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# On zeros and deficiencies of differences of meromorphic functions

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**Abstract**

For a transcendental entire function $f(z)$ in the complex plane, we study its divided differences $G_n(z)$. We partially prove a conjecture posed by Bergweiler and Langley under the additional condition that the lower order of $f(z)$ is smaller than $\frac{1}{2}$. Furthermore, we prove that if zero is a deficient value of $f(z)$, then $\delta(0, G) < 1$, where $G(z) = (f(z + c) - f(z))/f(z)$.

**MSC:** 30D35

**Keywords:** complex difference; zero; deficiency

## 1 Introduction and main results

In this paper, we assume that the reader is familiar with the standard notations of Nevanlinna theory of meromorphic functions (see [1, 2] or [3]). In particular, for a meromorphic function $f(z)$ in the complex plane $\mathbb{C}$, we use $\rho(f)$ and $\mu(f)$ to denote its order and lower order respectively, and $\lambda(f)$ to denote the exponent of the convergence of the zero-sequences.

Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$. The forward differences $\Delta^n f(z)$ are defined in the standard way by

$$\Delta f(z) = f(z+1) - f(z), \qquad \Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z), \quad n = 0, 1, \ldots. \tag{1.1}$$

The divided differences are defined by

$$G_1(z) = \frac{\Delta f(z)}{f(z)}, \qquad G_n(z) = \frac{\Delta^n f(z)}{f(z)}, \quad n = 0, 1, \ldots. \tag{1.2}$$

Recently, a number of papers including [4–10] have focused on the complex difference equations and differences. In [5] Bergweiler and Langley investigated the existence of zeros of $\Delta f(z)$ and $G_1(z)$. Their result may be viewed as discrete analogs of the following theorem on the zeros of $f'(z)$.

**Theorem A** ([11–13]) *Let $f(z)$ be transcendental and meromorphic in the complex plane with*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0. \tag{1.3}$$

*Then $f'$ has infinitely many zeros.*

Theorem A is sharp, as shown by $e^z$, $\tan z$ and examples of arbitrary order greater than 1 constructed in [14]. For $f(z)$ as in the hypotheses of Theorem A, it follows from Hurwitz's theorem that, if $z_1$ is a zero of $f'(z)$ then $f(z+c)-f(z)$ has a zero near $z_1$ for all sufficiently small $c \in \mathbb{C}\backslash\{0\}$. Thus it is natural to ask, for such functions $f(z)$, whether $f(z+c)-f(z)$ must always have infinitely many zeros or not. In [5], Bergweiler and Langley answered this problem and obtained Theorem B and Theorem C.

**Theorem B** ([5]) *Let $f(z)$ be a transcendental entire function of order $\rho(f) < \frac{1}{2}$. If $G_n(z)$ defined by* (1.2) *is transcendental, then $G_n(z)$ has infinitely many zeros. In particular, if $\rho(f) < \min\{\frac{1}{n}, \frac{1}{2}\}$, then $G_n(z)$ is transcendental and has infinitely many zeros.*

**Remark 1** Recently, Theorem B was extended by Langley to the case $\rho(f) = \frac{1}{2}$ in [15]. We only state the following situation which was proved in [5].

**Theorem C** ([5]) *There exists $\delta_0 \in (0, \frac{1}{2})$ with the following property. Let $f(z)$ be a transcendental entire function with order $\rho(f) \le \rho < \frac{1}{2} + \delta_0 < 1$. Then $G_1(z)$ defined by* (1.2) *has infinitely many zeros.*

In [5], Bergweiler and Langley conjecture that the conclusion of Theorem C holds for $\rho(f) < 1$. In this paper, we will prove this conjecture under the additional condition that $\mu(f) < \frac{1}{2}$.

**Theorem 1** *Let $n \in \mathbf{N}$ and let $f$ be a transcendental entire function of order $\rho(f) < 1$. If $\mu(f) < \frac{1}{2}$ and $\mu(f) \neq \frac{1}{n}, \frac{2}{n}, \ldots, \frac{[\frac{n}{2}]}{n}$, then $G_n(z)$ defined by* (1.2) *is transcendental and has infinitely many zeros.*

Using Theorem 1, we easily obtain the following corollary.

