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Periodic boundary value problems for second-order functional differential equations with impulse

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Abstract
In this paper, we study the existence of multiple positive solutions of the second-order periodic boundary value problems for functional differential equations with impulse. The proof of our main results is based upon the fixed point index theorem in cones.

Keywords: boundary value problem; impulsive functional differential equations; multiple solutions; cones

1 Introduction

In this paper, we will consider the existence of positive solutions for boundary value problems of second order impulsive functional differential equations of the form

$$\begin{cases} (\rho(t)u'(t))' + f(t,u_t) = 0, \quad t \in J, t \neq t_k, k = 1,\ldots,m, \\ -\Delta u'(t_k) = I_k(u_{t_k}), \quad k = 1,2,\ldots,m, \\ u_0 = \varphi, \qquad u(T) = A, \end{cases} \tag{1.1}$$

where $J = [0,T]$, $f : J \times C_\tau \to R$ is a continuous function, $\varphi \in C_\tau$ (C_τ be given in Section 2), $\tau \geq 0$, $\rho(t) \in C(J,(0,\infty))$, $u_t \in C_\tau$, $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau,0]$. $I_k \in C(C_\tau,R)$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $J' = (0,T)\backslash\{t_1,\ldots,t_m\}$. $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u'(t_k^+)$ ($u'(t_k^-)$) denote the right limit (left limit) of $u'(t)$ at $t = t_k$, and $A \in R = (-\infty,+\infty)$.

Impulsive differential equations describe processes which experience a sudden change of their state at certain moments. The theory of impulse differential equations has been a significant development in recent years and played a very important role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, chemical technology and biotechnology; see [1–3].

Many papers have been published about the existence analysis of periodic boundary value problems of first and second order for ordinary or functional or integro-differential equations with impulsive. We refer the readers to the papers [4–23]. For instance, in [6], He and Yu investigated the following problem:

$$\begin{cases} u'(t) = f(t,u(t),u_t), \quad t \neq t_k, 0 < t < T, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ u(t) = u(0), \quad t \in [-\tau,0), \\ u(0) = u(T). \end{cases}$$

By using the coincidence degree, Dong [4] studied the following periodic boundary value problems (PBVP) for first-order functional differential equations with impulse:

$$\begin{cases} u'(t) = f(t, u_t), & t \neq t_k, 0 < t < T, \\ \Delta u(t_k) = I_k(u(t_k)), & k = 1, 2, \ldots, m, \\ u(0) = u(T). \end{cases} \tag{1.2}$$

It is remarkable that the author required $u(t) = u(t + 1)$ for $t \in [-\tau, 0]$. The author also obtained the existence of one solution of PBVP (1.2).

To study periodic boundary value problems for first and second order functional differential equations with impulse, the approaches used in [4, 6–8, 13–16, 20–23] are the monotone iterative technique and the method of upper and lower solutions. What they obtained is the existence of at least one solution if there is a pair of upper and lower solutions. However, in some cases it is difficult to find upper and lower solutions for general differential equations.

As we know, the fixed point theorem of cone expression and compression is extensively used to study the existence of multiple solutions of boundary value problems for second-order differential equations. In paper [5], Ma considers the following periodic boundary value problem:

$$\begin{cases} u''(t) + f(t, u_t) = 0, & 0 < t < T, \\ u_0 = \varphi, & u(T) = A, \end{cases} \tag{1.3}$$

and he obtained some sufficient conditions for the existence of at least one positive solution of the PBVP (1.3).

In [19], by applying the fixed point theorem of cone expression and compression, Liu investigates the existence of multiple positive solutions of the following problem:

$$\begin{cases} u'(t) = f(t, u_t), & t \neq t_k, t \in (0, T), \\ \Delta u(t_k) = I_k(u(t_k)), & k = 1, 2, \ldots, m, \\ u(t) = \varphi(t), & t \in [-\tau, 0], \\ u(0) = u(T). \end{cases}$$

Motivated by the results in [5, 19, 24], the aim of this paper is to consider the existence of multiple positive solutions for the PBVP (1.1) by using some properties of the Green function and the fixed point index theorem in cones.

This paper will be divided into three sections. In Section 2, we provide some preliminaries and establish several lemmas which will be used throughout Section 3. In Section 3, we shall give the existence theorems of multiplicity positive solutions of PBVP (1.1).

2 Preliminary and lemmas

Let $C_\tau := \{\varphi : [-\tau, 0] \to R;\ \varphi(t)$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\varphi(\bar{t}^+)$ and $\varphi(\bar{t}^-)$ exist and $\varphi(\bar{t}^-) = \varphi(\bar{t})\}$, then C_τ is a normed space with the norm

$$\|\varphi\|_{[-\tau,0]} = \sup_{\theta \in [-\tau,0]} |\varphi(\theta)|, \quad \forall \varphi \in C_\tau.$$

Let $J^* = [-\tau, T]$, $C(J^*)$ and $C^1(J^*)$ represent the set of a continuous and continuously differentiable on J^*, respectively. Moreover, for $u \in C(J^*)$ we define $\|u\|_{J^*} = \sup_{t\in J^*} |u(t)|$. Furthermore, we denote:

$PC(J^*) = \{u : u(t)$ is a map from J^* into R such that $u(t)$ is continuous at $t \neq t_k$, and $u(t_k^+)$, $u(t_k^-)$ exist and $u(t_k^-) = u(t_k)$ for $k = 1, 2, \ldots, m$; $u(t) = \varphi(t)$ for $t \in [-\tau, 0]\}$.

