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Uniqueness results for fully anti-periodic fractional boundary value problems with nonlinearity depending on lower-order derivatives

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Abstract

We investigate the uniqueness of solutions for fully anti-periodic fractional boundary value problems of order $2 < q \leq 3$ with nonlinearity depending on lower-order fractional derivatives. Our results are based on some standard fixed point theorems. The paper concludes with illustrative examples.

MSC: 34A12; 34A40

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1 Introduction

In this article, we show the existence of solutions for a fully fractional-order anti-periodic boundary value problem of the form

$${}^c D^q x(t) = f(t, x(t), {}^c D^r x(t)), \quad t \in [0, T], T > 0, 2 < q \leq 3, 0 < r \leq 1, \quad (1.1)$$

$$x(0) = -x(T), \quad {}^c D^p x(0) = -{}^c D^p x(T), \quad (1.2)$$

$${}^c D^{p+1} x(0) = -{}^c D^{p+1} x(T), \quad 0 < p < 1,$$

where ${}^c D^q$ denotes the Caputo fractional derivative of order q and f is a given continuous function.

As a second problem, we will discuss the existence of solutions for the following fractional differential equation with the boundary conditions (1.2):

$${}^c D^q x(t) = f(t, x(t), {}^c D^r x(t), {}^c D^{r+1} x(t)), \quad t \in [0, T], T > 0, 2 < q \leq 3, 0 < r \leq 1. \quad (1.3)$$

The present work is motivated by a recent paper [1] in which the problem (1.1)-(1.2) was discussed with the nonlinearity of the type $f(t, x)$. Thus the present paper generalizes the results obtained in [1].

In the last few decades, fractional calculus has evolved as an attractive field of research in view of its extensive applications in basic and technical sciences. Examples can be found

in physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [2–5].

The subject of boundary value problems of differential equations, having an enriched history, has been progressing at the same pace as before. In the context of fractional boundary value problems, there has been a much development in the last ten years; for instance, see [6–25] and the references cited therein.

In view of the importance of anti-periodic boundary conditions in the mathematical modeling of a variety of physical processes [26–28], the study of anti-periodic boundary value problems has received considerable attention. Some recent work on anti-periodic boundary value problems of fractional order can be found in a series of papers [29–34] and the references therein.

2 Preliminaries

We begin this section with some basic concepts [3, 4].

Definition 2.1 The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2 For a function $g \in AC^{n-1}([0, \infty), \mathbb{R})$, the Caputo derivative of fractional order q is defined as

$${}^c D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q .

Lemma 2.1 [1] For any $y \in C[0, T]$, the unique solution of the linear fractional equation ${}^c D^q x(t) = y(t)$, $0 < t < T$, $2 < q \leq 3$ with anti-periodic boundary conditions (1.2) is given by

$$x(t) = \int_0^T G(t,s)y(s) ds,$$

where $G(t,s)$ is the Green's function (depending on q and p) given by

$$G(t,s) = \begin{cases} \frac{2(t-s)^{q-1} - (T-s)^{q-1}}{2\Gamma(q)} + \frac{\Gamma(2-p)(T-2t)(T-s)^{q-p-1}}{2T^{1-p}\Gamma(q-p)} \\ + \frac{(\Gamma(2-p))^2 T^{p-1} (T-s)^{q-p-2}}{4\Gamma(q-p-1)\Gamma(3-p)} \left\{ \frac{(T^2-2t^2)\Gamma(3-p)}{\Gamma(2-p)} - 2T^2 + 4tT \right\}, & s \leq t, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{\Gamma(2-p)(T-2t)(T-s)^{q-p-1}}{2T^{1-p}\Gamma(q-p)} \\ + \frac{(\Gamma(2-p))^2 T^{p-1} (T-s)^{q-p-2}}{4\Gamma(q-p-1)\Gamma(3-p)} \left\{ \frac{(T^2-2t^2)\Gamma(3-p)}{\Gamma(2-p)} - 2T^2 + 4tT \right\}, & t < s. \end{cases}$$

3 Uniqueness of solutions

This section is devoted to the uniqueness of solutions for the problems at hand by means of Banach's contraction principle.

3.1 Uniqueness result for the problem (1.1)-(1.2)

For $0 < r \leq 1$, let us define a space $\mathcal{C} = \{x : x, {}^c D^r x \in C([0, T])\}$, where $C([0, T])$ denotes the space of all continuous functions defined on $[0, T]$. Note that the space \mathcal{C} endowed with the norm defined by $\|x\| = \sup\{|x(t)| + |{}^c D^r x(t)|, t \in [0, T]\}$ is a Banach space.