**Corollary 1** *Let $n \in \mathbf{N}$ and let $f$ be a transcendental entire function of order $\rho(f) < 1$. If $\mu(f) < \min\{\frac{1}{2}, \frac{1}{n}\}$, then $G_n(z)$ is transcendental and has infinitely many zeros.*

In [5], Bergweiler and Langley also proved that, for a transcendental meromorphic $f(z)$ of order $\rho(f) < 1$, if $f(z)$ has finitely many poles $z_j$, $z_k$ such that $z_j - z_k = c$, then $g(z) = f(z+c) - f(z)$ has infinitely many zeros (see [5], Theorem 1.4). Furthermore, for a transcendental entire function $f$ of order $\rho(f) < 1$, Chen and Shon proved that $\lambda(g) = \rho(g) = \rho(f)$, and if $f(z)$ has finitely many zeros $z_j$, $z_k$ such that $z_j - z_k = c$, then $G(z) = g(z)/f(z)$ has infinitely many zeros and $\lambda(G) = \rho(G) = \rho(f)$ (see [7]). This result implies that zero is not the Borel exceptional value of $G(z)$.

In [15], Langley investigated the deficiency of divided difference $G_1(z)$ defined by (1.2). He obtained that if $f(z)$ is a transcendental entire function of order $\rho(f) < 1$ and $\mu(G_1) < \frac{1}{2}$, then $\delta(0, G_1) < 1$. In particular, if $\rho(f) < \frac{1}{2}$, then $\delta(0, G_1) < 1$ (see [15], Theorem 1.4). The proof of his result depends on $\cos \pi\rho$ theorem which is invalid for $\mu(G_1) \ge \frac{1}{2}$.

In this paper, we consider some more general cases. For $c \in \mathbb{C}\backslash\{0\}$, we define

$$g(z) = f(z+c) - f(z), \qquad G(z) = \frac{g(z)}{f(z)}. \tag{1.4}$$

We get the following results on the deficiency $\delta(0, G)$.

**Theorem 2** *Let $f(z)$ be a transcendental entire function of order $\rho(f) < 1$. Suppose that $f(z)$ has at most finitely many zeros $z_j$, $z_k$ such that $z_j - z_k = c$. If $G(z)$ is defined by* (1.4), *then the following two statements hold*:

(i) *If $\delta(0, G) = 1$, then there exists a set $E \subset (0, \infty)$ of positive upper logarithmic density such that $m(r, \frac{1}{f}) = o(\log M(r,f))$, as $r \to \infty$, $r \in E$, where $M(r,f) = \max_{|z|=r} |f(z)|$.*

(ii) *If zero is a deficient value of $f(z)$, then $\delta(0, G) < 1$.*

It is clear that, for a given transcendental entire function $f(z)$, all but countably many $c \in \mathbb{C}$ such that $f(z)$ has at most finitely many zeros $z_j$, $z_k$ such that $z_j - z_k = c$. Furthermore, we know that, for an entire function $f(z)$, if $f(z)$ has a finite deficient value then $\mu(f) > \frac{1}{2}$. Hence, Theorem 2 implies that, for some particular functions $f(z)$ of order $\rho(f) > \frac{1}{2}$, we obtain a similar conclusion.

**Example** There is an example for Theorem 2. Let $\frac{1}{2} < \mu < 1$. Set

$$f(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z}{k^{\frac{1}{\mu}}}\right).$$

Then $\mu(f) = \rho(f) = \mu$ and $\delta(0,f) = 1 - \sin \mu\pi > 0$ (see [3, p.252]). If we let $c = 1$, then it follows from Theorem 2 that $\delta(0, G_1) < 1$.

