_1,\ldots,a_r) &= \left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_{n-1}(x-1|a_1,\ldots,a_r)\\ &\quad+\frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{D}_{n-l}(x-1|a_1,\ldots,a_r)\\ &\quad-\frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jc_l\widehat{D}_{n-l}(x-1|a_1,\ldots,a_r,a_j).\end{aligned} \tag{36}$$

Proof For $n \geq 1$, we have

$$\begin{aligned}D_n(y|a_1,\ldots,a_r) &= \left\langle \sum_{l=0}^{\infty}D_l(y|a_1,\ldots,a_r)\frac{t^l}{l!}\middle| x^n\right\rangle\\ &= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y\middle| x^n\right\rangle\\ &= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y\right)\middle| x^{n-1}\right\rangle\\ &= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\partial_t(1+t)^y\right)\middle| x^{n-1}\right\rangle\\ &\quad+\left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)(1+t)^y\middle| x^{n-1}\right\rangle\\ &= yD_{n-1}(y-1|a_1,\ldots,a_r)\\ &\quad+\left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)(1+t)^y\middle| x^{n-1}\right\rangle.\end{aligned}$$

Observe that

$$\begin{aligned}\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) &= \sum_{j=1}^{r}\prod_{i\neq j}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\frac{\frac{1}{1+t}((1+t)^{a_j}-1)-\ln(1+t)(a_j(1+t)^{a_j-1})}{((1+t)^{a_j}-1)^2}\\ &= \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\sum_{j=1}^{r}\left(\frac{1}{\ln(1+t)}-\frac{a_j(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\\ &= \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\frac{\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)}{t}.\end{aligned}$$

Since

$$\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}\right) = -\frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots$$

is a series with order ≥ 1, we have

$$\begin{aligned}
&\left\langle \left(\partial_t \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
&= \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| \frac{\sum_{j=1}^{r} (\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j}-1})}{t} x^{n-1} \right\rangle \\
&= \frac{1}{n} \sum_{j=1}^{r} \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| \left(\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j}-1} \right) x^n \right\rangle \\
&= \frac{r}{n} \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| \frac{t}{\ln(1+t)} x^n \right\rangle \\
&\quad - \frac{1}{n} \sum_{j=1}^{r} a_j \left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y+a_j-1} \middle| \frac{t}{\ln(1+t)} x^n \right\rangle \\
&= \frac{r}{n} \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \\
&\quad - \frac{1}{n} \sum_{j=1}^{r} a_j \left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y+a_j-1} \middle| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \\
&= \frac{r}{n} \sum_{l=0}^{n} c_l \binom{n}{l} \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| x^{n-l} \right\rangle \\
&\quad - \frac{1}{n} \sum_{j=1}^{r} a_j \sum_{l=0}^{n} c_l \binom{n}{l} \left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y+a_j-1} \middle| x^{n-l} \right\rangle \\
&= \frac{r}{n} \sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}(y-1|a_1,\ldots,a_r) - \frac{1}{n} \sum_{j=1}^{r} \sum_{l=0}^{n} \binom{n}{l} a_j c_l D_{n-l}(y+a_j-1|a_1,\ldots,a_r,a_j).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
D_n(x|a_1,\ldots,a_r) &= x D_{n-1}(x-1|a_1,\ldots,a_r) \\
&\quad + \frac{r}{n} \sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}(x-1|a_1,\ldots,a_r) \\
&\quad - \frac{1}{n} \sum_{j=1}^{r} \sum_{l=0}^{n} \binom{n}{l} a_j c_l D_{n-l}(x+a_j-1|a_1,\ldots,a_r,a_j),
\end{aligned}$$

which is the identity (35).