In view of Lemma 2.1, let us define an operator $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{C}$ associated with the problem (1.1)-(1.2) as

$$\begin{aligned}
 (\mathcal{G}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^c D^r x(s)) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^c D^r x(s)) ds \\
 &\quad + \mu(t) \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} f(s, x(s), {}^c D^r x(s)) ds \\
 &\quad + \nu(t) \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} f(s, x(s), {}^c D^r x(s)) ds,
 \end{aligned} \tag{3.1}$$

where

$$\mu(t) = \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}}, \quad \nu(t) = \frac{\Gamma(2-p)(T^2 - 2t^2 - \frac{2\Gamma(2-p)T^2}{\Gamma(3-p)} + \frac{4t\Gamma(2-p)}{\Gamma(3-p)})}{4T^{1-p}}.$$

Observe that the problem (1.1)-(1.2) has a solution only if the operator \mathcal{G} has a fixed point.

Before proceeding further, let us introduce some notations:

$$N = \max\{N_1, N_2\}, \tag{3.2}$$

where

$$\begin{aligned}
 N_1 &= T^q \left[\frac{3}{2\Gamma(q+1)} + \frac{\Gamma(2-p)}{2\Gamma(q-p+1)} + \frac{\Gamma(2-p)}{4\Gamma(q-p)} \left[1 - \frac{2\Gamma(2-p)}{\Gamma(3-p)} + 2 \left(\frac{\Gamma(2-p)}{\Gamma(3-p)} \right)^2 \right] \right], \\
 N_2 &= T^{q-r} \left[\frac{1}{\Gamma(q-r+1)} + \frac{\Gamma(2-p)}{\Gamma(2-r)\Gamma(q-p+1)} \right. \\
 &\quad \left. + \frac{\Gamma(2-p)}{\Gamma(q-p)} \left[\frac{\Gamma(2-p)}{\Gamma(3-p)\Gamma(2-r)} - \frac{1}{2\Gamma(3-r)} \right] \right].
 \end{aligned}$$

Theorem 3.1 Assume that $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the condition

$$|f(t, x, \bar{x}) - f(t, y, \bar{y})| \leq L(|x - y| + |\bar{x} - \bar{y}|), \quad \forall t \in [0, T], x, y, \bar{x}, \bar{y} \in \mathbb{R}$$

with $L < \frac{1}{N}$, where N is given by (3.2). Then the anti-periodic boundary value problem (1.1)-(1.2) has a unique solution.

Proof Let us set $\sup_{t \in [0, T]} |f(t, 0, 0)| = M < \infty$ and $R \geq MN(1 - LN)^{-1}$ to show that $\mathcal{G}B_R \subset B_R$, where $B_R = \{x \in \mathcal{C} : \|x\| \leq R\}$. For $x \in B_R$, we have

$$\begin{aligned}
 |(\mathcal{G}x)(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s), {}^c D^r x(s))| ds \\
 &\quad + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x(s), {}^c D^r x(s))| ds
 \end{aligned}$$

$$\begin{aligned}
 & + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s, x(s), {}^cD^r x(s))| ds \\
 & + |\nu(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} |f(s, x(s), {}^cD^r x(s))| ds \\
 \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (|f(s, x(s), {}^cD^r x(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) ds \\
 & + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (|f(s, x(s), {}^cD^r x(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) ds \\
 & + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} (|f(s, x(s), {}^cD^r x(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) ds \\
 & + |\nu(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} (|f(s, x(s), {}^cD^r x(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) ds \\
 \leq & (LR + M) \left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \right. \\
 & \left. + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} ds + |\nu(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} ds \right] \\
 \leq & (LR + M)N_1 \leq (LR + M)N \leq R.
 \end{aligned}$$