The paper is organized as follows. In Section 2, we shall collect some notations and give some lemmas which will be used later. In Section 3, we shall prove Theorem 1. In Section 4, we shall prove Theorem 2.

## 2 Preliminaries and lemmas

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function in the complex plane. Then (see [16, p.13]) we have

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \limsup_{r\to\infty} \frac{\log\log \mu(r)}{\log r} = \limsup_{r\to\infty} \frac{\log \nu(r)}{\log r}$$

and

$$\mu(f) = \liminf_{r\to\infty} \frac{\log\log M(r)}{\log r} = \liminf_{r\to\infty} \frac{\log\log \mu(r)}{\log r} = \liminf_{r\to\infty} \frac{\log \nu(r)}{\log r},$$

where $M(r) = \max\{|f(z)| : |z| = r\}$, $\mu(r) = \max_{0\leq n<\infty} |a_n| r^n$ is the maximum term and $\nu(r) = \max\{m : |a_m| r^m = \mu(r)\}$ is the central index. It is well known that $\nu(r)$ is a nondecreasing and right continuous function. Furthermore, if $f(z)$ is transcendental entire, then $\nu(r) \to \infty$ as $r \to \infty$.

For a set $E \subset [0, \infty)$, we define its Lebesgue measure by $m(E)$ and its logarithmic measure by $m_l(E) = \int_E \frac{dt}{t}$.

We also define the upper and lower logarithmic density of $E \subset [0, \infty)$, respectively, by

$$\overline{\log dens} E = \overline{\lim_{r\to\infty}} \frac{m_l(E \cap [0, r])}{\log r},$$

and

$$\underline{\log dens}E = \varliminf_{r\to\infty} \frac{m_l(E\cap[0,r])}{\log r}.$$

Following Hayman [17, pp.75-76], we say that a set $E$ is an $\varepsilon$-set if $E$ is a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If $E$ is an $\varepsilon$-set, then the set of $r \geq 1$ for which the circle $S(0,r)$ meets has finite logarithmic measure and hence zero upper logarithmic density. Moreover, for almost all real $\theta$, the intersection of $E$ with the ray $\arg z = \theta$ is a bounded set.

The following lemma contains a basic property of meromorphic functions of finite order.

**Lemma 2.1** ([18]) *Let $f(z)$ be a meromorphic function with $\rho(f) < \infty$. Then, for given real constants $c > 0$ and $H$ $(> \rho(f))$, there exists a set $E \subset (0,\infty)$ such that $\underline{\log dens}E \geq 1 - \frac{\rho(f)}{H}$, where $E = \{t \mid T(te^c,f) \leq e^k T(r,f)\}$ and $k = cH$.*

The following lemma is a version of the celebrated $\cos\pi\rho$ theorem of [19].

**Lemma 2.2** ([19]) *Let $f(z)$ be a transcendental entire function with lower order $0 \leq \mu(f) < 1$. Then, for each $\alpha \in (\mu(f),1)$, there exists a set $E \subset [0,\infty)$ such that $\overline{\log dens}E \geq 1 - \frac{\mu(f)}{\alpha}$, where $E = \{r \in [0,\infty) : A(r) > B(r)\cos\pi\alpha\}$, $A(r) = \inf_{|z|=r} \log|f(z)|$, and $B(r) = \sup_{|z|=r} \log|f(z)|$.*

We collect some important properties of the differences of meromorphic functions in the following lemmas.

**Lemma 2.3** ([5]) *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$ which satisfies* (1.3). *Then, with the notation* (1.1) *and* (1.2), *$\Delta f(z)$ and $G_1(z)$ are both transcendental.*

**Lemma 2.4** ([6]) *Let $f(z)$ be a meromorphic function of finite order $\rho(f) = \rho$ and let $c$ be a non-zero finite complex number. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\left(r^{\rho-1+\varepsilon}\right) \tag{2.1}$$

*and*

$$T\left(r, f(z+c)\right) = T(r,f) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r). \tag{2.2}$$

**Lemma 2.5** ([5]) *Let $n \in \mathbf{N}$ and let $f(z)$ be a transcendental meromorphic function of order smaller than* 1 *in the complex plane $\mathbb{C}$. Then there exists an $\varepsilon$-set $E_n$ such that*

$$\Delta^n f(z) \sim f^{(n)}(z), \quad z \to \infty, z \in \mathbb{C}\backslash E_n. \tag{2.3}$$

## 3 Proof of Theorem 1

In order to prove Theorem 1, we need one more lemma. This lemma can be proved in a similar way to the proof of Lemma 4 in [20]; we shall omit the proof.