Next, for $n \geq 1$ we have

$$\begin{aligned}
\widehat{D}_n(y|a_1,\ldots,a_r) &= \left\langle \sum_{l=0}^{\infty} \widehat{D}_l(y|a_1,\ldots,a_r) \frac{t^l}{l!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^y \middle| x^n \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \partial_t \left(\prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^y \right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \left(\partial_t (1+t)^y \right) \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
&= y \widehat{D}_{n-1}(y-1 | a_1, \ldots, a_r) \\
&\quad + \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\partial_t \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \\
&= \partial_t \left(\prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \prod_{j=1}^{r} (1+t)^{a_j} \right) \\
&= \left(\partial_t \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) \prod_{j=1}^{r} (1+t)^{a_j} \\
&\quad + \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \left(\partial_t \prod_{j=1}^{r} (1+t)^{a_j} \right) \\
&= \frac{1}{1+t} \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) \frac{\sum_{j=1}^{r} \left(\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} \\
&\quad + \frac{1}{1+t} \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) \sum_{j=1}^{r} a_j.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\left\langle \left(\partial_t \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
&= \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| \frac{\sum_{j=1}^{r} \left(\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} x^{n-1} \right\rangle \\
&\quad + \left(\sum_{j=1}^{r} a_j \right) \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
&= \left(\sum_{j=1}^{r} a_j \right) \widehat{D}_{n-1}(y-1 | a_1, \ldots, a_r) \\
&\quad + \frac{1}{n} \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| \sum_{j=1}^{r} \left(\frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) x^n \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{j=1}^{r} a_j\right)\widehat{D}_{n-1}(y-1|a_1,\ldots,a_r) + \frac{r}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle| \frac{t}{\ln(1+t)}x^n\right\rangle \\
&\quad - \frac{1}{n}\sum_{j=1}^{r} a_j\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle| \frac{t}{\ln(1+t)}x^n\right\rangle \\
&= \left(\sum_{j=1}^{r} a_j\right)\widehat{D}_{n-1}(y-1|a_1,\ldots,a_r) + \frac{r}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle| \sum_{l=0}^{\infty} c_l\frac{t^l}{l!}x^n\right\rangle \\
&\quad - \frac{1}{n}\sum_{j=1}^{r} a_j\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle| \sum_{l=0}^{\infty} c_l\frac{t^l}{l!}x^n\right\rangle \\
&= \left(\sum_{j=1}^{r} a_j\right)\widehat{D}_{n-1}(y-1|a_1,\ldots,a_r) \\
&\quad + \frac{r}{n}\sum_{l=0}^{n} c_l\binom{n}{l}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle| x^{n-l}\right\rangle \\
&\quad - \frac{1}{n}\sum_{j=1}^{r} a_j\sum_{l=0}^{n} c_l\binom{n}{l}\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle| x^{n-l}\right\rangle \\
&= \left(\sum_{j=1}^{r} a_j\right)\widehat{D}_{n-1}(y-1|a_1,\ldots,a_r) + \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{D}_{n-l}(y-1|a_1,\ldots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jc_l\widehat{D}_{n-l}(y-1|a_1,\ldots,a_r,a_j).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\widehat{D}_n(x|a_1,\ldots,a_r) &= \left(x+\sum_{j=1}^{r} a_j\right)\widehat{D}_{n-1}(x-1|a_1,\ldots,a_r) \\
&\quad + \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{D}_{n-l}(x-1|a_1,\ldots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jc_l\widehat{D}_{n-l}(x-1|a_1,\ldots,a_r,a_j),
\end{aligned}
$$

which is the identity (36). □

3.7 Relations including the Stirling numbers of the first kind

Theorem 7 *For $n-1 \geq m \geq 1$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)D_l(a_1,\ldots,a_r) \\
&\quad = \sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l(-1|a_1,\ldots,a_r) \\
&\qquad + \frac{1}{n}\sum_{l=0}^{n-m-1}\binom{n}{l+1}S_1(n-l-1,m)
\end{aligned}
$$

$$\times\left(r\sum_{i=0}^{l+1}\binom{l+1}{i}c_iD_{l+1-i}(-1|a_1,\ldots,a_r)\right.$$