Using the facts ${}^cD^r b = 0$ (b is a constant), ${}^cD^r t = \frac{t^{1-r}}{\Gamma(2-r)}$, ${}^cD^r t^2 = \frac{2t^{2-r}}{\Gamma(3-r)}$, ${}^cD^{r+1} t^2 = \frac{2t^{1-r}}{\Gamma(2-r)}$, for $0 < r < 1$, we get

$$\begin{aligned}
 ({}^cD^r \mathcal{G}x)(t) = & \int_0^t \frac{(t-s)^{q-r-1}}{\Gamma(q-r)} f(s, x(s), {}^cD^r x(s)) ds \\
 & - \frac{\Gamma(2-p)t^{1-r}}{\Gamma(2-r)T^{1-p}} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} f(s, x(s), {}^cD^r x(s)) ds \\
 & + \Gamma(2-p)T^{p-1} \left[\frac{Tt^{1-r}\Gamma(2-p)}{\Gamma(3-p)\Gamma(2-r)} - \frac{t^{2-r}}{\Gamma(3-r)} \right] \\
 & \times \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} f(s, x(s), {}^cD^r x(s)) ds.
 \end{aligned}$$

As in the previous step, it can be shown that

$$|({}^cD^r \mathcal{G}x)(t)| \leq (LR + M)N_2 \leq (LR + M)N \leq R.$$

Thus we get $\mathcal{G}x \in B_R$. Hence $\mathcal{G}B_R \subset B_R$. Next, for $x_1, x_2 \in \mathcal{C}$ and for each $t \in [0, T]$, we obtain

$$\begin{aligned}
 & |(\mathcal{G}x_1)(t) - (\mathcal{G}x_2)(t)| \\
 \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x_1, {}^cD^r x_1) - f(s, x_2, {}^cD^r x_2)| ds \\
 & + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x_1, {}^cD^r x_1) - f(s, x_2, {}^cD^r x_2)| ds \\
 & + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s, x_1, {}^cD^r x_1) - f(s, x_2, {}^cD^r x_2)| ds
 \end{aligned}$$

$$\begin{aligned}
 & + |v(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} |f(s, x_1, {}^c D^r x_1) - f(s, x_2, {}^c D^r x_2)| ds \\
 \leq & L \|x_1 - x_2\| \left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \right. \\
 & \left. + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} ds + |v(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} ds \right] \\
 < & LN_1 \|x_1 - x_2\| \\
 \leq & LN \|x_1 - x_2\|.
 \end{aligned}$$

In a similar manner, we find that

$$|({}^c D^r \mathcal{G}x_1)(t) - ({}^c D^r \mathcal{G}x_2)(t)| \leq LN_2 \|x_1 - x_2\| \leq LN \|x_1 - x_2\|.$$

By the given assumption, $L < 1/N$, it follows that the operator \mathcal{G} is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). \square

3.2 Uniqueness result for the problem (1.3)-(1.2)

Here, we study the uniqueness of solutions for the problem of (1.3)-(1.2). For that, let $\bar{C} = \{x : x, {}^c D^r x, {}^c D^{r+1} x(t) \in C([0, T])\}$ be a Banach space endowed with the norm $\|x\| = \sup\{|x(t)| + |{}^c D^r x(t)| + |{}^c D^{r+1} x(t)|, t \in [0, T]\}$, $0 < r \leq 1$.

Relative to the problem (1.3)-(1.2), we define an operator $\bar{\mathcal{G}} : \bar{C} \rightarrow \bar{C}$ as

$$\begin{aligned}
 (\bar{\mathcal{G}}x)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(s)) ds \\
 & - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(s)) ds \\
 & + \mu(t) \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(s)) ds \\
 & + v(t) \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(s)) ds.
 \end{aligned} \tag{3.3}$$

In what follows, we set

$$\bar{N} = \max\{N, N_3\}, \tag{3.4}$$

where N is given by (3.2) and

$$N_3 = T^{q-r-1} \left[\frac{1}{\Gamma(q-r)} + \frac{\Gamma(2-p)}{\Gamma(2-r)\Gamma(q-p)} \right].$$

Theorem 3.2 *Let $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and there exists a positive number $\bar{L} < 1/\bar{N}$ such that*

$$|f(t, x, \bar{x}, \bar{\bar{x}}) - f(t, y, \bar{y}, \bar{\bar{y}})| \leq \bar{L} (|x - y| + |\bar{x} - \bar{y}| + |\bar{\bar{x}} - \bar{\bar{y}}|), \quad \forall t \in [0, T], x, y, \bar{x}, \bar{y}, \bar{\bar{x}}, \bar{\bar{y}} \in \mathbb{R}.$$

Then the anti-periodic boundary value problem (1.3)-(1.2) has a unique solution on $[0, T]$.