**Lemma 3.1** *Let $T(r)$ $(>1)$ be a nonconstant increasing function of finite lower order $\mu$ in $r \in (0,\infty)$, i.e.*

$$\liminf_{r\to\infty} \frac{\log T(r)}{\log r} = \mu < \infty. \tag{3.1}$$

*For any given $\mu_1 > 0$ such that $\mu < \mu_1$, define*

$$E(\mu_1) = \{r \geq 1 : T(r) < r^{\mu_1}\}.$$

*Then $\overline{\log dens} E(\mu_1) > 0$.*

*Proof of Theorem* 1 Since $f$ is a transcendental entire function of order $\rho(f) < 1$, by Lemma 2.5, we know that there exists an $\varepsilon$-set $E_n$, such that

$$\Delta^n f(z) \sim f^{(n)}(z), \quad z \to \infty, z \in \mathbb{C}\backslash E_n. \tag{3.2}$$

From the Wiman-Valiron theory ([21] or [2]), there is a subset $F \subset [1,\infty)$ with $m_l(F) < \infty$ such that for all $z$ satisfying $|z| = r \notin F$ and $|f(z)| = M(r,f)$,

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu(r)}{z}\right)^n \bigl(1 + o(1)\bigr), \quad |z| = r \to \infty. \tag{3.3}$$

Set $H = \{|z| = r : z \in E_n\}$. We know that $m_l(H) < \infty$. Therefore, by (3.2) and (3.3), for all $z$ satisfying $|z| = r \notin (F \cup H)$ and $|f(z)| = M(r,f)$, we get

$$G_n(z) = \left(\frac{\nu(r)}{z}\right)^n \bigl(1 + o(1)\bigr), \quad |z| = r \to \infty. \tag{3.4}$$

Now we divide the proof into two steps. Firstly, we prove that $G_n(z)$ is transcendental. To do this, we assume contrarily that $G_n(z)$ is a rational function and seek a contradiction. By using Lemma 3.1 to $\nu(r)$ with lower order $\mu(f)$, we see that, for any given $\varepsilon$, $0 < \varepsilon < 1 - \mu(f)$, there exists a sequence $\{t_j\} \in E(\varepsilon)\backslash(F \cup H)$ such that the following inequalities:

$$t_j^{(\mu(f)-1-\varepsilon)n} < \left(\frac{\nu(t_j)}{t_j}\right)^n < t_j^{(\mu(f)-1+\varepsilon)n} \tag{3.5}$$

hold for all sufficiently large $j$, where $E(\varepsilon) = \{r \geq 1 : \nu(r) < r^{\mu(f)+\varepsilon}\}$. Since $(\mu(f) - 1 + \varepsilon)n < 0$ and $G_n(z)$ is a rational function, we deduce from (3.4) and (3.5) that

$$G_n(z) = \beta z^{-k}\bigl(1 + o(1)\bigr), \quad z \to \infty, \tag{3.6}$$

where $\beta$ $(\neq 0)$ is a constant and $k$ is a positive integer. By using (3.4), (3.5) and (3.6), we get

$$t_j^{(n\mu(f)+k-n-n\varepsilon)} < |\beta|\bigl|1 + o(1)\bigr| < t_j^{(n\mu(f)+k-n+n\varepsilon)}, \quad t_j \to \infty.$$

Since $\varepsilon$ can be arbitrary small, we must have $n\mu(f) + k - n = 0$, *i.e.*,

$$\mu(f) = 1 - \frac{k}{n}.$$

This contradicts the assumption that $\mu(f) \neq \frac{1}{n}, \frac{2}{n}, \ldots, \frac{[\frac{n}{2}]}{n}$. Thus $G_n(z)$ must be transcendental.