$$\left.-\sum_{j=1}^{r}\sum_{i=0}^{l+1}\binom{l+1}{i}a_jc_iD_{l+1-i}(a_j-1|a_1,\ldots,a_r,a_j)\right),\tag{37}$$

$$\begin{aligned}&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{D}_l(a_1,\ldots,a_r)\\&=\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l(-1|a_1,\ldots,a_r)\\&\quad+\frac{1}{n}\sum_{l=0}^{n-m-1}\binom{n}{l+1}S_1(n-l-1,m)\\&\quad\times\left(r\sum_{i=0}^{l+1}\binom{l+1}{i}c_i\widehat{D}_{l+1-i}(-1|a_1,\ldots,a_r)\right.\\&\quad\left.-\sum_{j=1}^{r}\sum_{i=0}^{l+1}\binom{l+1}{i}a_jc_i\widehat{D}_{l+1-i}(-1|a_1,\ldots,a_r,a_j)\right)\\&\quad+\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\sum_{j=1}^{r}a_j\widehat{D}_l(-1|a_1,\ldots,a_r).\end{aligned}\tag{38}$$

Proof We shall compute

$$\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m\middle|x^n\right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}&\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m\middle|x^n\right\rangle\\&=\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|\left(\ln(1+t)\right)^m x^n\right\rangle\\&=\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|\sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n\right\rangle\\&=\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|x^{n-l-m}\right\rangle\\&=\sum_{l=0}^{n-m}m!\binom{n}{l+m}S_1(l+m,m)D_{n-l-m}(a_1,\ldots,a_r)\\&=\sum_{l=0}^{n-m}m!\binom{n}{l}S_1(n-l,m)D_l(a_1,\ldots,a_r).\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m \middle| x^n\right\rangle \\
&= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m\right) \middle| x^{n-1}\right\rangle \\
&= \left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)\left(\ln(1+t)\right)^m \middle| x^{n-1}\right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\partial_t\left(\left(\ln(1+t)\right)^m\right) \middle| x^{n-1}\right\rangle.
\end{aligned} \tag{39}$$

The second term of (39) is equal to

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\partial_t\left(\left(\ln(1+t)\right)^m\right) \middle| x^{n-1}\right\rangle \\
&= m\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| \left(\ln(1+t)\right)^{m-1}x^{n-1}\right\rangle \\
&= m\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| \sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)t^{l+m-1}x^{n-1}\right\rangle \\
&= m\sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)(n-1)_{l+m-1} \\
&\quad \times \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| x^{n-l-m}\right\rangle \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l+m-1}S_1(l+m-1,m-1)D_{n-l-m}(-1|a_1,\ldots,a_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l(-1|a_1,\ldots,a_r).
\end{aligned}$$

The first term of (39) is equal to

$$\begin{aligned}
&\left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)\left(\ln(1+t)\right)^m \middle| x^{n-1}\right\rangle \\
&= \left\langle \partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \left(\ln(1+t)\right)^m x^{n-1}\right\rangle \\
&= \left\langle \partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \sum_{l=0}^{n-m-1}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^{n-1}\right\rangle \\
&= \sum_{l=0}^{n-m-1}\frac{m!}{(l+m)!}S_1(l+m,m)(n-1)_{l+m}\left\langle \partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| x^{n-l-m-1}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{n-m-1} m! \binom{n-1}{l+m} S_1(l+m,m) \\
&\quad \times \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{-1} \middle| \frac{\sum_{j=1}^{r} (\frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1})}{t} x^{n-l-m-1} \right\rangle \\
&= m! \sum_{l=0}^{n-m-1} \frac{1}{n-l-m} \binom{n-1}{l+m} S_1(l+m,m) \\
&\quad \times \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{-1} \middle| \sum_{j=1}^{r} \left(\frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1} \right) x^{n-l-m} \right\rangle \\
&= \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-1-l,m) \left(r \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{-1} \middle| \frac{t}{\ln(1+t)} x^{l+1} \right\rangle \right. \\
&\quad \left. - \left(\sum_{j=1}^{r} a_j \right) \left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1} (1+t)^{a_j-1} \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) \middle| \frac{t}{\ln(1+t)} x^{l+1} \right\rangle \right) \\
&= \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1,m) \left(r \left\langle \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{-1} \middle| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right\rangle \right. \\
&\quad \left. - \left(\sum_{j=1}^{r} a_j \right) \left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1} (1+t)^{a_j-1} \prod_{i=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) \middle| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right\rangle \right) \\
&= \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1,m) \left(r \sum_{i=0}^{l+1} \binom{l+1}{i} c_i D_{l+1-i}(-1|a_1,\ldots,a_r) \right. \\
&\quad \left. - \sum_{j=1}^{r} a_j \sum_{i=0}^{l+1} \binom{l+1}{i} c_i D_{l+1-i}(a_j-1|a_1,\ldots,a_r,a_j) \right).
\end{aligned}
$$