$PC^1(J^*) = \{u \in PC(J^*) : u|_{(t_k,t_{k+1})} \in C^1(t_k, t_{k+1})$, $u'(t_k^-)$ and $u'(t_k^+)$ exist; and $u'(t_k^-) = u'(t_k)$ for $k = 1, 2, \ldots, m\}$. $PC^+(J^*) = \{u \in PC(J^*) : u(t) \geq 0, t \in [-\tau, T]\}$.

Clearly, $PC(J^*)$ is a Banach space with the norm $\|u\|_{J^*} = \sup_{t\in J^*} |u(t)|$ for $u(t) \in PC(J^*)$. $PC^1(J^*)$ is also a Banach space with the norm $\|u\|_1 = \max\{\|u\|_{J^*}, \|u'\|_{J^*}\}$.

We need to assume the following conditions:

(A1) $1 - \int_0^T G(s,s)\,ds > 0$; $\varphi(\theta) \geq 0$ for $\theta \in [-\tau, 0]$, and $A \geq \phi(0)$;

(A2) $f(t,u) \geq 0$ for $t \in [0,T]$ and $u \in PC^+(J^*)$;

(A3) I_k $(k = 1, 2, \ldots, m)$ are continuous and $I_k(u) \geq 0$ for $u \in PC^+(J^*)$.

Lemma 2.1 (see [25]) *Let E be a Banach spaces and $K \subset E$ be a cone in E. Let $r > 0$ and $\Omega_r = \{x \in K : \|x\| < r\}$. Assume that $S : \overline{\Omega}_r \to K$ is a completely continuous operator such that $Sx \neq x$ for $x \in \partial\Omega_r$.*

(i) *If $\|Sx\| \leq \|x\|$ for $x \in \partial\Omega_r$, then $i(S, \Omega_r, K) = 1$.*

(ii) *If $\|Sx\| \geq \|x\|$ for $x \in \partial\Omega_r$, then $i(S, \Omega_r, K) = 0$.*

Lemma 2.2 *For any $y, a_k \in PC(J,R)$, and $\eta, A \in R$. Then the problem*

$$\begin{cases} (\rho(t)u'(t))' + y(t) = 0, & t \in J, t \neq t_k, k = 1, 2, \ldots, m, \\ -\Delta u'(t_k) = a_k, & k = 1, 2, \ldots, m, \\ u(0) = \eta, & u(T) = A \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \eta + \frac{(A-\eta)}{\phi(T)}\phi(t) + \int_0^T G(t,s)y(s)\,ds + \sum_{0<t_k<T} G(t,t_k)\rho(t_k)a_k, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\phi(T)} \begin{cases} (\phi(T) - \phi(t))\phi(s), & 0 \leq s \leq t \leq T, \\ \phi(t)(\phi(T) - \phi(s)), & 0 \leq t \leq s \leq T, \end{cases} \qquad \phi(t) = \int_0^t \frac{ds}{\rho(s)} < +\infty. \tag{2.3}$$

Proof Integrating the first equation of (2.1) over the interval $[0,t]$ for $t \in [0,T)$, we get

$$u(t) = u(0) + \rho(0)u'(0)\phi(t) - \int_0^t \bigl[\phi(t) - \phi(s)\bigr]y(s)\,ds - \sum_{0<t_k<t} \bigl[\phi(t) - \phi(t_k)\bigr]\rho(t_k)a_k. \tag{2.4}$$

It follows from the boundary conditions $u(0) = \eta$, $u(T) = A$ and (2.4) that

$$u'(0) = \frac{1}{\rho(0)\phi(T)}\left[(A-\eta) + \int_0^T \bigl[\phi(T) - \phi(s)\bigr]y(s)\,ds + \sum_{0<t_k<T} \bigl[\phi(T) - \phi(t_k)\bigr]\rho(t_k)a_k\right].$$

Together with (2.4), we obtain (2.2). □

By the standard discussion, we have the following lemma which will be used later.

Lemma 2.3 *The Green function $G(t,s)$ defined in* (2.3) *has the following properties*:

(i) $G(t,s) \geq 0$, $\forall t,s \in [0,T]$;

(ii) $G(t,s) \leq G(s,s) < +\infty$, $\forall t,s \in [0,T]$;

(iii) $0 < \sigma G(s,s) \leq G(t,s)$, $\forall t \in [\alpha,\beta]$, $s \in [0,T]$, *where* $\alpha \in (0,t_1]$, $\beta \in [t_m,T)$, *and*

$$0 < \sigma := \min\left\{\frac{\phi(T)-\phi(\beta)}{\phi(T)-\phi(\alpha)}, \frac{\phi(\alpha)}{\phi(\beta)}\right\} < 1.$$

Form Lemma 2.2, the problem (1.1) is equivalent to the integral equation:

$$(Su)(t) := u(t) = \begin{cases} \varphi(0) + \frac{(A-\varphi(0))}{\phi(T)}\phi(t) + \int_0^T G(t,s)f(s,u_s)\,ds \\ \quad + \sum_{0<t_k<T} G(t,t_k)\rho(t_k)I_k(u_{t_k}), & t \in [0,T], \\ \varphi(t), & t \in [-\tau,0]. \end{cases} \tag{2.5}$$

Definition 2.4 A function $u \in PC^1(J^*) \cap C^2(J)$ is called a positive solution of PBVP (1.1), if it satisfies the PBVP (1.1) and $u(t) \geq 0$ on J^*, and $u(t) \not\equiv 0$ on J.