In the second step, we prove that $G_n(z)$ has infinitely many zeros. If this is not true, assume that $G_n(z)$ has only finitely many zeros. Then $1/G_n(z)$ is also transcendental having finitely many poles. By using (2.2) and the Jensen formula, for $0 < \varepsilon < 1 - \rho(f)$, we have

$$T(r, G_n) \leq (n+2)T(r,f) + o(1) + O(\log r), \quad r \to \infty. \tag{3.7}$$

Thus, It follows from (3.7) that $\mu(\frac{1}{G_n}) < \frac{1}{2}$. Using Lemma 2.2 to $\frac{1}{G_n}$, we deduce that there exists a subset $F_1 \subset [1, \infty)$ with $\overline{\log dens}F_1 > 0$ such that

$$\log |G_n| < -cT(r, G_n) < -n \log r, \quad |z| = r \in F_1, \tag{3.8}$$

where $c$ is a positive real number. Hence, by (3.4) and (3.8), we get

$$\bigl|G_n(z)\bigr||z|^n = \bigl(\nu(r)\bigr)^n \bigl|1 + o(1)\bigr| < 1, \quad |z| = r \in F_1 \backslash (F \cup H). \tag{3.9}$$

Since $\nu(r) \to \infty$ as $r \to \infty$, (3.9) gives a contradiction. Therefore, $G_n(z)$ must have infinitely many zeros and the proof of Theorem 1 is completed. □

## 4 Proof of Theorem 2

To prove Theorem 2, we first prove the following lemma.

**Lemma 4.1** *Let $f(z)$ be a transcendental entire function of order $\rho(f) < 1$. Suppose that $f(z)$ has at most finitely many zeros $z_k$, $z_j$ satisfying $z_k - z_j = c$ and the deficiency $\delta(0, G) = 1$. Then, for any given $0 < \varepsilon < 1 - \rho(f)$, there exists a constant $r_0(\varepsilon)$ $(> 0)$ such that the following inequalities*:

$$(1 - \varepsilon)T(r, G) \leq m(r,f) - m\left(r, \frac{1}{f}\right) \leq (1 + \varepsilon)T(r, G) \tag{4.1}$$

*hold for all $r \geq r_0(\varepsilon)$.*

*Proof* Let $\varepsilon > 0$ such that $\rho(f) + \varepsilon < 1$. Using (2.1), we get

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O(1), \quad r \to \infty. \tag{4.2}$$

Hence, it follows from (4.2), (1.4) and the Jensen formula that

$$m(r, G) = O(1), \qquad T(r, G) = N(r, G) + O(1), \quad r \to \infty. \tag{4.3}$$

So, by (4.3) and the Jensen formula, we get

$$T(r, G) \leq N\left(r, \frac{1}{f}\right) + O(1), \qquad T(r, G) \leq m(r,f) - m\left(r, \frac{1}{f}\right) + O(1), \quad r \to \infty. \tag{4.4}$$

It follows from Theorem 5 in [7] that $G(z)$ is transcendental. Hence, we obtain

$$(1-\varepsilon)T(r,G) \le m(r,f) - m\left(r,\frac{1}{f}\right), \quad \text{for all } r \ge r_1(\varepsilon). \tag{4.5}$$

On the other hand, from our hypothesis, we can easily deduce that

$$n\left(r,\frac{1}{G}\right) = n\left(r,\frac{1}{g}\right) + O(1). \tag{4.6}$$

Hence, by (4.6), we have

$$\left|N\left(r,\frac{1}{G}\right) - N\left(r,\frac{1}{g}\right)\right| = \left|N\left(r_2,\frac{1}{G}\right) - N\left(r_2,\frac{1}{g}\right)\right| = O(1) \tag{4.7}$$

for sufficiently large $r_2$. Since $\delta(0,G) = 1$, we have

$$N\left(r,\frac{1}{G}\right) = o\bigl(T(r,G)\bigr), \quad r \to \infty. \tag{4.8}$$