Therefore, we have, for $n-1 \geq m \geq 1$,

$$
\begin{aligned}
& m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) D_l(a_1,\ldots,a_r) \\
&= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) D_l(-1|a_1,\ldots,a_r) \\
&\quad + \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1,m) \\
&\quad \times \left(r \sum_{i=0}^{l+1} \binom{l+1}{i} c_{l+1-i} D_i(-1|a_1,\ldots,a_r) \right. \\
&\quad \left. - \sum_{j=1}^{r} \sum_{i=0}^{l+1} a_j \binom{l+1}{i} c_{l+1-i} D_i(a_j-1|a_1,\ldots,a_r,a_j) \right).
\end{aligned}
$$

Thus, we get (37).

Next, we shall compute

$$
\left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j}-1} \right) \bigl(\ln(1+t)\bigr)^m \middle| x^n \right\rangle
$$

in two different ways. On the one hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m \middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \bigl(\ln(1+t)\bigr)^m x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n\right\rangle \\
&= \sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| x^{n-l-m}\right\rangle \\
&= \sum_{l=0}^{n-m} m!\binom{n}{l+m}S_1(l+m,m)\widehat{D}_{n-l-m}(a_1,\ldots,a_r) \\
&= \sum_{l=0}^{n-m} m!\binom{n}{l}S_1(n-l,m)\widehat{D}_{l}(a_1,\ldots,a_r).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m \middle| x^n\right\rangle \\
&= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m\right) \middle| x^{n-1}\right\rangle \\
&= \left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)\bigl(\ln(1+t)\bigr)^m \middle| x^{n-1}\right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\partial_t\bigl(\bigl(\ln(1+t)\bigr)^m\bigr) \middle| x^{n-1}\right\rangle. \qquad (40)
\end{aligned}$$

The second term of (40) is equal to

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\partial_t\bigl(\bigl(\ln(1+t)\bigr)^m\bigr) \middle| x^{n-1}\right\rangle \\
&= m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| \bigl(\ln(1+t)\bigr)^{m-1}x^{n-1}\right\rangle \\
&= m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| \sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)t^{l+m-1}x^{n-1}\right\rangle \\
&= m\sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)(n-1)_{l+m-1} \\
&\quad \times \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| x^{n-l-m}\right\rangle
\end{aligned}$$

$$= m!\sum_{l=0}^{n-m}\binom{n-1}{l+m-1}S_1(l+m-1,m-1)\widehat{D}_{n-l-m}(-1|a_1,\ldots,a_r)$$
$$= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l(-1|a_1,\ldots,a_r).$$

The first term of (40) is equal to

$$\left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)\left(\ln(1+t)\right)^m \middle| x^{n-1}\right\rangle$$
$$= \left\langle \partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| \left(\ln(1+t)\right)^m x^{n-1}\right\rangle$$
$$= \left\langle \partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| \sum_{l=0}^{n-m-1}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^{n-1}\right\rangle$$
$$= \sum_{l=0}^{n-m-1}\frac{m!}{(l+m)!}S_1(l+m,m)(n-1)_{l+m}\left\langle \partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle| x^{n-l-m-1}\right\rangle.$$