Define a cone K in $PC(J^*)$ as follows:

$$K = \bigl\{x \in PC(J^*) : x \geq 0 \text{ on } J^* \text{ and } x(t) \geq \sigma \|x\|_{[0,T]}, \forall t \in [\alpha,\beta]\bigr\}. \tag{2.6}$$

Lemma 2.5 *The operator $S : K \to K$ is completely continuous.*

Proof It is easy to show that $S(K) \subseteq K$ holds. Let B be a bounded subset in $PC(J^*)$. By virtue of the Ascoli-Arzela theorem, we only show that $S(B)$ is bounded in $PC(J^*)$ and $S(B)$ is equicontinuous. For any $u \in C_\tau$ and $\theta \in [-\tau,0]$, $t \in [0,T]$, we have $u_t(\theta) = u(t+\theta)$. Therefore, the set $\{u_t : u \in B, t \in [0,T]\}$ is uniformly bounded with respect to $t \in [0,T]$ on C_τ. Then there exist two constants $L_1 > 0$, $L_2 > 0$ such that

$$\max_{u\in B, s\in[0,T]}\bigl\{\bigl|f(s,u_s)\bigr|\bigr\} < L_1, \qquad \max_{u\in B, 0<t_k<T}\bigl\{\bigl|I_k(u_{t_k})\bigr|\bigr\} < L_2, \quad k = 1,2,\ldots,m. \tag{2.7}$$

Taking

$$L = \max_{t,s\in[0,T]}\bigl\{\bigl|G(t,s)\bigr|\bigr\}, \qquad L_0 = \max_{t,t_k\in[0,T]}\bigl\{\bigl|G(t,t_k)\bigr|\bigr\}, \quad k = 1,2,\ldots,m. \tag{2.8}$$

It follows from (2.5), (2.7), and (2.8) that $S(B)$ is bounded in $PC(J^*)$.

Let $u \in B$ and $t,t' \in [-\tau,T]$ with $t < t'$. There are three possibilities:

Case I. If $0 \leq t < t' \leq T$, then

$$\begin{aligned} \bigl|(Su)(t) - (Su)(t')\bigr| \leq{}& \int_0^T \bigl|G(t,s) - G(t',s)\bigr|f(s,u_s)\,ds \\ &+ \frac{A-\phi(0)}{\phi(T)}\bigl|\phi(t) - \phi(t')\bigr| \\ &+ \sum_{0<t_k<T}\bigl|G(t,t_k) - G(t',t_k)\bigr|\rho(t_k)I_k(u_{t_k}). \end{aligned}$$

Case II. If $-\tau \leq t < t' \leq 0$, then we have $|(Su)(t) - (Su)(t')| = |\varphi(t) - \varphi(t')|$.

Case III. If $-\tau \leq t < 0 < t' \leq T$, then

$$\begin{aligned}
\left|(Su)(t) - (Su)(t')\right| &\leq \left|(Su)(t') - (Su)(0)\right| + \left|(Su)(0) - (Su)(t)\right| \\
&\leq \int_0^T \left|G(t', s) - G(0, s)\right| f(s, u_s)\, ds \\
&\quad + \frac{A - \varphi(0)}{\phi(T)} \left|\phi(t) - 0\right| + \left|\varphi(0) - \varphi(t)\right| \\
&\quad + \sum_{0 < t_k < T} \left|G(t', t_k) - G(0, t_k)\right| \rho(t_k) I_k(u_{t_k}).
\end{aligned}$$

Clearly, in either case, it follows from the continuity of $G(t,s)$ and the uniform continuity of φ in $[-\tau, 0]$ that for any $\varepsilon > 0$, there exists a positive constant δ, independent of t, t' and u, whenever $|t - t'| < \delta$, such that $|(Su)(t) - S(u)(t')| \leq \varepsilon$ holds. Therefore, $S(B)$ is equicontinuous. □

3 The main results

In this section, we shall consider the existence of multiple positive solutions for the periodic boundary value problems (1.1).

For convenience sake, we set

$$f_\infty := \lim_{v \in PC^+, \|v\|_{[-\tau,0]} \to \infty} \frac{f(t,v)}{\|v\|_{[-\tau,0]}}, \qquad I_k^\infty := \lim_{v \in PC^+, \|v\|_{[-\tau,0]} \to \infty} \frac{I_k(v)}{\|v\|_{[-\tau,0]}}, \quad k = 1, 2, \ldots, m.$$

For the first theorem we need the following hypotheses:

(C1) There exists a constant $a_1 > 0$ such that for $v \in PC^+(J^*) : \|v\|_{[-\tau,0]} \leq a_1$,

$$\begin{aligned}
&f(t,v) \geq M_0 \max\left\{\|v\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\right\}, \quad \forall t \in [0,T], \\
&I_k(v) \geq M_k \max\left\{\|v\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\right\}, \quad k = 1, 2, \ldots, m,
\end{aligned}$$

where M_0 and M_k are two positive constants satisfying:

$$\sigma \left\{ M_0 \int_\alpha^\beta G\left(\frac{1}{2}, s\right) ds + \sum_{\alpha < t_k < \beta} G\left(\frac{1}{2}, t_k\right) \rho(t_k) M_k \right\} \geq 1. \tag{3.1}$$