Since

$$\frac{N(r,\frac{1}{g})}{T(r,G)} \le \frac{N(r,\frac{1}{G})}{T(r,G)} \cdot \frac{N(r,\frac{1}{g})}{N(r,\frac{1}{g}) - O(1)} \tag{4.9}$$

and $g(z)$ is transcendental of order $\rho(g) < 1$ (see [7] and (4.8)), we get

$$\lim_{r\to\infty} N\left(r,\frac{1}{g}\right) = \infty, \qquad N\left(r,\frac{1}{g}\right) = o\bigl(T(r,G)\bigr), \quad r \to \infty. \tag{4.10}$$

Hence, by using (4.3), (4.10) and the Jensen formula, we see that the following inequalities

$$N\left(r,\frac{1}{f}\right) = m(r,f) - m\left(r,\frac{1}{f}\right) + O(1) \le (1+\varepsilon)T(r,G) \tag{4.11}$$

hold for all $r \ge r_3(\varepsilon)$. Let $r_0 = \max\{r_1(\varepsilon), r_2, r_3(\varepsilon)\}$. By (4.5) and (4.11), we see that (4.1) holds for all $r \ge r_0$. The proof of Lemma 4.1 is completed. □

*Proof of Theorem* 2 Firstly, we prove that (i) holds. Assume that $\delta(0,G) = 1$. Let $0 < \varepsilon < 1 - \rho(f)$ be a given constant. Since $N(r,G) = N(r,\frac{1}{f}) + O(1)$, by using (4.3) and (4.10), we get

$$N\left(r,\frac{1}{g}\right) = o\left(N\left(r,\frac{1}{f}\right)\right). \tag{4.12}$$

Rotate the zeros of $f(z)$ and $g(z)$ to the negative axis and form the canonical products $f_1(z)$ and $g_1(z)$. Obviously, $\rho(f_1) = \lambda(f_1) = \lambda(f) = \rho(f) < 1$. For an entire function $h(z)$, define

$$m_0(r,h) = \min_{|z|=r} \bigl|h(z)\bigr|.$$

By using the standard estimates for entire functions (see [22] or [12]), we have

$$\log m_0(r,g_1) \le \log m_0(r,g) \le \log M(r,g) \le \log M(r,g_1), \tag{4.13}$$

$$\log m_0(r,f_1) + \log M(r,f_1) \le \log m_0(r,f) + \log M(r,f), \tag{4.14}$$

and

$$\log M(r,g_1) = r\int_0^\infty \frac{N(t,\frac{1}{g})}{(r+t)^2}\,dt, \qquad \log M(r,f_1) = r\int_0^\infty \frac{N(t,\frac{1}{f})}{(r+t)^2}\,dt. \tag{4.15}$$

Hence, by (4.12), (4.13) and (4.15), we see that

$$\log M(r,g) = o\bigl(\log M(r,f_1)\bigr), \quad r \to \infty. \tag{4.16}$$

To finish the proof of (i), we need to consider the following two cases.

Case 1. $\mu(f_1) < \frac{1}{2}$. By Lemma 2.2, there exists a set $E_1 \subset (0,\infty)$ with $\overline{\log dens} E_1 \ge 1 - \frac{\mu(f_1)}{\alpha_1}$, $\alpha_1 = \frac{\mu(f_1)+\frac{1}{2}}{2}$, such that

$$\cos \pi\alpha_1 \log M(r,f_1) < \log m_0(r,f_1) \tag{4.17}$$

holds for all $r \in E_1$. Set $E = E_1$. It follows from (4.14) and (4.17) that

$$\log M(r,f_1) \le 2\log M(r,f), \quad r \in E_1. \tag{4.18}$$

Case 2. $\frac{1}{2} \le \mu(f_1) < 1$. By Lemma 2.2, there exists a set $E_2 \subset (0,\infty)$ with $\overline{\log dens} E_2 \ge 1 - \frac{\mu(f_1)}{\alpha_2}$ such that