From the proof of (36), we recall

$$\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)$$
$$= \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)\frac{\sum_{j=1}^{r}(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1})}{t}$$
$$+ \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)\sum_{j=1}^{r}a_j.$$

Hence, the first term of (39) is equal to

$$\sum_{l=0}^{n-m-1} m!\binom{n-1}{l+m}S_1(l+m,m)$$
$$\times \left(\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| \frac{\sum_{j=1}^{r}(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1})}{t}x^{n-l-m-1}\right\rangle\right.$$
$$\left.+ \left(\sum_{j=1}^{r}a_j\right)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| x^{n-l-m-1}\right\rangle\right)$$
$$= m!\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)$$
$$\times \left(\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| \frac{\sum_{j=1}^{r}(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1})}{t}x^{l}\right\rangle\right.$$

$$
\begin{aligned}
&+ \left(\sum_{j=1}^{r} a_j\right)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| x^l\right\rangle\Bigg) \\
&= m!\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m) \\
&\quad\times\Bigg(\frac{1}{l+1}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| \sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)x^{l+1}\right\rangle \\
&\quad+ \left(\sum_{j=1}^{r} a_j\right)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| x^l\right\rangle\Bigg) \\
&= m!\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m) \\
&\quad\times\Bigg(\frac{r}{l+1}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| \frac{t}{\ln(1+t)}x^{l+1}\right\rangle \\
&\quad- \frac{1}{l+1}\left(\sum_{j=1}^{r} a_j\right)\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| \frac{t}{\ln(1+t)}x^{l+1}\right\rangle \\
&\quad+ \left(\sum_{j=1}^{r} a_j\right)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| x^l\right\rangle\Bigg) \\
&= m!\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m) \\
&\quad\times\Bigg(\frac{r}{l+1}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| \sum_{i=0}^{l+1}c_i\frac{t^i}{i!}x^{l+1}\right\rangle \\
&\quad- \frac{1}{l+1}\left(\sum_{j=1}^{r} a_j\right)\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| \sum_{i=0}^{l+1}c_i\frac{t^i}{i!}x^{l+1}\right\rangle \\
&\quad+ \left(\sum_{j=1}^{r} a_j\right)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1}\middle| x^l\right\rangle\Bigg) \\
&= m!\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m) \\
&\quad\times\Bigg(\frac{r}{l+1}\sum_{i=0}^{l+1}\binom{l+1}{i}c_i\widehat{D}_{l+1-i}(-1|a_1,\ldots,a_r) \\
&\quad- \frac{1}{l+1}\sum_{j=1}^{r}a_j\sum_{i=0}^{l+1}\binom{l+1}{i}c_i\widehat{D}_{l+1-i}(-1|a_1,\ldots,a_r,a_j)+\sum_{j=1}^{r}a_j\widehat{D}_l(-1|a_1,\ldots,a_r)\Bigg) \\
&= \frac{m!}{n}\sum_{l=0}^{n-m-1}\binom{n}{l+1}S_1(n-l-1,m) \\
&\quad\times\Bigg(r\sum_{i=0}^{l+1}\binom{l+1}{i}c_i\widehat{D}_{l+1-i}(-1|a_1,\ldots,a_r)
\end{aligned}
$$

$$- \sum_{j=1}^{r} \sum_{i=0}^{l+1} \binom{l+1}{i} a_j c_i \widehat{D}_{l+1-i}(-1|a_1, \ldots, a_r, a_j) \Bigg)$$
$$+ m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \sum_{j=1}^{r} a_j \widehat{D}_l(-1|a_1, \ldots, a_r).$$

Therefore, we get (38). □

3.8 Relations with the falling factorials

Theorem 8

$$D_n(x|a_1, \ldots, a_r) = \sum_{m=0}^{n} \binom{n}{m} D_{n-m}(a_1, \ldots, a_r)(x)_m, \tag{41}$$

$$\widehat{D}_n(x|a_1, \ldots, a_r) = \sum_{m=0}^{n} \binom{n}{m} \widehat{D}_{n-m}(a_1, \ldots, a_r)(x)_m. \tag{42}$$

Proof For (13) and (23), assume that $D_n(x|a_1, \ldots, a_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (12), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^{r} (\frac{e^{a_j \ln(1+t)} - 1}{\ln(1+t)})} t^m \middle| x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| t^m x^n \right\rangle \\
&= \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-m} \right\rangle \\
&= \binom{n}{m} D_{n-m}(a_1, \ldots, a_r).
\end{aligned}$$

Thus, we get the identity (41).