(C2) There exists a constant $b_1 > 0$ satisfying:

$$b_1 > \max\left\{\|\varphi\|_{[-\tau,0]}, a_1, (1-D)^{-1} A\right\} \quad \bigl(D \text{ be given in (3.2)}\bigr)$$

such that

$$f(t,v) \leq M_0^* \|v\|_{[-\tau,0]}, \quad \forall t \in [0,T] \quad \text{and} \quad I_k(v) \leq M_k^* \|v\|_{[-\tau,0]}, \quad k = 1, 2, \ldots, m$$

for $v \in PC^+(J^*) : \|v\|_{[-\tau,0]} \leq b_1$, where $M_0^* > 0$, $M_k^* > 0$ are constants satisfying:

$$D := \max_{t \in [0,T]} \left\{ M_0^* \int_0^T G(t,s)\, ds + \sum_{k=1}^m G(t, t_k) \rho(t_k) M_k^* \right\} < 1. \tag{3.2}$$

Theorem 3.1 *Assume that* (C1), (C2), $f_\infty = \infty$, *and* $I_k^\infty = \infty$ *hold. Then PBVP* (1.1) *has at least two positive solutions* u_*, u_{**} *with* $0 < \|u_*\|_{[-\tau,T]} < b_1 < \|u_{**}\|_{[-\tau,T]}$.

Proof For any $u \in K$, we have $u(t) \geq \sigma \|u\|_{[0,T]}$, $t \in [\alpha, \beta]$. It follows from the definitions of $PC(J^*)$ and u_t that

$$\|u_s\|_{[-\tau,0]} = \sup_{\theta \in [-\tau,0]} \left|u(s+\theta)\right| \geq u(s) \geq \sigma \|u\|_{[0,T]}.$$

Then for $u \in K$ with $\|u\|_{[-\tau,T]} = a_1$, it follows from (2.5) and assumption (C1) that

$$\begin{aligned}
(Su)\left(\frac{1}{2}\right) &= \varphi(0) + \frac{(A-\varphi(0))}{\phi(T)}\phi\left(\frac{1}{2}\right) + \int_0^T G\left(\frac{1}{2}, s\right) f(s, u_s)\, ds \\
&\quad + \sum_{0<t_k<T} G\left(\frac{1}{2}, t_k\right)\rho(t_k) I_k(u_{t_k}) \\
&\geq \int_0^T G\left(\frac{1}{2}, s\right) f(s, u_s)\, ds + \sum_{0<t_k<T} G\left(\frac{1}{2}, t_k\right)\rho(t_k) I_k(u_{t_k}) \\
&\geq \int_0^T G\left(\frac{1}{2}, s\right) M_0 \max\left\{\|u_s\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\right\} ds \\
&\quad + \sum_{0<t_k<T} G\left(\frac{1}{2}, t_k\right)\rho(t_k) M_k \max\left\{\|u_{t_k}\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\right\} \qquad (3.3) \\
&\geq \int_\alpha^\beta G\left(\frac{1}{2}, s\right) M_0 \max\left\{\|u_s\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\right\} ds \\
&\quad + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2}, t_k\right)\rho(t_k) M_k \max\left\{\|u_{t_k}\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\right\} \\
&\geq \int_\alpha^\beta G\left(\frac{1}{2}, s\right) M_0 \max\left\{\sigma \|u\|_{[0,T]}, \|\varphi\|_{[-\tau,0]}\right\} ds \\
&\quad + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2}, t_k\right)\rho(t_k) M_k \max\left\{\sigma \|u\|_{[0,T]}, \|\varphi\|_{[-\tau,0]}\right\} \\
&\geq \sigma \left\{ M_0 \int_\alpha^\beta G\left(\frac{1}{2}, s\right) ds + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2}, t_k\right)\rho(t_k) M_k \right\} \|u\|_{[-\tau,T]} \\
&\geq \|u\|_{[-\tau,T]} = a_1. \qquad (3.4)
\end{aligned}$$

Now if we set $\Omega_{a_1} = \{u \in K : \|u\|_{[-\tau,T]} < a_1\}$, then (3.4) shows that $\|Su\|_{[-\tau,T]} \geq \|u\|_{[-\tau,T]}$ for $u \in \partial\Omega_{a_1}$. Thus, Lemma 2.1 yields

$$i(S, \Omega_{a_1}, K) = 0. \qquad (3.5)$$

For $u \in K$ with $\|u\|_{[-\tau,T]} = b_1$, from (2.5) and assumption (C2), we have

$$\begin{aligned}
\left|Su(t)\right| &= \begin{cases} |\varphi(0) + \frac{(A-\varphi(0))}{\phi(T)}\phi(t) + \int_0^T G(t,s) f(s,u_s)\, ds \\ \quad + \sum_{0<t_k<T} G(t,t_k)\rho(t_k) I_k(u_{t_k})|, & t \in [0,T], \\ |\varphi(t)|, & t \in [-\tau, 0] \end{cases} \\
&\leq \begin{cases} \int_0^T G(t,s) M_0^* \|u_s\|_{[-\tau,0]}\, ds + \sum_{0<t_k<T} G(t,t_k)\rho(t_k) M_k^* \|u_{t_k}\|_{[-\tau,0]} + A, \\ \|\varphi\|_{[-\tau,0]} \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\le \begin{cases} \max_{t\in[0,T]}\{M_0^*\int_0^T G(t,s)\|u_s\|_{[-\tau,0]}\,ds \\ \quad + \sum_{0<t_k<T} G(t,t_k)\rho(t_k)M_k^*\|u_{t_k}\|_{[-\tau,0]}\} + A, \\ \|\varphi\|_{[-\tau,0]} \end{cases} \\
&\le \begin{cases} Db_1 + A, \\ \|\varphi\|_{[-\tau,0]} \end{cases} \\
&< b_1.
\end{aligned} \tag{3.6}$$