$$-\log m_0(r,f_1) < c\log M(r,f_1), \quad r \in E_2, \tag{4.19}$$

where $0 < c = -\cos \pi\alpha_2 < 1$ and $\alpha_2 = \frac{\mu(f_1)+1}{2}$. Set $E = E_2$. It follows from (4.14) and (4.19) that

$$\log M(r,f_1) \le \frac{2}{1-c}\log M(r,f), \quad r \in E_2. \tag{4.20}$$

Hence, by using (4.16), (4.18) and (4.20), we deduce that there exists a set $E \subset (0,\infty)$ with $\overline{\log dens} E > 0$, such that

$$\log M(r,g) = o\bigl(\log M(r,f)\bigr), \quad r \to \infty, r \in E. \tag{4.21}$$

Using (4.1) and noting that $T(r,f) - T(r,g) - O(1) \le T(r,G)$, we see that the following inequalities

$$m\left(r,\frac{1}{f}\right) - (1-\varepsilon)O(1) \le \varepsilon T(r,f) + (1-\varepsilon)T(r,g) \tag{4.22}$$

hold for all $r \geq r_0(\varepsilon)$. Hence, there exists a constant $r_1(\varepsilon) > r_0(\varepsilon)$ satisfying $M(r_1,f) > 1$ and $M(r_1,g) > 1$. By using (4.21) and (4.22), we deduce that

$$m\left(r,\frac{1}{f}\right) - (1-\varepsilon)O(1) \leq \varepsilon \log M(r,f) + (1-\varepsilon)\big(o\big(\log M(r,f)\big)\big) \tag{4.23}$$

hold for $r \in E$ and $r \geq r_1(\varepsilon)$. As $\varepsilon$ can be arbitrarily small, we get

$$m\left(r,\frac{1}{f}\right) = o\big(\log M(r,f)\big), \quad r \to \infty, r \in E. \tag{4.24}$$

This gives (i).

In order to prove (ii), we suppose contrarily that $\delta(0,G) = 1$ and seek a contradiction. From (i), we know that there exists a set $E \subset (0,\infty)$ with $\overline{\log dens}E > 0$ satisfying (4.24). Assume that the zero is a deficient value of $f(z)$ with deficiency $\delta(0,f) = \delta > 0$. It follows from definition of deficiency, we easily find that, for all sufficiently large $r$

$$m\left(r,\frac{1}{f}\right) > \frac{\delta}{2} m(r,f). \tag{4.25}$$

By (4.24) and (4.25), we have

$$T(r,f) = m(r,f) = o\big(\log M(r,f)\big), \quad r \to \infty, r \in E. \tag{4.26}$$

Obviously, for an entire function $f(z)$, the following inequalities (see [3] or [18])

$$T(r,f) \leq \log M(r,f) \leq 3T(2r,f) \tag{4.27}$$

hold for all sufficiently large $r$. Set

$$E_3 = \big\{t | T(4t,f) \leq 4^{H_0} T(r,f)\big\}. \tag{4.28}$$

By using Lemma 2.5, we have

$$\underline{\log dens}E_3 \geq 1 - \frac{\rho(f)}{H_0},$$

where $H_0 = (\overline{\log dens}E)^{-1}\rho(f) + 1 > \rho(f)$. Set $E_4 = E \cap E_3$. By a simple computation, we can get

$$\overline{\log dens}E_4 \geq \overline{\log dens}E - \frac{\rho(f)}{H_0} > 0.$$

Hence, by (4.26), (4.27) and (4.28), we obtain

$$1 \leq 4^{H_0} 3o(1), \quad r \to \infty, r \in E_4. \tag{4.29}$$

Obviously, (4.29) gives a contradiction and the proof of Theorem 2 is completed. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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