Similarly, for (13) and (23), assume that $\widehat{D}_n(x|a_1, \ldots, a_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (12), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^{r} (\frac{e^{a_j \ln(1+t)} - 1}{e^{a_j \ln(1+t)} \ln(1+t)})} t^m \middle| x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| t^m x^n \right\rangle \\
&= \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-m} \right\rangle \\
&= \binom{n}{m} \widehat{D}_{n-m}(a_1, \ldots, a_r).
\end{aligned}$$

Thus, we get the identity (42). □

3.9 Relations with higher-order Frobenius-Euler polynomials

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

(see *e.g.* [11, 12]).

Theorem 9

$$D_n(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j \times (1-\lambda)^{-j}S_1(n-j-l,m)D_l(a_1,\dots,a_r)\right)H_m^{(s)}(x|\lambda), \tag{43}$$

$$\widehat{D}_n(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j \times (1-\lambda)^{-j}S_1(n-j-l,m)\widehat{D}_l(a_1,\dots,a_r)\right)H_m^{(s)}(x|\lambda). \tag{44}$$

Proof For (13) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right), \tag{45}$$

assume that $D_n(x|a_1,\dots,a_r) = \sum_{m=0}^{n} C_{n,m}H_m^{(s)}(x|\lambda)$. By (12), similarly to the proof of (37), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{(\frac{e^{\ln(1+t)}-\lambda}{1-\lambda})^s}{\prod_{j=1}^{r}(\frac{e^{a_j\ln(1+t)}-1}{\ln(1+t)})}\big(\ln(1+t)\big)^m \middle| x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m(1-\lambda+t)^s \middle| x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| \sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\lambda)^{s-i}t^ix^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|\big(\ln(1+t)\big)^m x^{n-i}\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)D_l(a_1,\dots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\lambda)^{-i}S_1(n-i-l,m)D_l(a_1,\dots,a_r).
\end{aligned}$$

Thus, we get the identity (43).

Next, for (14) and (45), assume that $\widehat{D}_n(x|a_1,\ldots,a_r)=\sum_{m=0}^{n}C_{n,m}H_m^{(s)}(x|\lambda)$. By (12), similarly to the proof of (38), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{(\frac{e^{\ln(1+t)}-\lambda}{1-\lambda})^s}{\prod_{j=1}^{r}(\frac{e^{a_j\ln(1+t)}-1}{e^{a_j\ln(1+t)}\ln(1+t)})}\bigl(\ln(1+t)\bigr)^m\middle| x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m\middle|(1-\lambda+t)^s x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m\middle|\sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\lambda)^{s-i}t^i x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|\bigl(\ln(1+t)\bigr)^m x^{n-i}\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{D}_l(a_1,\ldots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\lambda)^{-i}S_1(n-i-l,m)\widehat{D}_l(a_1,\ldots,a_r).
\end{aligned}$$

Thus, we get the identity (44). □

3.10 Relations with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see *e.g.* [5, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r=\sum_{n=0}^{\infty}\frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see *e.g.* [13, (2.1)], [14, (6)]).