Proof We define $B_{\bar{R}} = \{x \in \bar{C} : \|x\| \leq \bar{R}\}$, $\bar{R} \geq \frac{\bar{M}\bar{N}}{1-\bar{L}\bar{N}}$, $\bar{M} = \sup_{t \in [0, T]} |f(t, 0, 0, 0)| < \infty$ and show that $\bar{G}B_{\bar{R}} \subset B_{\bar{R}}$. In view of the given assumption, we have

$$\begin{aligned}
 &|f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(t))| \\
 &\leq |f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(t)) - f(t, 0, 0, 0)| + |f(t, 0, 0, 0)| \\
 &\leq \bar{L}\bar{R} + \bar{M}, \quad x \in B_{\bar{R}}.
 \end{aligned} \tag{3.5}$$

Thus

$$\begin{aligned}
 |(\bar{G}x)(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(t))| ds \\
 &\quad + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(t))| ds \\
 &\quad + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(t))| ds \\
 &\quad + |\nu(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} |f(s, x(s), {}^c D^r x(s), {}^c D^{r+1} x(t))| ds \\
 &\leq (\bar{L}\bar{R} + \bar{M}) \left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \right. \\
 &\quad \left. + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} ds + |\nu(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} ds \right] \\
 &\leq (\bar{L}\bar{R} + \bar{M})N \\
 &\leq (\bar{L}\bar{R} + \bar{M})\bar{N} \leq \bar{R}.
 \end{aligned}$$

Further, it can be shown in a similar way that

$$\begin{aligned}
 |({}^c D^r \bar{G}x)(t)| &\leq (\bar{L}\bar{R} + \bar{M})N_2 \leq (\bar{L}\bar{R} + \bar{M})\bar{N} \leq \bar{R}, \\
 |({}^c D^{r+1} \bar{G}x)(t)| &\leq (\bar{L}\bar{R} + \bar{M})N_3 \leq (\bar{L}\bar{R} + \bar{M})\bar{N} \leq \bar{R}.
 \end{aligned}$$

Next, for $x_1, x_2 \in \bar{C}$ and for each $t \in [0, T]$, we obtain

$$\begin{aligned}
 &|(\bar{G}x_1)(t) - (\bar{G}x_2)(t)| \\
 &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x_1, {}^c D^r x_1, {}^c D^{r+1} x_1) - f(s, x_2, {}^c D^r x_2, {}^c D^{r+1} x_2)| ds \\
 &\quad + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x_1, {}^c D^r x_1, {}^c D^{r+1} x_1) - f(s, x_2, {}^c D^r x_2, {}^c D^{r+1} x_2)| ds \\
 &\quad + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s, x_1, {}^c D^r x_1, {}^c D^{r+1} x_1) - f(s, x_2, {}^c D^r x_2, {}^c D^{r+1} x_2)| ds \\
 &\quad + |\nu(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} |f(s, x_1, {}^c D^r x_1, {}^c D^{r+1} x_1) - f(s, x_2, {}^c D^r x_2, {}^c D^{r+1} x_2)| ds \\
 &\leq L \|x_1 - x_2\| \left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \right.
 \end{aligned}$$

$$\begin{aligned}
 & + |\mu(t)| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} ds + |\nu(t)| \int_0^T \frac{(T-s)^{q-p-2}}{\Gamma(q-p-1)} ds \Big] \\
 & < \bar{L}N_1 \|x_1 - x_2\| \\
 & \leq \bar{L}\bar{N} \|x_1 - x_2\|.
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 |({}^c D^r \bar{G}x_1)(t) - ({}^c D^r \bar{G}x_2)(t)| & \leq LN_2 \|x_1 - x_2\| \leq \bar{L}\bar{N} \|x_1 - x_2\|, \\
 |({}^c D^{r+1} \bar{G}x_1)(t) - ({}^c D^{r+1} \bar{G}x_2)(t)| & \leq \bar{L}N_3 \|x_1 - x_2\| \leq \bar{L}\bar{N} \|x_1 - x_2\|.
 \end{aligned}$$

Since $\bar{L} < 1/\bar{N}$, therefore, the operator \bar{G} is a contraction. Thus, it follows by the contraction mapping principle that the problem (1.3)-(1.2) has a unique solution on $[0, T]$. \square