Set $\Omega_{b_1} = \{u \in K : \|u\|_{[-\tau,T]} < b_1\}$. Then (3.6) shows that $\|Su\|_{[-\tau,T]} < \|u\|_{[-\tau,T]}$ for $u \in \partial\Omega_{b_1}$. Hence, Lemma 2.1 implies that

$$i(S,\Omega_{b_1},K) = 1. \tag{3.7}$$

According to $f_\infty = \infty$ and $I_k^\infty = \infty$, choose a constant b^* such that

$$b^* > \max\{\|\varphi\|_{[-\tau,0]}, b_1\} \tag{3.8}$$

and

$$\begin{aligned}
&f(t,v) \ge \overline{M}_0\|v\|_{[-\tau,0]}, \quad \forall t\in[0,T], \forall v \in PC^+(J^*), \sigma b^* \le \|v\|_{[-\tau,0]}, \\
&I_k(v) \ge \overline{M}_k\|v\|_{[-\tau,0]}, \quad \forall v \in PC^+(J^*), \sigma b^* \le \|v\|_{[-\tau,0]}, k = 1,2,\ldots,m,
\end{aligned}$$

where $\overline{M}_0 > 0$, $\overline{M}_k > 0$ are constants satisfying:

$$\sigma\left\{\overline{M}_0\int_\alpha^\beta G\left(\frac{1}{2},s\right)ds + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2},t_k\right)\rho(t_k)\overline{M}_k\right\} > 1. \tag{3.9}$$

For $u \in K$ with $\|u\|_{[-\tau,T]} = b^*$, from (2.5), Lemma 2.5 and (3.9), by using the same method to get (3.3), we can get

$$\begin{aligned}
(Su)\left(\frac{1}{2}\right) &= \varphi(0) + \frac{(A-\varphi(0))}{\phi(T)}\phi\left(\frac{1}{2}\right) + \int_0^T G\left(\frac{1}{2},s\right)f(s,u_s)\,ds \\
&\quad + \sum_{0<t_k<T} G\left(\frac{1}{2},t_k\right)\rho(t_k)I_k(u_{t_k}) \\
&\ge \int_\alpha^\beta G\left(\frac{1}{2},s\right)\overline{M}_0\|u_s\|_{[-\tau,0]}\,ds + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2},t_k\right)\rho(t_k)\overline{M}_k\|u_{t_k}\|_{[-\tau,0]} \\
&\ge \sigma\left\{\int_\alpha^\beta G\left(\frac{1}{2},s\right)\overline{M}_0\,ds + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2},t_k\right)\rho(t_k)\overline{M}_k\right\}\|u\|_{[-\tau,T]} > b^*.
\end{aligned} \tag{3.10}$$

Now if we set $\Omega_{b^*} = \{u \in K : \|u\|_{[-\tau,T]} < b^*\}$. Then (3.10) shows that $\|Su\|_{[-\tau,T]} > \|u\|_{[-\tau,T]}$ for $u \in \partial\Omega_{b^*}$. Thus, an application of Lemma 2.1 again shows that

$$i(S,\Omega_{b^*},K) = 0. \tag{3.11}$$

Since $a_1 < b_1 < b^*$, it follows from (3.5), (3.7), (3.11), and the additivity of the fixed index that

$$i(S,\Omega_{b_1}\backslash\overline{\Omega}_{a_1},K) = 1, \qquad i(S,\Omega_{b^*}\backslash\overline{\Omega}_{b_1},K) = -1.$$

Thus, S has a fixed point u_* in $\Omega_{b_1}\backslash\overline{\Omega}_{a_1}$, and a fixed point u_{**} in $\Omega_{b^*}\backslash\overline{\Omega}_{b_1}$. They are positive solutions of the PBVP (1.1) and

$$0 < \|u_*\|_{[-\tau,T]} < b_1 < \|u_{**}\|_{[-\tau,T]}.$$

The proof is complete. □

For the second theorem we need the following hypotheses:

(C3) There exists a constant $a_2 > 0$ satisfying

$$a_2 > \max\bigl\{\|\varphi\|_{[-\tau,0]}, \bigl(1-D^*\bigr)^{-1}A\bigr\} \quad \bigl(D^* \text{is given in (3.12)}\bigr),$$

such that for $v \in PC^+(J^*) : \|v\|_{[-\tau,0]} \leq a_2$,

$$f(t,v) \leq \lambda_0^*\|v\|_{[-\tau,0]}, \quad \forall t \in [0,T] \quad \text{and} \quad I_k(v) \leq \lambda_k^*\|v\|_{[-\tau,0]}, \quad k = 1,2,\ldots,m,$$

where λ_0^*, λ_k^* are positive constants satisfying:

$$D^* := \max_{t\in[0,T]}\left\{\lambda_0^*\int_0^T G(t,s)\,ds + \sum_{k=1}^m G(t,t_k)\rho(t_k)\lambda_k^*\right\} < 1. \tag{3.12}$$

(C4) $H_k : R^+ \to R^+$ ($k = 1,2,\ldots,m$) are continuous nonincreasing functions such that $|I_k(v)| \leq H_k(\|v\|_{[-\tau,0]})$ for $v \in PC^+(J^*)$.