Theorem 10

$$\begin{aligned}
&D_n(x|a_1,\ldots,a_r) \\
&\quad=\sum_{m=0}^{n}\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)D_l(a_1,\ldots,a_r)\right)\mathfrak{B}_m^{(s)}(x), \qquad (46)\\
&\widehat{D}_n(x|a_1,\ldots,a_r) \\
&\quad=\sum_{m=0}^{n}\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{D}_l(a_1,\ldots,a_r)\right)\mathfrak{B}_m^{(s)}(x). \qquad (47)
\end{aligned}$$

Proof For (13) and

$$\mathfrak{B}_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right), \tag{48}$$

assume that $D_n(x|a_1,\dots,a_r) = \sum_{m=0}^n C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (12), similarly to the proof of (37), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{(\frac{e^{\ln(1+t)}-1}{\ln(1+t)})^s}{\prod_{j=1}^r(\frac{e^{a_j\ln(1+t)}-1}{\ln(1+t)})}\big(\ln(1+t)\big)^m \middle| x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| \left(\frac{t}{\ln(1+t)}\right)^s x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| \sum_{i=0}^\infty \mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle \prod_{j=1}^r\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| x^{n-i}\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i} m!\binom{n-i}{l}S_1(n-i-l,m)D_l(a_1,\dots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)D_l(a_1,\dots,a_r).
\end{aligned}$$

Thus, we get the identity (46).

Next, for (13) and (48), assume that $\widehat{D}_n(x|a_1,\dots,a_r) = \sum_{m=0}^n C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (12), similarly to the proof of (38), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{(\frac{e^{\ln(1+t)}-1}{\ln(1+t)})^s}{\prod_{j=1}^r(\frac{e^{a_j\ln(1+t)}-1}{e^{a_j\ln(1+t)}\ln(1+t)})}\big(\ln(1+t)\big)^m \middle| x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| \left(\frac{t}{\ln(1+t)}\right)^s x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| \sum_{i=0}^\infty \mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| x^{n-i}\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i} m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{D}_l(a_1,\dots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{D}_l(a_1,\dots,a_r).
\end{aligned}$$

Thus, we get the identity (47). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea. [3]Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan. [4]Department of Applied Mathematics, Pukyong National University, Pusan, 608-739, Republic of Korea.

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References
1. Kim, T: An invariant p-adic integral associated with Daehee numbers. Integral Transforms Spec. Funct. **13**, 65-69 (2002)
2. Kim, DS, Kim, T, Rim, S-H: On the associated sequence of special polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **23**, 355-366 (2013)
3. Ozden, H, Cangul, IN, Simsek, Y: Remarks on q-Bernoulli numbers associated with Daehee numbers. Adv. Stud. Contemp. Math. (Kyungshang) **18**, 41-48 (2009)
4. Park, J-W, Rim, S-H, Kwon, J: The twisted Daehee numbers and polynomials. Adv. Differ. Equ. **2014**, 1 (2014)
5. Roman, S: The Umbral Calculus. Dover, New York (2005)
6. Kim, T: On Euler-Barnes multiple zeta functions. Russ. J. Math. Phys. **10**, 261-267 (2003)
7. Kim, T: Barnes-type multiple q-zeta functions and q-Euler polynomials. J. Phys. A **43**, 255201 (2010)
8. Bayad, A, Kim, T, Kim, WJ, Lee, SH: Arithmetic properties of q-Barnes polynomials. J. Comput. Anal. Appl. **15**, 111-117 (2013)
9. Comtet, L: Advanced Combinatorics. Reidel, Dordrecht (1974)
10. Komatsu, T: Poly-Cauchy numbers. Kyushu J. Math. **67**, 143-153 (2013)
11. Kim, DS, Kim, T: Some identities of Frobenius-Euler polynomials arising from umbral calculus. Adv. Differ. Equ. **2012**, 196 (2012)
12. Kim, DS, Kim, T, Lee, S-H: Poly-Bernoulli polynomials arising from umbral calculus. arXiv:1306.6697
13. Carlitz, L: A note on Bernoulli and Euler polynomials of the second kind. Scr. Math. **25**, 323-330 (1961)
14. Liang, H, Wuyungaowa: Identities involving generalized harmonic numbers and other special combinatorial sequences. J. Integer Seq. **15**, Article 12.9.6 (2012)

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