4 Examples

(a) Consider the anti-periodic fractional boundary value problem given by

$$\begin{aligned}
 {}^c D^{\frac{5}{2}}x(t) & = L \left(\frac{|x(t)|}{1+|x(t)|} + \frac{1}{\sqrt{1+t}} \left(\frac{|{}^c D^{\frac{1}{2}}x(t)|}{1+|{}^c D^{\frac{1}{2}}x(t)|} \right) \right) + \sqrt{1+\sin^3 t}, \quad L > 0, t \in [0, 1], \\
 x(0) & = -x(1), \quad {}^c D^{\frac{1}{2}}x(0) = -{}^c D^{\frac{1}{2}}x(1), \quad {}^c D^{\frac{3}{2}}x(0) = -{}^c D^{\frac{3}{2}}x(1),
 \end{aligned} \tag{4.1}$$

where $q = 5/2, p = 1/2, r = 1/2, T = 1$, and

$$f(t, x, \bar{x}) = L \left(\frac{|x|}{1+|x|} + \frac{1}{\sqrt{1+t}} \left(\frac{|\bar{x}|}{1+|\bar{x}|} \right) \right) + \sqrt{1+\sin^3 t}, \quad \bar{x} = {}^c D^{\frac{1}{2}}x(t).$$

Clearly

$$\begin{aligned}
 |f(t, x, \bar{x}) - f(t, y, \bar{y})| & \leq L \left(\left| \frac{x}{1+x} - \frac{y}{1+y} \right| + \left| \frac{\bar{x}}{1+\bar{x}} - \frac{\bar{y}}{1+\bar{y}} \right| \right) \leq L(|x-y| + |\bar{x}-\bar{y}|), \\
 N_1 & = \frac{4}{5\sqrt{\pi}} + \frac{7\sqrt{\pi}}{36}, \quad N_2 = \frac{5}{3}.
 \end{aligned}$$

With $L < 1/N$ ($N = N_2 > N_1$), all the assumptions of Theorem 3.1 hold. Therefore, the problem (4.1) has a unique solution on $[0, 1]$.

(b) Consider the following anti-periodic fractional boundary value problem:

$$\begin{aligned}
 {}^c D^{\frac{5}{2}}x(t) & = \bar{L} \left(\frac{|x(t)|}{1+|x(t)|} + \tan^{-1} |{}^c D^{\frac{1}{2}}x(t)| + \cos t \left(\frac{|{}^c D^{\frac{3}{2}}x(t)|}{1+|{}^c D^{\frac{3}{2}}x(t)|} \right) \right) \\
 & + e^{-t}, \quad \bar{L} > 0, t \in [0, 1], \\
 x(0) & = -x(1), \quad {}^c D^{\frac{1}{2}}x(0) = -{}^c D^{\frac{1}{2}}x(1), \quad {}^c D^{\frac{3}{2}}x(0) = -{}^c D^{\frac{3}{2}}x(1),
 \end{aligned} \tag{4.2}$$

where $q = 5/2, p = 3/4, r = 1/2, T = 1$. With $\bar{x} = {}^c D^{\frac{1}{2}}x(t), \bar{\bar{x}} = {}^c D^{\frac{3}{2}}x(t)$, we can write

$$f(t, x, \bar{x}, \bar{\bar{x}}) = \bar{L} \left(\frac{|x|}{1+|x|} + \tan^{-1} |\bar{x}| + \cos t \left(\frac{|\bar{\bar{x}}|}{1+|\bar{\bar{x}}|} \right) \right) + e^{-t}.$$

Furthermore, we have

$$|f(t, x, \bar{x}, \bar{x}) - f(t, y, \bar{y}, \bar{y})| \leq \bar{L}(|x - y| + |\bar{x} - \bar{y}| + |\bar{x} - \bar{y}|),$$
$$N_1 = \frac{4}{5\sqrt{\pi}} - \frac{73\Gamma(1/4)}{525\Gamma(3/4)}, \quad N_2 = \frac{1}{2} + \frac{61\Gamma(1/4)}{105\sqrt{\pi}\Gamma(3/4)}, \quad N_3 = 1 + \frac{2\Gamma(1/4)}{3\sqrt{\pi}\Gamma(3/4)}.$$

Clearly all the assumptions of Theorem 3.2 are satisfied with $\bar{L} < 1/\bar{N}$ ($\bar{N} = N_3 > N_2 > N_1$). Hence, the problem (4.2) has a unique solution on $[0, 1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors, AA, BA, NM, and SKN, contributed to each part of this work equally and read and approved the final version of the manuscript.

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