(C5) There exists a constant b_2 with $b_2 > a_2$ such that for any $v \in PC^+(J^*) : \sigma b_2 \leq \|v\|_{[-\tau,0]} \leq b_2$,

$$f(t,v) \geq \lambda_0 \max\bigl\{\|v\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\bigr\}, \quad \forall t \in [0,T]$$

and

$$I_k(v) \geq \lambda_k \max\bigl\{\|v\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\bigr\}, \quad k = 1,2,\ldots,m,$$

where $\lambda_0 > 0$, $\lambda_k > 0$, and

$$\sigma\left\{\lambda_0\int_\alpha^\beta G\left(\frac{1}{2},s\right)ds + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2},t_k\right)\rho(t_k)\lambda_k\right\} \geq 1.$$

Theorem 3.2 *Assume that* (C3)-(C5), *and* $f_\infty = 0$ *hold. Then PBVP* (1.1) *has at least two positive solutions* u_*, u_{**} *with* $0 < \|u_*\|_{[-\tau,T]} < b_2 < \|u_{**}\|_{[-\tau,T]}$.

Proof For any $u \in K$ with $\|u\|_{[-\tau,T]} = a_2$. According to (2.5) and assumption (C3), we have

$$\begin{aligned}
\bigl|Su(t)\bigr| &= \begin{cases} |\varphi(0) + \frac{(A-\varphi(0))}{\phi(T)}\phi(t) + \int_0^T G(t,s)f(s,u_s)\,ds \\ \quad + \sum_{0<t_k<T} G(t,t_k)\rho(t_k)I_k(u_{t_k})|, & t \in [0,T], \\ |\varphi(t)|, & t \in [-\tau,0], \end{cases} \\
&\leq \begin{cases} \max_{t\in[0,T]}\{\lambda_0^*\int_0^T G(t,s)\|u_s\|_{[-\tau,0]}\,ds \\ \quad + \sum_{0<t_k<T} G(t,t_k)\rho(t_k)\lambda_k^*\|u_{t_k}\|_{[-\tau,0]}\} + A, \\ \|\varphi\|_{[-\tau,0]}, \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\leq \begin{cases} D^* a_2 + A, \\ \|\varphi\|_{[-\tau,0]} \end{cases} \\
&< a_2.
\end{aligned} \tag{3.13}$$

Set $\Omega_{a_2} = \{u \in K : \|u\|_{[-\tau,T]} < a_2\}$. Then (3.13) shows that $\|Su\|_{[-\tau,T]} < \|u\|_{[-\tau,T]}$ for $u \in \partial\Omega_{a_2}$. Thus Lemma 2.1 implies

$$i(S, \Omega_{a_2}, K) = 1. \tag{3.14}$$

On the other hand, it follows from (2.5) and assumption (C5) that

$$\begin{aligned}
(Su)\left(\frac{1}{2}\right) &= \varphi(0) + \frac{(A - \varphi(0))}{\phi(T)}\phi\left(\frac{1}{2}\right) + \int_0^T G\left(\frac{1}{2}, s\right) f(s, u_s)\, ds \\
&\quad + \sum_{0<t_k<T} G\left(\frac{1}{2}, t_k\right)\rho(t_k) I_k(u_{t_k}) \\
&\geq \int_0^T G\left(\frac{1}{2}, s\right)\lambda_0 \max\left\{\|u_s\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\right\} ds \\
&\quad + \sum_{0<t_k<T} G\left(\frac{1}{2}, t_k\right)\rho(t_k)\lambda_k \max\left\{\|u_{t_k}\|_{[-\tau,0]}, \|\varphi\|_{[-\tau,0]}\right\} \\
&\geq \int_\alpha^\beta G\left(\frac{1}{2}, s\right)\lambda_0 \max\left\{\sigma\|u\|_{[0,T]}, \|\varphi\|_{[-\tau,0]}\right\} ds \\
&\quad + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2}, t_k\right)\rho(t_k)\lambda_k \max\left\{\sigma\|u\|_{[0,T]}, \|\varphi\|_{[-\tau,0]}\right\} \\
&\geq \sigma\left\{\lambda_0 \int_\alpha^\beta G\left(\frac{1}{2}, s\right) ds + \sum_{\alpha<t_k<\beta} G\left(\frac{1}{2}, t_k\right)\rho(t_k)\lambda_k\right\}\|u\|_{[-\tau,T]} \\
&\geq \|u\|_{[-\tau,T]} = b_2 \quad \text{for any } u \in K \text{ with } \|u\|_{[-\tau,T]} = b_2.
\end{aligned} \tag{3.15}$$

Set $\Omega_{b_2} = \{u \in K : \|u\|_{[-\tau,T]} < b_2\}$. Then (3.15) implies that $\|Su\|_{[-\tau,T]} \geq \|u\|_{[-\tau,T]}$ for $u \in \partial\Omega_{b_2}$. Thus an application of Lemma 2.1 again shows that

$$i(S, \Omega_{b_2}, K) = 0. \tag{3.16}$$

In view of assumption (C4) and $f_\infty = 0$, there are two possibilities:

Case 1. Suppose that f is unbounded, then there exists a constant d_1 satisfying:

$$d_1 > \max\left\{\|\varphi\|_{[-\tau,0]}, b_2\right\},$$

such that

$$f(t, \nu) \leq \varepsilon_0\|\nu\|_{[-\tau,0]} \quad \text{for } \nu \in PC^+\left(J^*\right) : \|\nu\|_{[-\tau,0]} \geq d_1, \forall t \in [0, T] \tag{3.17}$$

and

$$\left|I_k(\nu)\right| \leq H_k\left(\|\nu\|_{[-\tau,0]}\right) \leq H_k(d_1), \quad k = 1, 2, \ldots, m,$$

where ε_0 is a positive constant satisfying

$$\max_{t\in[0,T]}\left\{\varepsilon_0 d_1\int_0^T G(t,s)\,ds+\sum_{k=1}^{m}G(t,t_k)\rho(t_k)H_k(d_1)\right\}+A\le d_1. \tag{3.18}$$

If $u\in K$ with $\|u\|_{[-\tau,T]}=d_1$, then it follows from (2.5), (3.17), and (3.18) that

$$\begin{aligned}\left|Su(t)\right| &\le \begin{cases}\max_{t\in[0,T]}\{\varepsilon_0 d_1\int_0^T G(t,s)\,ds+\sum_{0<t_k<T}G(t,t_k)\rho(t_k)H_k(d_1)\}+A,\\ \|\varphi\|_{[-\tau,0]}\end{cases}\\ &\le d_1.\end{aligned} \tag{3.19}$$

Case 2. Suppose that f is bounded. Then there exists a constant N such that

$$\left|f(t,v)\right|\le N,\quad t\in[0,T],\forall v\in PC^+\left(J^*\right).$$

Taking

$$d_2\ge\max\left\{\|\varphi\|_{[-\tau,0]},\max_{t\in[0,T]}\left\{N\int_0^T G(t,s)\,ds+\sum_{0<t_k<T}G(t,t_k)\rho(t_k)H_k(0)\right\}+A,b_2\right\}.$$

For $u\in K$ and $\|u\|_{[-\tau,T]}=d_2$, from (2.5), we have

$$\begin{aligned}\left|Su(t)\right| &\le \begin{cases}\max_{t\in[0,T]}\{N\int_0^T G(t,s)\,ds+\sum_{0<t_k<T}G(t,t_k)\rho(t_k)H_k(0)\}+A,\\ \|\varphi\|_{[-\tau,0]}\end{cases}\\ &\le d_2.\end{aligned}$$

Choose $d=\max\{d_1,d_2\}$. Hence, in either case, we always may set

$$\Omega_d=\left\{u\in K:\|u\|_{[-\tau,T]}<d\right\},$$

such that $\|Su\|_{[-\tau,T]}\le\|u\|_{[-\tau,T]}$ for $u\in\partial\Omega_d$. Thus, Lemma 2.1 yields

$$i(S,\Omega_d,K)=1. \tag{3.20}$$

Since $a_2<b_2<d$, it follows from (3.14), (3.16), (3.20), and the additivity of the fixed point index that

$$i(S,\Omega_{b_2}\backslash\overline{\Omega}_{a_2},K)=-1,\qquad i(S,\Omega_d\backslash\overline{\Omega}_{b_2},K)=1.$$

Thus, S has a fixed point u_* in $\Omega_{b_2}\backslash\overline{\Omega}_{a_2}$, and a fixed point u_{**} in $\Omega_d\backslash\overline{\Omega}_{b_2}$. They are positive solutions of the PBVP (1.1) and

$$0<\|u_*\|_{[-\tau,T]}<b_2<\|u_{**}\|_{[-\tau,T]}.$$

The proof is complete. □

Example 3.3 Consider the following PBVP:

$$\begin{cases} (e^t u'(t))' = (1+t^2)F(\sup_{\theta\in[-1,0]}|u_t(\theta)|), \\ -\Delta u'(\frac{1}{2}) = I(\sup_{\theta\in[-1,0]}|u(\frac{1}{2}+\theta)|), \\ u(t) = \frac{1}{20}(1+\sin t), \quad t\in[-1,0], \qquad u(1)=2, \end{cases} \tag{3.21}$$

where

$$F(x) = \begin{cases} 1-\sqrt{x}, & x\in[0,0.25), \\ 2x, & x\in[0.25,20), \\ \frac{1}{10}x^2, & x\in[20,+\infty), \end{cases} \qquad I(x) = \begin{cases} x+0.8, & x\in[0,0.6), \\ \frac{7}{3}x, & x\in[0.6,21), \\ \frac{1}{147}x^3-14, & x\in[21,+\infty). \end{cases} \tag{3.22}$$

Then PBVP (3.21) has at least two positive solutions u_*, u_{**} with $0 < \|u_*\|_{[-1,1]} < 10 < \|u_{**}\|_{[-1,1]}$.

Proof PBVP (3.21) can be regarded as a PBVP of the form (1.1), where

$$f(t,x) = \bigl(1+t^2\bigr)F\bigl(\|x\|\bigr), \qquad I_1(x) = I\bigl(\|x\|\bigr), \qquad \varphi(t) = \frac{1}{20}(1+\sin t), \tag{3.23}$$

and $\rho(t) = e^t$, $A = 2$, $\tau = -1$, $T = 1$.

First we have $1-\int_0^1 G(s,s)\,ds \approx 0.8980$. By choosing $\alpha = 0.25$, $\beta = 0.75$, we get $\sigma = 0.2543 < 1$. On the other hand, it follows from (3.21) and (3.22) that $f_\infty = \infty$ and $I_1^\infty = \infty$ are satisfied. Finally, we show that (C1) and (C2) hold. Choose $M_0 = 10$, $M_1 = 16$, $a_1 = \frac{1}{25}$, $b_1 = 10$, $M_0^* = 2$, $M_1^* = \frac{7}{3}$. By calculation, we get

$$\sigma\left\{M_0\int_{0.25}^{0.75} G\left(\frac{1}{2},s\right)ds + G\left(\frac{1}{2},\frac{1}{2}\right)\rho\left(\frac{1}{2}\right)M_1\right\} \approx 1.0868 > 1, \qquad D\approx 0.7754$$

and

$$\max\bigl\{\|\varphi\|_{[-\tau,0]}, a_1, (1-D)^{-1}A\bigr\} \approx 8.9048 < b_1 = 10.$$

Then it is not difficult to see that the conditions (C1) and (C2) hold.

By Theorem 3.1, PBVP (3.21) has at least two positive solutions u_*, u_{**} with $0 < \|u_*\|_{[-1,1]} < 10 < \|u_{**}\|_{[-1,1]}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Bainov, DD, Simeonov, PS: Systems with Impulse Effect. Ellis Horwood, Chichester (1989)
2. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
3. Guo, D: Multiple positive solutions for *n*th-order impulsive integro-differential equations in Banach spaces. Nonlinear Anal. **60**, 955-976 (2005)
4. Dong, Y: Periodic boundary value problems for functional differential equations with impulses. J. Math. Anal. Appl. **210**, 170-182 (1997)
5. Ma, R: Positive solutions for boundary value problems of functional differential equations. Appl. Math. Comput. **193**, 66-72 (2007)
6. He, Z, Yu, J: Periodic boundary value problem for first-order impulsive ordinary differential equations. J. Math. Anal. Appl. **272**, 67-78 (2002)
7. Zhao, Y, Chen, H: Multiplicity of solutions to two-point boundary value problems for second-order impulsive differential equation. Appl. Math. Comput. **206**, 925-931 (2008)
8. Huseynov, A: Positive solutions of a nonlinear impulsive equation with periodic boundary conditions. Appl. Math. Comput. **217**, 247-259 (2010)
9. Mohamed, M, Ahmad, HS, Noorani, MSM: Periodic boundary value problems for systems of first-order differential equations with impulses. Differential and Difference Equations with Applications. Springer Proceedings in Mathematics and Statistics, vol. 47, pp. 525-534 (2013)
10. Jankowski, T: Positive solutions for second order impulsive differential equations involving Stieltjes integral conditions. Nonlinear Anal. **74**, 3775-3785 (2011)
11. Tian, Y, Jiang, D, Ge, W: Multiple positive solutions of periodic boundary value problems for second order impulsive differential equations. Appl. Math. Comput. **200**, 123-132 (2008)
12. Zhang, D, Dai, B: Infinitely many solutions for a class of nonlinear impulsive differential equations with periodic boundary conditions. Comput. Math. Appl. **61**, 3153-3160 (2011)
13. Ding, W, Han, M, Mi, J: Periodic boundary value problem for the second-order impulsive functional differential equations. Comput. Math. Appl. **50**, 491-507 (2005)
14. Li, J, Nieto, JJ, Shen, J: Impulsive periodic boundary value problems of first-order differential equations. J. Math. Anal. Appl. **325**, 226-236 (2007)
15. Liang, R, Shen, J: Periodic boundary value problem for the first-order impulsive functional differential equations. J. Comput. Appl. Math. **202**, 498-510 (2007)
16. Liu, Y: Further results on periodic boundary value problems for nonlinear first-order impulsive functional differential equations. J. Math. Anal. Appl. **327**, 435-452 (2007)
17. Nieto, JJ: Basic theory for nonresonance impulsive periodic problems of first order. J. Math. Anal. Appl. **205**, 423-433 (1997)
18. Chen, L, Sun, J: Nonlinear boundary value problem of first-order impulsive integro-differential equations. J. Comput. Appl. Math. **202**, 392-401 (2007)
19. Liu, Y: Periodic boundary value problems for first order functional differential equations with impulse. J. Comput. Appl. Math. **223**, 27-39 (2009)
20. Jiang, D, Nieto, JJ, Zuo, W: On monotone method for first and second-order periodic boundary value problems and periodic solutions of functional differential equations. J. Math. Anal. Appl. **289**, 691-699 (2004)
21. Yang, X, Shen, J: Nonlinear boundary value problems for first-order impulsive functional differential equations. Appl. Math. Comput. **189**, 1943-1952 (2007)
22. Thaiprayoon, C, Samana, D, Tariboon, J: Periodic boundary value problems for second-order impulsive integro-differential equations with integral jump conditions. Bound. Value Probl. **2012**, 122 (2012)
23. Wang, G, Zhang, L, Song, G: New existence results and comparison principles for impulsive integral boundary value problem with lower and upper solutions in reversed order. Adv. Differ. Equ. **2011**, Article ID 783726 (2011)
24. Zhao, Y, Chen, H: Existence of multiple positive solutions for singular functional differential equation with sign-changing nonlinearity. J. Comput. Appl. Math. **234**, 1543-1550 (2010)
25. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cone. Academic Press, Orlando (1988)

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